בוחן מד"ר א 104285   3 ביולי 2014

3 שעות, 7 שאלות, 14 נקודות לכל אחת, 3 נקודות על השתתפות

רשמו תשובות סופיות ודרך פתרון (בקצרה).

תשובה סופית בלבד לא תזכה בנקודות!.

חומר פתוח. טעות חישוב מחשבון או מחשב זאת טעות שלכם.

ת.ז.   בהצלחה!   קוד גרסה: 212AB

PROBLEM	POINTS (MAX 14)
1	
2	
3	
4	
5	
6	
7	
TO ADD	3
GRADE FOR THE TEST (MAX 101)	
MIDTERM	
HW	
GRADE FOR THE COURSE	

.1

א. יהי $x = x(t)$ **פתרון של המשוואה**

$$x' = -ln\left((x-1)^2+1\right) \qquad (1)$$

המוגדר בקטע $t \in (\alpha, \beta)$. **הוכיחו כי יש פתרון** $\tilde{x}(t)$ **של אותה משוואה (1) המוגדר לכל** $t \in \mathbf{R}$ **וכך ש־** $\tilde{x}(t) = x(t)$ **לכל** $t \in (\alpha, \beta)$.

ב. יהי $x = x_1(t)$ **פתרון של המשוואה (1) המקיים את התנאי** $x(0) = a$ **ומוגדר לכל** $t \in \mathbf{R}$ **ויהי** $x = x_2(t)$ **פתרון של אותה משוואה (1) המקיים את התנאי** $x_2(0) = x_1(T) - \epsilon$ **ומוגדר לכל** $t \in \mathbf{R}$.

הוכיחו כי לכל $0 < \epsilon < 1$ **ולכל** $a > 1$ **יש** $T^* > 0$ **כך ש־ אם** $T > T^*$ **אז יש** $t^* > 0$ **כך ש־** $x_2(t^*) = 0$.

הוכחה של א:

The function $-ln\ ((x-1)^2+1)$ belongs to $C^1(\mathbf{R})$, therefore we can apply the theorems for equations of the form $x' = f(x) \in C^1(\mathbf{R})$. According to these theorems, if $\int_{x_0}^{\pm\infty} \frac{dx}{f(x)}$ diverges for any x_0 such that the interval $[x_0, \pm\infty]$ contains no singular points, then any solution can be prolonged to the whole t-axes. In our case the only singular point is $x = 1$ and the integrals $\int_{x_0}^{\infty} \frac{dx}{f(x)}$ with $x_0 > 1$ and $\int_{x_0}^{-\infty} \frac{dx}{f(x)}$ with $x_0 < -1$ diverge, because $ln\ ((x-1)^2+1) < x$ for sufficiently big $|x|$.

הוכחה של ב:

We have the only singular point $x = 1$ and we see from the equation that any non-constant solution is a decreasing function. The solution $x_1(t)$ satisfying $x_1(0) = a > 1$ tends to the singular point $x = 1$ as $t \to \infty$. Therefore there exists T^* such that $1 < x_1(t) < 1 + \epsilon/2$ if $t > T^*$. It follows that for $T > T^*$ we have $0 < x_2(0) < 1$. Since $x_2(0) < 1$ the solution $x_2(t)$ tends to $-\infty$ as $t \to \infty$ and since $x_2(0) > 0$ we have a unique point t^* such that $x_2(t^*) = 0$.

2. יהי $x = x(t)$ פתרון של המשוואה

$$x' = \frac{(x^4-1)(t-1)}{t^2+1}$$

המקיים את התנאי $x(0)=0$ ומוגדר בקטע מקסימלי אפשרי (t^-, t^+).

א. הוכיחו כי $t^+ = \infty,\ t^- = -\infty$

ב. יהי $lim_{t\to\infty} x(t) = a,\ lim_{t\to-\infty} x(t) = b$. הקיפו היגדים נכונים:

$a = 1$ $\qquad -1 < a < 1$ $\qquad a = -1$

$b = 1$ $\qquad 0 \le b < 1$ $\qquad -1 < b < 0$ $\qquad b = -1$

Part A. We have constant solutions $x(t) \equiv 1$ and $x(t) \equiv -1$. By the uniqueness theorem, for OUR solution (satisfying $x(0) = 0$) we have $-1 < x(t) < 1$ for any $t \in (t^-, t^+)$. Therefore there exist *finite* limits of $x(t)$ as $t \to t^{\pm}$. If t^+ or t^- is finite it would contradict to the theorem on prolongation of solutions.

