ORDINARY DIFFERENTIAL EQUATIONS: 104285

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Contents

CHAPTER 1

Few examples

1.1. A body moves with a constant velocity

Assume that a body moves along a straight line with a constant velocity v, see fig. 1.1. Let x be the coordinate on the line and let t be the time. Let $x(t)$ be the coordinate of the body at the time t. Then

$$x'(t) = v. \tag{1.1.1}$$

It is an equation: the unknown is the function $x(t)$. It is a differential equation since it involves the derivative of the unknown function. It is a differential equation of order 1 since it involves the first derivative and does not involve higher order derivatives. It is an ordinary differential equation (ODE) because the unknown function depends on only one variable t. [1] Usually, though not always, this variable in ODEs is the time.

Definition. A solution of an ODE is a function satisfying this equation and defined on on open interval (a, b) (the case $a = -\infty$ and/or $b = \infty$ is not excluded). The set of all solutions on (a, b) is called the general solution on (a, b).

It is clear that equation (1.1.1) has infinitely many solutions defined on $(-\infty, \infty)$, namely

$$x(t) = vt + C, \quad C \in \mathbb{R}. \tag{1.1.2}$$

It is easy to prove that there are no other solutions defined on $(-\infty, \infty)$, so that (1.1.2) is the general solution of (1.1.1). It is parameterized by a "free" constant $C \in \mathbb{R}$.

Certainly the reason for infinitely many solutions is the unknown initial position of the body. If we know the location of the body at any fixed time t_0 we will know its location at any time t. Mathematically the initial position of the body is the initial condition for equation (1.1.1), it is

$$x(t_0) = x_0, \tag{1.1.3}$$

where t_0 and x_0 are given numbers. Substituting $t = t_0$ to (1.1.2) we obtain

$$C = x_0 - vt_0$$

and consequently (1.1.1) has unique solution satisfying the initial condition (1.1.3):

$$x(t) = x_0 + v(t - t_0).$$

[1] if the unknown function depends on several variables and there are derivatives with respect to ≥ 2 variables then the equation is called partial differential equation (PDE). An example of PDE is the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ with respect to the unknown function $u = u(x, y)$.

1.2. Simplest version of the two body problem

I hope that some of the students will have to solve, in few years, the following two body problem: what to do if your girlfriend (boyfriend) is offered a good job in Tel Aviv (or USA) and you – in Haifa, or vise a versa. I hope you will solve this problem somehow. I also hope that you will not deal with a three body problem in this sense.

In this course we will deal with the following two body problem: there are two bodies on a straight line, the big one does not move and located all the time at the point $x = 0$, the small body moves along the straight line under the force of attraction to the big body. We assume that this force is $F = F(x) > 0$, where $x = x(t)$ is the coordinate of the small body at the time t, i.e. the force depends on the distance between the bodies only. We also assume that the initial velocity v_0 of the small body is directed opposite to the force F. See fig. 1.2.

The principal questions to be answered are as follows: will the bodies meet or the small body moves to ∞ as time $t \to \infty$? Is anything else possible?

To answer these and many other questions one has to solve the ODE describing the movement of the small body. If m is its mass then $f = f(x) = F(x)/m$ is its acceleration and we obtain

$$x''(t) = -f(x(t)). \tag{1.2.1}$$

The sign $-$ in the equation corresponds to the fact that the direction of the force is opposite to the positive direction of the x-axes and consequently the acceleration is negative. We obtain a second order ODE since the equation involved the second derivative and does not involve higher order derivatives. In what follows we will give a way of solving equations of this form. The equation has infinitely many solutions because it does not contain information about initial position of the small body and about its initial velocity. The initial conditions for the second order ODE are conditions of the form

$$x(t_0) = x_0,\ x'(t_0) = v_0, \tag{1.2.2}$$

where t_0, x_0 and v_0 are given numbers. If the initial conditions (1.2.2) are fixed and the function $F(x)$ is "good enough" (in what follows we will explain what does it mean) then we have uniqueness of solution of (4.5.1) in the following sense: two solutions defined on the same time-interval $t \in (a, b)$ and satisfying the same initial conditions (1.2.2) are the same solution.

1.2.1. Particular case: a stone thrown up. In this case $f(x) = g$ so we have the equation

$$x''(t) = -g.$$

This equation can be easily solved. The general solution defined for $t \in \mathbb{R}$ is

$$x(t) = C_1 + C_2 t - \frac{1}{2} g t^2, \quad C_1, C_2 \in \mathbb{R}.$$

It is parameterized by two constants C_1, C_2 which are uniquely determined by the initial conditions (1.2.2). For example, if $t_0 = 0$ then $C_1 = x_0$ and $C_2 = v_0$.

1.2.2. Particular case: a rocket launched vertically up. In this case the force is

$$F(x) = \frac{kMm}{x^2}$$

where x is the distance from the rocket to the center of the Earth, M is the mass of the Earth, m is the mass of the rocket, and k is a certain gravitation constant. See fig. 1.3. Consequently the acceleration is

$$f(x) = \frac{kM}{x^2}.$$

Let R be the radius of the Earth. Then $f(R) = \frac{kM}{R^2}$. On the other hand we know that $f(R)$ is the acceleration at the surface of the Earth, therefore $f(R) == g$ and it follows $kM = gR^2$. We obtain $f(x) = \frac{gR^2}{x^2}$. Therefore the ODE describing the movement of the rocket is as follows:

$$x''(t) = -\frac{gR^2}{x^2(t)}, \tag{1.2.3}$$

where $x(t)$ is the distance from the rocket to the center of the Earth at time t. If the rocket is launched from the surface of the Earth with initial velocity v_0 we have the initial conditions $x(t_0) = R,\ x'(t_0) = v_0$. Under which condition on v_0 the rocket will return to the Earth? Under which conditions it will reach the height 10000 km? Which velocity will it have at this height? Will something principally change if in the gravitation law we had x^r with $r \neq 2$ in the denominator? In what follows we will answer all these and many other related questions.

1.3. Interest rate and interest yield

In 1998 for those who had a saving account in Bank of America the interest rate was 6 % and the interest yield 6.18 % a year. What does it mean? [2]

Assume you put to your saving account A dollars. How much money will be in the account in a year, $1.06A$ or more? It should be more. If they give 6 % for one year they must give 3 % for 1/2 year, i.e. in 1/2 year there should be $1.03A$ dollars. And in 1/2 year more there should be $1.03A$ plus 3 % of $1.03A$ which gives $1.0609A$. But you can count differently to get even more than that assuming that you open a new account and put to it all what you had in the previous account every 3 months, or better every month, every week, day, hour, minute, second. The limit of this procedure is an ODE. If your account has $x(t)$ dollars at tine t it should have $x(t)(1+0.06\epsilon)$ dollars at time $t+\epsilon$ where t and ϵ are measures in years. Therefore $x(t+\epsilon) = x(t) + 0.06\epsilon x(t)$. Write this in the form $\frac{x(t+\epsilon)-x(t)}{\epsilon} = 0.06x(t)$ and take the limit as $\epsilon \to 0$. We obtain the first order ODE

$$x'(t) = 0.06x(t) \tag{1.3.1}$$

It is easy to check that it has solutions

$$x(t) = Ce^{0.06t}, \quad C \in \mathbb{R}$$

and one can prove that there are no other solutions. We have the initial condition $x(t_0) = A$ where t_0 is the time the account was opened. This initial condition

[2] in 2013 the interest rate and the interest yield are the same 0.2 %.

determines $C = 1000e^{-0.06t_0}$. Therefore there is unique solution satisfying the initial condition

$$x(t) = 1000e^{0.06(t-t_0)}.$$

It means that in one year your account, when $t - t_0 = 1$, your account should have

$$x(t_0 + 1) = Ae^{0.06} \approx 1,0618A$$

dollars. It is 6.18 % more than the initial deposit A. This is why the interest yield is 6.18 %.

1.4. Growth of population

If there are no wars or another crime then the growth (or decrease) of population is described by the same equation as in section 1.3, the equation

$$x'(t) = kx(t) \tag{1.4.1}$$

where $x(t)$ is the number of citizens at time t (measured, for example, in years) and k is a very important constant (if $k > 0$ the population increases, if $k < 0$ it decreases). If b people a year die because of crime the equation is different:

$$x'(t) = kx(t) - b. \tag{1.4.2}$$

To solve this equation let us note that one of solutions is

$$x^*(t) \equiv \frac{b}{k}.$$

Note also that if $x(t)$ is any solution then the function $y(t) = x(t) - x^*(t)$ satisfies the equation $y'(t) = ky(t)$. Therefore $x(t) - x^*(t) = Ce^{kt}$ for some $C \in \mathbb{R}$. It follows that the general solution of equation (1.4.2) is

$$x(t) = \frac{b}{k} + Ce^{kt}, \quad C \in \mathbb{R}.$$

The constant C can be found from the initial condition $x(t_0) = x_0$ with given t_0 and x_0.

1.5. Abnormal growth of population

Is it possible that the velocity of the growth of population (of cats or bacterium, etc.) is proportional to the square of the number of the members of population? Assume it is so, then the number $x(t)$ of the members of population changes according to the equation

$$x'(t) = kx^2(t), \quad k > 0. \tag{1.5.1}$$

This equation can be solved as follows. Note that any solution $x(t)$ is a growing function because its derivative is positive for all t. Therefore there exists the <u>inverse function</u> $t(x)$, see fig. 1.4. According to the theorem on the derivative of the inverse function we have

$$t'(x) = \frac{1}{kx^2}.$$

Integrating we obtain

$$t(x) = -\frac{1}{kx} + C$$

for some constant $C \in \mathbb{R}$. We can find C from the initial condition $x(t_0) = x_0$ or equivalently $t(x_0) = t_0$. Substituting $x = x_0$ we obtain $C = t_0 + \frac{1}{kx_0}$. Therefore

$$t(x) = -\frac{1}{kx} + t_0 + \frac{1}{kx_0}$$

and from here

$$x(t) = \frac{1}{k\left(t_0 - t + \frac{1}{kx_0}\right)}.$$

We see that

$$x(t) \to \infty \text{ as } t \to t_0 + \frac{1}{kx_0}.$$

It means that the number of members of population is infinite in a finite time. It means that the velocity of the growth of population can be proportional to the square of the members of population only for a finite time, smaller than $\frac{1}{kx_0}$.

Returning to math language we can state the following: equation (1.5.1) has no silutions satisfying the initial condition $x(t_0) = x_0$ and defined on an interval (a, b) for any $a < t_0$ and $b > \frac{1}{kx_0}$.

1.6. Growth of population in the case of not enough food

Let $x(t)$ be the number of fishes in a lake at time t. The function $x(t)$ changes according to equation $x'(t) = f(x(t))$ where $f = f(x)$ is a certain function. What we can say about the function $f(x)$? Certainly $f(0) = 0$. There is a number A, corresponding to "full lake", such that if $x > A$ then there is not enogh food and some fishes start to die. Therefore $f(x) < 0$ for $x > A$. If $x < A$ there is enough food and $f(x) > 0$. The simplest function satisfying the above conditions is $f(x) = kx(A - x)$, $k > 0$. We obtain the equation

$$x'(t) = kx(t)\left(A - x(t)\right). \tag{1.6.1}$$

It is a very important equation and later we will analyze it. Two solutions satisfying the initial condition $x(t_0) = x_0$, one with $x_0 < A$ and one with $x_0 > A$ are showed in fig. 1.5.

1.7. A fight between two armies

Assume that the loss of soldiers is proportional to the number of soldiers in the enemy army and to the number or weapon in the enemy army. Assume also that the number of weapon in each of the armies does not change during the fight (which is certainly not the case for now-days fights, but could be so 300 years ago). Let w_1 be the number of weapon in the first army, w_2 the number of weapon in the second army. Let s_1 and s_2 be the number of soldiers in the first and the second army in the beginning of the fight. Assume

$$w_2 = 2w_1,\ s_2 = \frac{1}{2}s_1.$$

Which of the armies will win the fight? In other words, what is more important - weapon of soldiers under the assumptions above?

To formulate the question mathematically denote by $x_1(t)$ and $x_2(t)$ the number of soldiers in the first and the second army at time t. Then

$$
\begin{aligned}
x_1'(t) &= -kw_2x_2(t) \\
x_2'(t) &= -kw_1x_1(t).
\end{aligned}
\tag{1.7.1}
$$

Here k is a certain positive constant. We obtain a system of ODEs: two ODEs for two unknown functions $x_1(t)$ and $x_2(t)$. There are infinitely many solutions, but there is a unique solution satisfying our initial conditions

$$x_1(t_0) = s_1,\ x_2(t_0) = s_2.$$

This system can be solved, but to answer the question which of the armies will win the fight it is not necessary to solve it, it is enough to understand the phase portrait of the system. What is the phase portrait will be explained later. From the phase portrait it is clear that there are three possibilities: (a) there exists t_1 such that $x_1(t), x_2(t) > 0$ for all $t_0 < t < t_1$ and $x_1(t_1) > 0, x_2(t_1) = 0$; (b) there exists t_1 such that $x_1(t), x_2(t) > 0$ for all $t_0 < t < t_1$ and $x_1(t_1) = 0, x_2(t_1) > 0$; and (c) there exists t_1 such that $x_1(t), x_2(t) > 0$ for all $t_0 < t < t_1$ and $x_1(t_1) = x_2(t_1) = 0$. In case (a) the first army wins, in case (b) the second army wins. If one understands the phase portrait it is easy to determine under which relation between w_1, w_2, s_1, s_2 one has the case (a), (b), or (c). The answer does not depend on the constant k.