Part B. The right answers is $a = b = -1$.

The proof is as follows. Since $-1 < x(t) < 1$ the function $x^4(t) - 1$ takes negative values for all t and it follows that $x(t)$ increases as $t < -1$ and decreases as $t > 1$ (we have maximum as $t = 1$). It follows that $-1 \le b < 0$ and $-1 \le a < 1$. We have to determine if $b = -1$ or $b > -1$, and we have to determine if $a = -1$ or $a > -1$. In this purpose we analyze the relation between $x(t)$ and t:

$$\int_0^{x(t)} \frac{ds}{s^4-1} = \int_0^t \frac{s-1}{s^4+1}.$$

Taking in this equation the limit as $t \to \pm\infty$ we obtain

$$\int_0^a \frac{dx}{x^4-1} = \int_0^\infty \frac{t-1}{t^2+1},\quad \int_0^b \frac{dx}{x^4+1} = \int_0^{-\infty} \frac{t-1}{t^4+1}.$$

Since $\int_0^{\pm\infty} \frac{t-1}{t^2+1} = \infty$ it follows $\int_0^a \frac{dx}{x^4-1} = \infty$ and $\int_0^b \frac{dx}{x^4-1} = \infty$ which implies $a = b = -1$.

.3 יהי $x=x(t)$ פתרון של המשוואה $x''=-sin\,x$ המקיים את התנאים

$$x(1)=\frac{\pi}{2},\ \ x'(1)=-1$$

ומוגדר לכל $t\in\mathbf{R}$. מצאו את הקואורדינטות (t^*,x^*) של אחת מנקודות המקסימום המקומי של $x(t)$.

<u>תשובה</u>:

$$x^*=\frac{2\pi}{3},\qquad t^*=\int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}}\frac{1}{\sqrt{1+2cos\,x}}dx+\int_{-\frac{2\pi}{3}}^{\frac{\pi}{2}}\frac{1}{\sqrt{1+2cos\,x}}dx$$

<u>דרך פתרון</u> :

The energy equation gives $\frac{(x'(t))^2}{2}-cos(x(t))=const=(t=1)=\frac{1}{2}$. From here it is easy to determine that the initial velocity is less than the critical velocity so that the solution $x(t)$ is a periodic function. Since $x'(0)<0$ there exists $t_1>0$ such that the solution $x(t)$ decreases for $t\in(0,t_1)$ and we have a point of local minimum at $t=t_1$. Let $x(t_1)=x_{min}$. Then $x'(t_1)=0$ and substituting t_1 to the energy equation we obtain $cos\,x_{min}=-\frac{1}{2}$ and it follows $x_{min}=-\frac{2\pi}{3}$. By the lemma on symmetries we have $x_{max}=x^*=\frac{2\pi}{3}$. The energy equation implies:

$$x'(t)=\sqrt{1+2cos\,x(t)},\quad t\in[0,t_1],\qquad x'(t)=-\sqrt{1+2cos\,x},\quad t\in[t_1,t^*]$$

For the inverse functions we have

$$t'(x)=\frac{1}{\sqrt{1+2cos\,x(t)}},\ \ x\in[\frac{\pi}{2},-\frac{2\pi}{3}],\quad t'(x)=-\frac{1}{\sqrt{1+2cos\,x(t)}},\ \ x\in[-\frac{2\pi}{3},\frac{2\pi}{3}]$$

It follows

$$t_1=t(-\frac{2\pi}{3})=t(\frac{\pi}{2})+\int_{\frac{\pi}{2}}^{-\frac{2\pi}{3}}-\frac{1}{\sqrt{1+2cos\,x}}dx$$

$$t^*-t_1=t(\frac{2\pi}{3})-t(-\frac{2\pi}{3})=\int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}}\frac{1}{\sqrt{1+2cos\,x}}dx$$

which implies the answer (one should use that $t(\frac{\pi}{2})=1$).

4. מצאו את אוסף כל הפתרונות הממשיים של המשוואה

$$P\left(\frac{d}{dt}\right)(x(t)) = sin\,t + cos(2t), \quad P(\lambda) = (\lambda^2+1)^2(\lambda^2-1).$$

בתוצאות הסופיות צריכים להופיע רק מספרים ממשיים..