1.8. Exercises

1. How much money you will have in your pension bank account at the age 70 if you deposit 1000 shekels a year, the interest rate is all the time 2 % a year, and the bank counts your (a) with the interest yield (b) without interest yield. Assume that the interest of a pension bank account is not subject to state taxes.

2. Draw the graph of the solution of equation (1.4.2) satisfying the initial condition $x(0) = 10^6$ for the following values of the parameters k and b:

(a) $k = 0.02,\ b = 10^4$ (b) $k = 0.01,\ b = 2 \cdot 10^4$ (c) $k = -0.02, b = 1000$

3. Follow the argument with the inverse function in section 1.5 to find the maximal time-interval for the solution of the equation $x'(t) = x^2(t) + 1$ satisfying the initial condition $x(0) = 0$. Draw the graph of this solution.

4. Follow the argument with the inverse function in section 1.5 to draw in the same (t, x)-plane the graph of 6 functions: solutions of the equation $x'(t) = x^2(t)$ satisfying the initial conditions:
$x(0) = 1,\ x(1) = 1,\ x(2) = 1,\ x(0) = -1,\ x(1) = -1,\ x(2) = -1.$

5. Let (t_1, x_1) be the coordinates of the inflection point (nikudat pitul) in the graph of the solution of equation (1.6.1) showed in fig. 1.5. Find x_1.

6. Write down a system of ODEs for the rotation of the Earth about the Sun.

CHAPTER 2

Existence theorem, uniqueness theorem and theorem on prolongation of solutions for first order ODEs

2.1. Existence theorem

Consider the general form of first order ODE with given initial condition:

$$x' = f(t, x), \quad x = x(t) \tag{2.1.1}$$

$$x(t_0) = x_0. \tag{2.1.2}$$

Remind that by definition a solution of ODE is a function satisfying this equation and defined on an open interval in the t-axes. Does (2.1.1) have at least one solution satisfying (2.1.2)? Some property of the function $f(t, x)$ should b required for the positive answer. This property is very simple.

Theorem 2.1.1 (Existence theorem). *Assume that the function $f(t, x)$ is continuous in some neighborhood of the point (t_0, x_0). Then (2.1.1) has a solution satisfying (2.1.2).*

The proof is not simple and will be given in the end of the course if time allows.

Remark 2.1.2. Note that if (2.1.1) has a solution $x(t)$ satisfying (2.1.2) then it has infinitely many solutions satisfying the same initial condition. In fact, let I be the interval in the t-axes on which $x(t)$ is defined. Then $t_0 \in I$. Obviously the restriction of $x(t)$ to any smaller interval $\widetilde{I} \subset I$ containing the point t_0 is a solution of the same equation satisfying the same initial condition. See fig. 2.1.

The existence theorem gives no information on the maximal interval of definition of a solution of (2.1.1) satisfying (2.1.2). Even if $f(t, x)$ is continuous and moreover infinitely-differentiable in the whole (t, x)-plane the maximal interval might be very small.

Example 2.1.3. Let $x_0 > 0$ and $k > 0$. The maximal interval of definition of a solution of equation $x' = kx^2$ satisfying the initial condition $x(0) = x_0$ is $\left(-\infty, \frac{1}{kx_0}\right)$ (see section 1.5).

Example 2.1.4. Let $a > 0$. The maximal interval of definition of a solution of equation $x' = a(x^2 + 1)$ satisfying the initial condition $x(0) = 0$ is $\left(-\frac{\pi}{2a}, \frac{\pi}{2a}\right)$ (see exercise 3, section 1.8).

On the other hand there are many cases that the maximal interval is the whole t-axes. For example it is so for the equations (1.1.1), (1.4.1), (1.4.2) whatever is the initial condition (and the parameters of the equations). In Chapter 3 we will show that the maximal interval of definition of solution of equation (1.6.1) satisfying the initial condition $x(0) = x_0$ is the whole t-axes if and only if $0 \leq x_0 \leq A$.

2.2. Uniqueness theorem

One could ask if equation (2.1.1) has unique solution satisfying the initial condition (2.1.2). In view of Remark 2.1.2 it is not a good question. In view of the same remark a good question is as follows:

QUESTION 2.2.1. Let $x(t)$ and $\widetilde{x}(t)$ be two solutions of the same equation (2.1.1) satisfying the same initial condition (2.1.2) and defined on intervals I and $\widetilde{I}$. Is it true that $x(t) = \widetilde{x}(t)$ for any $t \in I \cap \widetilde{I}$?

The simplest sufficient condition is as follows.

THEOREM 2.2.2 (Uniqueness theorem 1). *Assume that a function $f(t, x)$ is continuous in some neighborhood U of the point (t_0, x_0) and differentiable with respect to x at any point of U. Assume also that the derivative $\frac{\partial f}{\partial x}$ is continuous function in U. Then the answer to Question 2.2.1 is positive.*

The assumptions of this theorem can be weakened. We need the following definition.

DEFINITION 2.2.3. We will say that a function $f(t, x)$ satisfies the Lipschitz condition at the point (t_0, x_0) with respect to x if there exists a neighborhood W of (t_0, x_0) and a constant $C > 0$ such that

$$|f(t, x_1) - f(t, x_2)| \leq C|x_1 - x_2| \text{ for any points } (t, x_1), (t, x_2) \in W. \tag{2.2.1}$$

THEOREM 2.2.4 (Uniqueness theorem 2). *Assume that a function $f(t, x)$ is continuous in some neighborhood U of the point (t_0, x_0) and satisfies the Lipschitz condition with respect to x at the point (t_0, x_0). Then the answer to Question 2.2.1 is positive.*

Theorem 2.2.4 is stronger than Theorem 2.2.2: the assumptions of Theorem 2.2.2 imply the Lipschitz condition.

THEOREM 2.2.2 FROM THEOREM 2.2.4. Let U be the neighborhood of (t_0, x_0) in Theorem 2.2.2. For small enough $\epsilon > 0$ the square

$$D = \{(t, x) : t \in [t_0 - \epsilon, t_0 + \epsilon], x \in [x_0 - \epsilon, x_0 + \epsilon]\}$$

belongs to U. Since the function $\frac{\partial f}{\partial x}$ is continuous in U it is bounded in D, i.e. there exists C such that $|\frac{\partial f(t,x)}{\partial x}| < C$ for any $(t, x) \in D$. For any points $(t, x_1), (t, x_2) \in D$ we have

$$|f(t, x_1) - f(t, x_2)| = \left| \int_{x_1}^{x_2} \frac{\partial f}{\partial x} dx \right|$$

and it follows that (2.2.1) holds for any neighborhood W of (t_0, x_0) which is contained in D. Q.E.D.

EXAMPLE 2.2.5. It is easy to see that the function $f(x) = |x|$ satisfies the Liptschitz condition at $x_0 = 0$, but it is not differentiable at $x_0 = 0$.