תשובה:

$$x(t) = C_1 sin\,t + C_2 cos\,t + C_3 t \cdot sin\,t + C_4 t \cdot cos\,t + C_5 e^t + C_6 e^{-t} + x^*(t)$$

$$x^*(t) = \frac{t^2}{16} sin\,t - \frac{1}{45} cos(2t)$$

דרך פתרון :

The general solution of the homogeneous equation is the first part of the answer (without $x^*(t)$ because $P(\lambda)$ has roots ± 1 of multiplicity 1 and roots $\pm i$ of multiplicity 2. A partial solution $x^*(t)$ is $x^*(t) = Im(x_1^*(t)) + Re(x_2^*(t))$ where $x_1^*(t)$ is a partial solution of the equation $P\left(\frac{d}{dt}\right)(x(t)) = e^{it}t$ and $x_2^*(t)$ is a partial solution of the equation $P\left(\frac{d}{dt}\right)(x(t)) = e^{2it}$. Since i is a root of $P(\lambda)$ of multiplicity 2 and $2i$ is not a root of $P(\lambda)$ we have

$$x_1^*(t) = \frac{t^2 e^{it}}{P''(i)}, \quad x_2^*(t) = \frac{e^{2it}}{P(2i)}.$$

We have $P(2i) = -45$. To compute $P''(i)$ it is worth to express $P(\lambda)$ in the form $P(\lambda) = (\lambda - i)^2 \cdot Q(\lambda)$, where $Q(\lambda) = (\lambda + i)^2(\lambda^2 - 1)$. We have $P''(i) = 2!Q(i) = 16$. Therefore

$$x^*(t) = Im\left(\frac{t^2 e^{it}}{16}\right) + Re\left(\frac{e^{2it}}{-45}\right)$$

and the answer follows.

5. נתון:

$$T = \begin{pmatrix} 0 & 0 & 2 & 1 \\ 1 & 3 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 5 \end{pmatrix}, \quad T^{-1}AT = J = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

מצאו דוגמא לבסיס של מרחב כל הפתרונות של המערכת

$$x' = Ax, \;\; x = x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix}.$$

תשובה:

$$\begin{pmatrix} 2sint + cost \\ 2cost \\ cost \\ 5cost \end{pmatrix}, \begin{pmatrix} 2cost - sint \\ -2sint \\ -sint \\ -5sint \end{pmatrix}, \begin{pmatrix} 0 \\ te^{3t} + 3e^{3t} \\ te^{3t} \\ te^{3t} \end{pmatrix}, \begin{pmatrix} 0 \\ e^{3t} \\ e^{3t} \\ e^{3t} \end{pmatrix}$$

דרך פתרון :

Let $x(t) = Ty(t)$. Then for $y(t)$ we have $y' = Jy$, i.e.

$$y_1' = 3y_1 + y_2, \;\; y_2' = 3y_2, \;\; y_3' = y_4, \;\; y_4' = -y_3.$$

It follows

$$\begin{pmatrix} y_3 \\ y_4 \end{pmatrix} = C_1 \begin{pmatrix} sint \\ cost \end{pmatrix} + C_2 \begin{pmatrix} cost \\ -sint \end{pmatrix}, \quad y_2 = C_3 e^{3t}$$

and for $y_1(t)$ we have the equation $y_1' = 3y_1 + C_3 e^{3t}$. Solving this equation by the method of variation of the constant we obtain $y_1 = C_3 te^{3t} + C_4 e^{3t}$. We obtain the general solution $y(t) = C_1 y^{(1)}(t) + \cdots + C_4 y^{(4)}(t)$, where

$$y^{(1)}(t) = \begin{pmatrix} 0 \\ 0 \\ sint \\ cost \end{pmatrix}, \; y^{(2)}(t) = \begin{pmatrix} 0 \\ 0 \\ cost \\ -sint \end{pmatrix}, \; y^{(3)}(t) = \begin{pmatrix} te^{3t} \\ e^{3t} \\ 0 \\ 0 \end{pmatrix}, \; y^{(4)}(t) = \begin{pmatrix} e^{3t} \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

It follows that $y^{(1)}(t), y^{(2)}(t), y^{(3)}(t), y^{(4)}(t)$ is a basis of the vector space of all solutions of the system $y' = Jy$. A basis for the system $x' = Ax$ is $x^{(1)}(t) = Ty^{(1)}(t), x^{(2)}(t) = Ty^{(2)}(t), x^{(3)}(t) = Ty^{(3)}(t), x^{(4)}(t) = Ty^{(4)}(t)$ and we obtain the answer above.