EXAMPLE 2.2.6. The continuous function $f(x) = \sqrt{|x|}$ does not satisfy the Liptschitz condition at $x_0 = 0$. In fact, if the Liptschitz condition was valid we would have a constant C such that $\sqrt{x} \leq Cx$ for sufficiently small $x > 0$ (tanking $x_1 = 0, x_2 = x > 0$). Certainly such C does not exist.

The next example shows that if the Liptschitz condition is not valid the answer to Question 2.2.1 on uniqueness might be negative.

EXAMPLE 2.2.7. It is easy to check that the equation $x' = \sqrt{|x|}$ has solutions

$$\text{(a)}\quad x(t) \equiv 0 \qquad \text{(b)}\quad x(t) = \begin{cases} 0, & t \leq 0 \\ \frac{1}{4}t^2, & t \geq 0 \end{cases} \qquad \text{(c)}\quad x(t) = \begin{cases} -\frac{1}{4}t^2, & t \leq 0 \\ \frac{1}{4}t^2, & t \geq 0 \end{cases}$$

See fig. 2.2. Each of this solutions satisfies the condition $x(0) = 0$.

If time allows we will prove the uniqueness Theorem 2.2.4 along with the existence Theorem 2.1.1 in the end of the course.

2.3. Illustrating example. From home to work with a stop

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I live in 4km from Technion. Assume I go home from Technion by a straight line with velocity $v = v(x)$ which depends on my current place $x = x(t)$ only as it is showed in fig. 2.3. Will I reach Mirkaz Ziv in a finite time? If yes, will I reach my home in a finite time or maybe I will be “freezed” at Ziv since my velocity there is 0? One can think about several solutions showed at fig. 2.4.a (I will never reach Ziv), fig. 2.4.b (I will reach Ziv and immediately after that will go home), at fig. 2.4.c (I will reach Ziv and will not move further), at fig. 2.4.d (I will reach Ziv, will spend night there, after that will go home). Which of these solutions holds?

We have the equation $x' = f(x)$ where $f(x)$ is a continuous function in fig. 2.3. We have $\underline{f(2) = 0}$ and it follows that the equation has the solution $\mathbf{x}(t) \equiv 2$ which means that I stay in Ziv all my life. This constant solution $\mathbf{x}(t) \equiv 2$ does not satisfy the initial condition $x(0) = 0$ corresponding to the fact that I start at Technion. But the existence of this constant solution (showed by dashed line in the figures) implies that solutions at fig. 2.4.b, 2.4,c, 2.4.d are impossible <u>provided we have uniqueness</u>. In fact, for any of these solutions there exists a time t_1 such that $x(t_1) = 2$ and we also have $\mathbf{x}(t_1) = 2$ because $\mathbf{x}(t) = 2$ for all t. Therefore we have contradiction to the uniqueness theorem at the point $(t = t_1, x = 2)$.

It follows that if the function $f(x)$ has continuous derivative then I will never reach Ziv (see Theorem 2.2.2). If it is not differentiable, but satisfies the Liptschitz condition I will neither reach Ziv (see Theorem 2.2.2). See fig. 2.5.

What happens if $f(x)$ satisfies the the Liptschitz condition at all points except the point $x = 2$? Take an example $f(x) = 2\sqrt{|4 - 2x|}$, see fig. 2.6. This example is almost the same as Example 2.2.7. Like in that example any of the solutions in figures fig. 2.4.b, 2.4,c is possible; fig. 2.4.d is also possible (see exercise 3 in section 2.5). Which of them holds? The couple (equation, initial condition) <u>does not know about that!</u> And what about fig. 2.4.a, is it possible in this case? A bit of analysis based on the inverse function, like in section 1.5 shows that it is impossible: if $f(x) = 2\sqrt{|4 - 2x|}$ I will reach Ziv in a finite time.

2.4. Theorem on prolongation of solutions

Let $x(t)$ be a solution of the equation $x' = f(t, x(t))$ defined on the time-interval $t \in (a, b)$. Does it have a prolongation to an interval $(\widetilde{a}, \widetilde{b})$ containing (a, b)? Prolongation means a solution $\widetilde{x}(t)$ of the same equation defined on $(\widetilde{a}, \widetilde{b})$ such that $x(t)$ is the restriction of $\widetilde{x}(t)$, i.e. $x(t) = \widetilde{x}(t)$ for any $t \in (a, b)$. We already had several examples showing that in some cases a solution $x(t)$ cannot be prolonged, i.e. the interval of its definition is maximal possible.

If there is a prolongation of $x(t)$ defined on (a, b) to $(a, \widetilde{b})$ with $\widetilde{b} > b$ we will say that it is prolongation to the right. Similarly if there is a prolongation to $(\widetilde{a}, b)$ with $\widetilde{a} < a$ we will say that it is prolongation to the left. See fig. 2.7.

It is clear that there is no prolongation to the right if $lim_{t \to b} x(t) = \pm\infty$ and there is no prolongation to the left if $lim_{t \to a} x(t) = \pm\infty$. See fig. 2.8. Are there other obstacles for prolongation or it is the only reason? It turns out that there are no other obstacles for prolongation.

THEOREM 2.4.1 (Theorem on prolongations of solutions to the right). *Let $x(t)$ be a solution of the equation $x' = f(t, x)$ defined on the interval $t \in (a, b)$. Assume that b is a finite number and assume that the function $f(t, x)$ is continuous in a neighborhood of any point of the vertical line $\{t = b, \forall x\}$ and satisfies the Liptschitz condition at any point of this line. Then the following holds:*

1. There exists $lim_{t \to b} x(t) = B$ where B is either a finite number or $B = \pm\infty$.

2. If B is a finite number then the solution $x(t)$ has a prolongation to the right.

Note that the first statement is not obvious at all: there are many differentiable functions $x(t)$ defined on (a, b) such that $\lim_{t \to b} x(t)$ does not exist, see fig. 2.9. But such functions cannot be solutions of the ODE $x' = f(t, x)$ if the assumptions of Theorem 2.4.1 hold.

Certainly for prolongations to the left we have a similar theorem.

THEOREM 2.4.2 (Theorem on prolongations of solutions to the left). *Let $x(t)$ be a solution of the equation $x' = f(t, x)$ defined on the interval $t \in (a, b)$. Assume that a is a finite number and assume that the function $f(t, x)$ is continuous in a neighborhood of any point of the vertical line $\{t = a, \forall x\}$ and satisfies the Liptschitz condition with respect to x at any point of this line. Then the following holds:*

1. There exists $lim_{t \to a} x(t) = A$ where A is either a finite number or $A = \pm\infty$.

2. If A is a finite number then the solution $x(t)$ has a prolongation to the left.

Below is the proof of Theorem 2.4.1 using the existence and uniqueness theorems and much of the INFI staff. The proof is illustrated in fig. 2.10. I leave some details of the proof to the reader who knows INFI well. All other students should understand the idea. The proof of Theorem 2.4.2 is similar.

PROOF OF THE FIRST STATEMENT OF THEOREM 2.4.1. Assume that the limit of $x(t)$ as $t \to b$ does not exist. Then there is an infinite sequence $t_i \to b$ such that $lim_{i \to \infty} x(t_i) \to B_1$ and an infinite sequence $s_i \to b$ such that $lim_{i \to \infty} x(s_i) \to B_2$ where $B_1 \neq B_2$. Here B_1 and B_2 are either finite numbers or $\pm\infty$. Take any B in the interval (B_1, B_2). Since $f(t, x)$ is continuous at the point (b, B) by the existence theorem the same equation has a solution $\widehat{x}(t)$ satisfying the condition $\widehat{x}(b) = B$.

This solution $\widehat{x}(t)$ is defined on some interval $t \in (b-\epsilon, b+\epsilon)$. Now, one can prove that there exists a sequence $r_i \to b$ such that $x(r_i) = \widehat{x}(r_i)$ and $x(r_i) \to B$ so that the point $(r_i, x(r_i))$ tends to the point (b, B). By the uniqueness theorem and the assumptions of Theorem 2.4.1 we must have $x(t) = \widehat{x}(t)$ for any $t \in (b-\epsilon, b)$. It is impossible. Q.E.D.

PROOF OF THE SECOND STATEMENT OF THEOREM 2.4.1. Let $lim_{t\to b}x(t) = B$ where B is a finite number. By the existence theorem the same equation has a solution $\widehat{x}(t)$ defined on some interval $t \in (b-\epsilon, b+\epsilon)$. Consider the function

$$\widetilde{x}(t) = \begin{cases} x(t), & t \in (a,b) \\ \widehat{x}(t), & t \in [b, b+\epsilon) \end{cases} .$$

It is easy to prove that $\widetilde{x}(t)$ is a solution of the same equation. It is the prolongation of the solution $x(t)$ to the right.

2.5. Exercises

1. Modify the equation and/or the initial condition in section 1.5 so that the maximal interval of definition of a solution satisfying the initial condition is $(-1/2, \infty)$.

2. Assume that a function $f(x)$ is differentiable and has continuous derivative on the set $\mathbb{R} - \{x_1, , ...x_n\}$ (at any point except a finite number of points). Assume that $\lim_{x\to x_i^+} f'(x) = a_i$ and $\lim_{x\to x_i^-} f'(x) = b_i \neq a_i$ for $i = 1, ..., n$ (limits from the left and from the right are different). Here a_i and b_i are finite number. Is it true that in this case $f(x)$ satisfies the Liptschitz condition at all points including the points $x_1, ..., x_n$? Prove or give a counterexample.

3. Prove that if $x(t)$ is a solution of an equation $x' = f(x)$ then $\widetilde{x}(t) = x(t-a)$ is also a solution of the same equation, for any a. Using this fact prove that the equation $x' = \sqrt{|x|}$ has infinitely many solutions satisfying the condition $x(0) = 0$ (except solutions given in Example 2.2.7) and draw the graphs of some of them.

4. Within the illustrating example in section 2.3 prove that if $f(x) = 2\sqrt{|4-2x|}$ then I will reach Ziv in a finite time.

5. Explain why the possibility to stop a car at a traffic red light does not contradict to the uniqueness theorem. Namely assume the following:

at time $t_0 = 0$ the distance to the traffic light is 0.02km ($x(0) = 0.02$) and the velocity is 50km/h ($x'(0) = 50$);

one starts to brake ($x'' = -a$) so that the acceleration a is negative and constant.

Find a and time t_1 such that the car stops at the traffic light in time t_1, i.e. $x(t_1) = 0,\ x'(t_1) = 0$. Determine how the velocity depends on the distance to the traffic light, i.e. find a function $f(x)$ such that the solution of the equation $x'' = -a$ with the a that you found implies $x' = f(x)$. Certainly you will have $f(0) = 0$ so that the equation $x' = f(x)$ has a constant solution $x(t) \equiv 0$. This means that there is no uniqueness. Explain why it does not contradict to the uniqueness Theorem 2.2.4.

6. (*) Cover all details of the proof of Theorem 2.4.1. The assumption that $f(t, x)$ satisfies the Liptschitz condition at all points of the vertical line $\ell = \{t = a, \forall x\}$ can be replaced by the assumption that $f(t, x)$ satisfies the Liptschitz condition at points of some subset $S \subset \ell$ having a certain property. Which property?

CHAPTER 3

Autonomous first order ODEs: $x' = f(x)$

As it was explained in the previous chapter, the simplest class of functions $f(x)$ satisfying the Liptschitz condition at all points $x \in \mathbb{R}$ is the class of differentiable for all x functions with continuous derivative.

NOTATION. The class of functions $f(x)$ which are twice differentiable at any $x \in \mathbb{R}$ and the derivative $f'(x)$ is continuous for any $x \in \mathbb{R}$ is denoted $C^1(\mathbb{R})$.

In all theorems in this chapter we will assume that $f(x) \in C^1(\mathbb{R})$ in order to use the uniqueness theorem.

In all theorems in this chapter the condition $f(x) \in C^1(\mathbb{R})$ can be replaced by a weaker condition that $f(x)$ satisfies the Liptschitz condition at any point $x \in \mathbb{R}$.

3.1. Non-constant solutions of first order autonomous ODEs are strictly monotonic functions

THEOREM 3.1.1. *Let $f(x) \in C^1(\mathbb{R})$. Any non-constant solution $x(t)$ of the equation $x'(t) = f(x(t))$ defined on any interval (a, b) satisfies $x'(t) \neq 0, t \in (a, b)$. Consequently any solution is a strictly monotonic function (strictly increasing or strictly decreasing).*

EXAMPLE 3.1.2. None of the functions t^2 and t^3 defined on an interval containing the point $t = 0$ can be a solution of an equation $x' = f(x) \in C^2(\mathbb{R})$.

PROOF. Assume, to get contradiction, that $x'(t_1) = 0$, where $t_1 \in (a, b)$. Let $x_1 = x(t_1)$. Then $f(x_1) = f(x(t_1)) = 0$. It follows that the equation has the constant solution $\widetilde{x}(t) \equiv x_1,\ t \in \mathbb{R}$. This constant solution $\widetilde{x}(t)$ and the solution $x(t)$ take the same values at the point t_1: $x(t_1) = \widetilde{x}(t_1) = x_1$. By the uniqueness theorem $x(t) = \widetilde{x}(t)$ for any $t \in (a, b)$. Therefore $x(t)$ is a constant solution: contradiction.

3.2. Constant solutions and singular (equilibrium) points

The constant solutions of autonomous equations $x' = f(x)$ are very important. They correspond to *singular = equilibrium* points defined as follows.

DEFINITION 3.2.1. A point $x^* \in \mathbb{R}$ is a singular, or equilibrium point of the equation $x' = f(x)$ if $f(x^*) = 0$.