6. תהי

$$A = \begin{pmatrix} 3 & a \\ 1 & -3 \end{pmatrix}, \quad x = \begin{pmatrix} x_1(t) \\ x_2(t)) \end{pmatrix}.$$

א. ציירו את התמונה הפזית של המערכת $x' = Ax$ ל־ $a = 0$ ול־ $a = -10$

ב. יהי $a = -10$. מצאו מספרים ממשיים r_1, r_2, r_3 כך ש־ $(r_1, r_2, r_3) \neq (0,0,0)$ וכך ש־

$$r_1 x_1^2(t) + r_2 x_1(t) x_2(t) + r_3 x_2^2(t) \equiv const$$

לכל פתרון של המערכת $x' = Ax$.
בתוצאות הסופיות צריכים להופיע רק מספרים ממשיים.

<u>תשובות:</u>

א. If $a = 0$ the phase portrait is a saddle with the stable line $span(0,1)$ and unstable line $span(6,1)$. The students should draw correct phase portrait with correct arrows. If $a = -10$ the phase portrait is a center with anticlockwise rotation. The students should draw the ellipses at least roughly correct.

ב.

$r_1 = 1$ $\qquad$ $r_2 = -6$ $\qquad$ $r_3 = 10$

<u>**ב:** דרך פתרון :</u>

The given requirement of r_1, r_2, r_3 is equivalent to the condition

$$\frac{d}{dt}\left(r_1 x_1^2(t) + r_2 x_1(t) x_2(t) + r_3 x_2^2(t)\right) \equiv 0$$

Differentiating and taking into account that $(x_1(t), x_2(t))$ is a solution of the system $x' = Ax$ we obtain

$$2r_1 x_1(3x_1 - 10x_2) + r_2 x_2(3x_1 - 10x_2) + r_2 x_1(x_1 - 3x_2) + 2r_3 x_2(x_1 - 3x_2) \equiv 0$$

for any solution $(x_1, x_2) = (x_1(t), x_2(t))$ of the system $x' = Ax$. This condition is equivalent to the system

$$6r_1 + r_2 = 0, \ -20r_1 + 2r_3 = 0, \ -10r_2 - 6r_3 = 0$$

We a priori know that this system has a non-trivial solution, and it remains to solve it.

7. מצאו פתרון של המשוואה

$$x' = \frac{x}{t} + \frac{x^{2/3}}{t^2}, \ \ t > 0$$

המקיים את התנאי $x(1) = 1$. <u>אין אינטגרלים בתשובה סופית</u>.

<u>תשובה</u>:

$$x(t) = \left(\frac{5}{4}t^{1/3} - \frac{1}{4t}\right)^3$$

<u>דרך פתרון :</u>

Let $y(t) = (x(t))^\alpha$. Then

$$y' = \alpha x^{\alpha-1}x' = \alpha x^{\alpha-1}\left(\frac{x}{t} + \frac{x^{2/3}}{t^2}\right) = \frac{\alpha y}{t} + \frac{\alpha x^{\alpha-1/3}}{t^2}$$

and we see that taking $\alpha = 1/3$, i.e. $y(t) = (x(t))^{1/3}$, we obtain a linear equation for $y(t)$:

$$y'(t) = \frac{1}{3}\left(\frac{y}{t} + \frac{1}{t^2}\right)$$

We have

$$y(t) = Ct^{1/3} + y^*(t)$$

where $y^*(t)$ is a partial solution which can be found by the method of variation of the constant:

$$y^*(t) = C(t)t^{1/3}, \ \ C'(t)t^{1/3} = \frac{1}{3t^2}.$$

We obtain

$$C'(t) = \frac{1}{3}t^{-7/3}, \ C(t) = -\frac{1}{4}t^{-4/3},$$

$$y^*(t) = -\frac{1}{4t}, \ y(t) = Ct^{1/3} - \frac{1}{4t}, \ x(t) = y^3(t) = \left(Ct^{1/3} - \frac{1}{4t}\right)^3$$

We have $x(1) = \left(C - \frac{1}{4}\right)^3 = 1$ whence $C = 5/4$.