The following statement is obvious:

PROPOSITION 3.2.2. *The equation $x' = f(x)$ has a constant solution $x(t) \equiv x^*$, $t \in (a, b)$ if and only if x^* is a singular point of this equation.*

3.3. The case of initial condition between two singular points: qualitative analysis

Let us obtain a qualitative information about the solution of equation $x' = f(x) \in C^1(\mathbb{R})$ satisfying the initial condition $x(t_0) = x_0$ where x_0 satisfies the following assumption:

$$\begin{gathered} x_1^* < x_0 < x_2^* \text{ where } x_1^* \text{ and } x_2^* \text{ are singular points} \\ \text{there are no singular points between } x_1^* \text{ and } x_2^* \end{gathered} \tag{3.3.1}$$

Note that this assumption implies that we have one of the following possibilities: either $f(x) > 0$ for all $x \in (x_1^*, x_2^*)$ or $f(x) > 0$ for all $x \in (x_1^*, x_2^*)$.

THEOREM 3.3.1. *Let $f(x) \in C^1(\mathbb{R})$ and let $x(t)$ be the solution of the equation $x' = f(x)$ satisfying the initial condition $x(t_0) = x_0$ and and defined on* <u>*maximal possible*</u> *interval $t \in (t^-, t^+)$. Assume that the point x_0 satisfies the assumption (3.3.1). Then the following holds:*

1. $t^- = -\infty$ and $t^+ = \infty$.

2. If $f(x) > 0$ for $x \in (x_1^*, x_2^*)$ then the solution $x(t)$ is a strictly increasing function and $lim_{t\to\infty} x(t) = x_2^*$, $lim_{t\to-\infty} x(t) = x_1^*$.

3. If $f(x) < 0$ for $x \in (x_1^*, x_2^*)$ then the solution $x(t)$ is a strictly decreasing function and $lim_{t\to\infty} x(t) = x_1^*$, $lim_{t\to-\infty} x(t) = x_2^*$.

EXAMPLE 3.3.2. The equation $x' = (x^2 - 1)(x^2 - 9)$ has 4 singular points $x = \pm 1, x = \pm 3$. By Theorem 3.3.1 the graphs of solutions of this equation satisfying the initial conditions

(a) $x(0) = -2$, (b) $x(0) = 0$, (c) $x(0) = 2$

have the form showed in fig. 3.1.

PROOF OF THEOREM 3.3.1. We will consider the case $f(x) > 0$, $x \in (x_1^*, x_2^*)$. The proof in the case $f(x) < 0$, $x \in (x_1^*, x_2^*)$ is similar.

In this case by Theorem 3.1.1 the solution $x(t)$ is an increasing function at any point $t \in (t^-, t^+)$. Let us show that $x(t)$ is a bounded function: $x_1^* < x(t) < x_2^*$, $t \in (t^-, t^+)$. In fact, if it was not so we would have a point t_1 such that $x(t_1) = x_1^*$ or a point t_2 such that $x(t_2) = x_2^*$. Since the equation has constant solutions $x(t) \equiv x_1^*$ and $x(t) \equiv x_2^*$, it contradicts to the uniqueness theorem.

Thus the solution $x(t)$ is an increasing function and $x_1^* < x(t) < x_2^*$, $t \in (t^-, t^+)$. It follows that there are limits $lim_{t\to t^+} x(t) = B \le x_2^*$ and $lim_{t\to t^-} x(t) = A \ge x_1^*$. Assume that t^+ is a finite number. Then by the prolongation Theorem 2.4.1 the solution $x(t)$ can be prolonged to an interval $(t^-, t^+ + \epsilon)$, $\epsilon > 0$ which contradicts to the fact that (t^-, t^+) is the maximal possible interval on which the solution with the initial condition $x(t_0) = x_0$ is defined. Therefore $t^+ = \infty$. In the same way the prolongation Theorem Theorem 2.4.2 implies $t^- = -\infty$.

It remains to prove that $B = x_2^*$ and $A = x_1^*$. The proof is as follows. Assume, to get contradiction, that $B < x_2^*$. Then the function $f(x)$ takes strictly positive values as $x \in [x_0, B]$. Consequently there exists $\epsilon > 0$ such that $f(x) > \epsilon$ for $x \in [x_0, B]$. Since $x(t) \in [x_0, B]$ for $t \in (t_0, \infty)$ and $x'(t) = f(x(t))$ we obtain that

$x'(t) > \epsilon$ for $t \in (t_0, \infty)$. It follows $lim_{t\to\infty}x(t) = \infty$ which is a contradiction. Therefore $B = x_2^*$. The proof that $A = x_1^*$ is similar.

3.4. The case of initial condition between two singular points: solving the equation

The equation $x' = f(x)$ can be solved exactly in the same way as in section 1.5, using the inverse function.

Assume that we have the initial condition $x(t_0) = x_0$ and x_0 is between two singular points x_1^*, x_2^* so that there are no other singular points between x_1^* and x_2^*. We know all qualitative information about $x(t)$ from Theorem 3.3.1. See fig. 3.2.a for the case $f(x_0) > 0$ and consequently $f(x) > 0$ for $x \in (x_1^*, x_2^*)$. since $x(t)$ is an increasing function the inverse function $t(x)$ is well-defined. see fig. 3.2.b. We have, by the theorem on the derivative of inverse function,

$$t'(x) = \frac{1}{f(x)}.$$

We also have

$$t(x_0) = x_0.$$

It follows

$$t(x) = t_0 + \int_{x_0}^{x} \frac{ds}{f(s)}. \tag{3.4.1}$$

In the case $f(x_0) < 0$ and consequently $f(x) < 0$ for $x \in (x_1^*, x_2^*)$ we have exactly the same formula by the same argument.

Can we say that (3.4.1) is a formula of solution? It is, even though it is a formula for the inverse function $f(x)$ rather than for $x(t)$ and even though the formula contains an integral which not always can be computed in elementary functions (depends on $f(x)$). In some cases the integral can be expressed in elementary functions, then we have a formula for the inverse function $t(x)$ without integrals. In some cases knowing the formula for $t(x)$ we can obtain a formula for $x(t)$.

EXAMPLE 3.4.1. By Theorem 3.3.1 the solution of the equation $x' = sin(x)$ satisfying the initial condition $x(10) = \frac{3\pi}{2}$ has the graph showed in fig. 3.3. The solution $x(t)$ satisfies the formula

$$t(x) = 10 + \int_{\frac{3\pi}{2}}^{x} \frac{ds}{sin(s)}.$$

This formula cannot be simplified, we cannot find a formula for $x(t)$. Nevertheless, we can compute (using basic computer programs) the time t for which the solution $x(t)$ takes any given value. For example

$$x(t_1) = \frac{7\pi}{4} \Leftrightarrow t_1 = 10 + \int_{\frac{3\pi}{2}}^{\frac{7\pi}{4}} \frac{dx}{sinx}.$$

From the graph of $x(t)$ we see that we must have $t_1 < 10$. It corresponds to the fact that $sinx$ is negative for $x \in [\frac{3\pi}{2}, \frac{7\pi}{4}]$. We also have

$$x(t_2) = \frac{5\pi}{4} \Leftrightarrow t_2 = 10 + \int_{\frac{3\pi}{2}}^{\frac{5\pi}{4}} \frac{dx}{sinx}.$$

From the graph of $x(t)$ we see that we must have $t_1 > 0$. It corresponds to the fact that $sinx$ is negative for $x \in [\frac{5\pi}{4}, \frac{3\pi}{2}]$ and the upper limit in the last integral is smaller than the bottom limit. It is nicer to write down the same integral in the form

$$t_2 = 10 - \int_{\frac{5\pi}{4}}^{\frac{3\pi}{2}} \frac{dx}{sinx} \quad \text{or} \quad t_2 = 10 + \int_{\frac{5\pi}{4}}^{\frac{3\pi}{2}} \frac{dx}{|sinx|}.$$

Example 3.4.2. The graph of solution $x(t)$ of the equation $x' = x^2 - 5$ satisfying the initial condition $x(3) = 2$ is showed in fig. 3.4 (according to Theorem 3.3.1). We have

$$t(x) = 3 + \int_2^x \frac{ds}{s^2 - 5}.$$

The integral can be computed:

$$\int \frac{ds}{s^2 - 5} = \frac{1}{2\sqrt{5}} ln \left| \frac{s - \sqrt{5}}{s + \sqrt{5}} \right|.$$

Therefore

$$t(x) = 3 + \frac{1}{2\sqrt{5}} \left(ln \left| \frac{x - \sqrt{5}}{x + \sqrt{5}} \right| - ln|R| \right), \quad R = \frac{2 - \sqrt{5}}{2 + \sqrt{5}}.$$

We know (and it is very important!) that $x(t) \in (-\sqrt{5}, \sqrt{5})$. It allows to write a formula <u>without</u> absolute value:

$$t = 3 + \frac{1}{2\sqrt{5}} ln \left(\frac{\sqrt{5} + 2}{\sqrt{5} - 2} \cdot \frac{\sqrt{5} - x}{\sqrt{5} + x} \right).$$

From here it is easy to obtain an explicit formula for $x(t)$:

$$x(t) = \sqrt{5} \cdot \frac{\sqrt{5} + 2 - (\sqrt{5} - 2) exp\left(2\sqrt{5}(t - 3)\right)}{\sqrt{5} + 2 + (\sqrt{5} - 2) exp\left(2\sqrt{5}(t - 3)\right)}.$$

3.5. Inflection points

An inflection point of a function $x(t)$ is a point t_1 where it changes from convex to concave or visa a versa. They always satisfy the condition $x''(t_1) = 0$.

Theorem 3.5.1. *Let t_1 be an inflection point of a non-constant solution $x(t)$ of equation $x' = f(x) \in C^1(\mathbb{R})$. Let $x_1 = x(t_1)$. Then $f(x_1) = 0$.*

Proof. We have $x''(t_1) = 0$. On the other hand

$$x''(t) = (x'(t))' = (f(x(t)))' = f'(x(t)) \cdot x'(t)$$

and substituting $t = t_1$ we obtain $f'(x_1) \cdot x'(t_1) = 0$. By Theorem 3.1.1 we have $x'(t_1) \neq 0$. Therefore $f'(x_1) = 0$.

Let us show how this theorem can be used.

Example 3.5.2. Consider, like in Example 3.4.1, the solution $x(t)$ of the equation $x' = sin(x)$ satisfying the initial condition $x(10) = \frac{3\pi}{2}$, see fig. 3.3. Let us find all inflections points in the graph of $x(t)$. By Theorem 3.5.1, if (t_1, x_1) is an inflection point in the graph of $x(t)$ then $cosx_1 = 0$. Therefore $x_1 = \pi/2 + \pi k$. But we know that $x(t)$ takes values between π and 2π only. Therefore there is only one inflection point with $x_1 = 3\pi/2$. We know (it initial condition) that if $x(t_1) = 3\pi/2$

then $t_1 = 10$. Therefore the graph of $x(t)$ contains one and only one inflection point with coordinates $t_1 = 10, x_1 = 3\pi/2$.

EXAMPLE 3.5.3. Consider, like in Example 3.4.2, the solution $x(t)$ of the equation $x' = x^2 - 5$ satisfying the initial condition $x(3) = 2$, see fig. 3.4. Let us find all inflections points in the graph of $x(t)$. By Theorem 3.5.1, if (t_1, x_1) is an inflection point in the graph of $x(t)$ then $x_1 = 0$. It follows that the graph of $x(t)$ contains one and only one inflection point with the coordinates

$$t_1 = 3 + \int_2^0 \frac{dx}{x^2 - 5} \qquad x_1 = 0.$$

Computing the integral we obtain

$$t_1 = 3 + \frac{1}{2\sqrt{5}} \left(1 + ln \frac{\sqrt{5} + 2}{\sqrt{5} - 2} \right), \quad x_1 = 0.$$

3.6. Solving the equation $x' = f(x)$

The relation (3.4.1) between t and x holds for any solution of the equation $x' = f(x)$ (not only in the case that x_0 is between two singular points) by exactly the same argument with the inverse function. The inverse function is defined because any solution is a monotonic function. Thus for any solution $x(t)$ we have

$$t = t_0 + \int_{x_0}^{x} \frac{ds}{f(s)}. \tag{3.6.1}$$

This equation holds for all $t \in (a, b)$ and $x = x(t)$ where (a, b) is the interval on which the solution $x(t)$ is defined.

3.7. The case $x_0 > x^*$ where x^* is the maximal singular point or $x_0 < x^*$ where x^* is the minimal singular point

The following theorem is illustrated in fig.3.5.

THEOREM 3.7.1. *Let $f(x) \in C^1(\mathbb{R})$ and let $x(t)$ be the solution of the equation $x' = f(x)$ satisfying the initial condition $x(t_0) = x_0$ and and defined on maximal possible interval $t \in (t^-, t^+)$.*

1. Assume that $x_0 > x^*$ where x^* is the maximal singular point and $f(x_0) > 0$ (and consequently $f(x) > 0$ for all $x > x^*$). Then $t^- = -\infty$, the solution $x(t)$ is an increasing function, $\lim_{t\to-\infty} x(t) = x^*$, $\lim_{t\to t^+} x(t) = \infty$ and

$$t^+ = t_0 + \int_{x_0}^{\infty} \frac{dx}{f(x)}.$$

2. Assume that $x_0 > x^*$ where x^* is the maximal singular point and $f(x_0) < 0$ (and consequently $f(x) < 0$ for all $x > x^*$). Then $t^+ = \infty$, the solution $x(t)$ is a decreasing function, $\lim_{t\to\infty} x(t) = x^*$, $\lim_{t\to t^-} x(t) = \infty$, and

$$t^- = t_0 + \int_{x_0}^{\infty} \frac{dx}{f(x)} = t_0 - \int_{x_0}^{\infty} \frac{dx}{|f(x)|}$$

3. Assume that $x_0 < x^*$ where x^* is the minimal singular point and $f(x_0) < 0$ (and consequently $f(x) < 0$ for all $x < x^*$). Then $t^- = -\infty$, the solution $x(t)$ is a decreasing function, $\lim_{t\to-\infty} x(t) = x^*$, $\lim_{t\to t^+} x(t) = -\infty$, and

$$t^+ = t_0 + \int_{x_0}^{-\infty} \frac{dx}{f(x)} = t_0 - \int_{-\infty}^{x_0} \frac{dx}{f(x)} = t_0 + \int_{-\infty}^{x_0} \frac{dx}{|f(x)|}$$

4. Assume that $x_0 < x^*$ where x^* is the minimal singular point and $f(x_0) > 0$ (and consequently $f(x) > 0$ for all $x < x^*$). Then $t^+ = \infty$, the solution $x(t)$ is an increasing function, $\lim_{t\to\infty} x(t) = x^*$, $\lim_{t\to t^-} x(t) = -\infty$, and

$$t^- = t_0 + \int_{x_0}^{-\infty} \frac{dx}{f(x)} = t_0 - \int_{-\infty}^{x_0} \frac{dx}{f(x)}$$

In cases 1. and 3. both possibilities $t^+ = \infty$ and t^+ is a finite number might hold; in cases 2. and 4. both possibilities $t^- = -\infty$ and t^- is a finite number might hold. It depends on the convergence of the integrals.

PROOF. We will prove only the first statement (fig. 3.5, a). The proof of the other statements is similar.

The proof of the statements that $x(t)$ is an increasing function, that $t^- = -\infty$ and $lim_{t\to-\infty} x(t) = x^*$ is similar to the proof of Theorem 3.3.1.

Let us prove that $lim_{t\to t^+} x(t) = \infty$ either in the case that t^+ is finite or in the case $t^+ = \infty$. If t^+ is finite it follows from the prolongation Theorem 2.4.1. For the case $t^+ = \infty$ the proof is as follows. Assume, to get contradiction, that $x(t) \to B < \infty$ as $t \to \infty$. Consider the function $f(x)$ on the closed interval $[x_0, B]$. There exists $\epsilon > 0$ such that $f(x) > \epsilon$ for $x \in [x_0, B]$. It follows that $x'(t) > \epsilon$ for all $t \in [0, \infty)$. Therefore $x(t) \to \infty$ as $t \to \infty$: contradiction.

It remains to prove that $t^+ = t_0 + \int_{x_0}^{\infty} \frac{dx}{f(x)}$. To obtain it, we use the fact that $x(t) \to \infty$ as $t \to t^+$ and take the limit in (3.6.1) as $t \to t^+$.

□

3.8. The case that there are no singular points

Exactly in the same way we can prove the following theorem illustrated in fig. 3.6.

THEOREM 3.8.1. *Let $f(x) \in C^1(\mathbb{R})$ such that $f(x) \neq 0$ for any $x \in \mathbb{R}$. let $x(t)$ be the solution of the equation $x' = f(x)$ satisfying the initial condition $x(t_0) = x_0$ and and defined on maximal possible interval $t \in (t^-, t^+)$.*

1. If $f(x) > 0$ for all x then $x(t)$ is an increasing function, $\lim_{t\to t^+} x(t) = \infty$, $\lim_{t\to t^-} x(t) = -\infty$, and

$$t^+ = t_0 + \int_{x_0}^{\infty} \frac{dx}{f(x)},$$

$$t^- = t_0 + \int_{x_0}^{-\infty} \frac{dx}{f(x)} = t_0 - \int_{-\infty}^{x_0} \frac{dx}{f(x)}.$$

2. If $f(x) < 0$ for all x then $x(t)$ is a decreasing function,
$\lim_{t\to t^+} x(t) = -\infty,\ \lim_{t\to t^-} x(t) = \infty$, *and*

$$t^+ = t_0 + \int_{x_0}^{-\infty} \frac{dx}{f(x)} = t_0 - \int_{-\infty}^{x_0} \frac{dx}{f(x)} = t_0 + \int_{-\infty}^{x_0} \frac{dx}{|f(x)|},$$
$$t^- = t_0 + \int_{x_0}^{\infty} \frac{dx}{f(x)} = t_0 - \int_{x_0}^{\infty} \frac{dx}{|f(x)|}.$$

In each of these cases any of the possibilities (a) both t^+ and t^- are finite, (b) $t^+ = \infty,\ t^- = -\infty$ (c) $t^+ = \infty$, t^- is finite, (d) $t^- = -\infty,\ t^+$ is finite might hold, it depends on the convergence of the integrals as $x \to \infty$ and $x \to -\infty$.

3.9. Phase portrait

Given equation of the form $x' = f(x)$ it is worth to draw its phase portrait. It is a straight line = the x-axes endowed with marked singular points, arrows between them, an arrow to the right from the maximal singular point, and an arrow to the left of the minimal singular point. Each arrow corresponds to increase of time. See fig. 3.7.

A phase portrait gives almost complete qualitative information about any solution. One should remember that we proved that

any solution of equation of the form $x' = f(x) \in C^1(\mathbb{R})$ is a monotonic function which tends, as $t \to t^+$ or $t \to t^-$ either to one of the singular point, or to ∞ or to $-\infty$.

Consider for example the phase portrait in fig. 3.7.(a) Let $x(t)$ be the solution satisfying the initial condition $x(t_0) = x_0$. We see from the phase portrait that if solutions depend on x_0 as it showed in fig. 3.7.,b. The only qualitative information which does not follow from the phase portrait is the question about t^+ (is it finite or not) if $x_0 > 6$ and about t^- (is it finite or not) if $x_0 < 1$.

3.10. Optimization by feedback versus rigid plan. Catastrophes

In this section there is a nice illustration, in terms of ODES, of the following universal life principle:

optimization by rigid plan leads to a catastrophe whereas optimization by feedback might give very good results.

The math illustration below is taken from the book "Catastrophe theory" by V. Arnol'd. It is about catching fishes in a lake. One can replace fishes and lake by certain words related to business, politics, family relations, love, studies, anything else. Parallel with fishes in a lake I will give the same illustration in terms of love between a boy and a girl, let us call then λ (girl) and μ (boy).

As we discussed in section 1.6, the simplest equation for multiplication of fishes in a lake is

$$x' = x(1-x), \quad x = x(t) \tag{3.10.1}$$

where t is time (for example in years) and $x(t)$ is the number of fishes in the lake at time t. Certainly it holds if people do not catch fishes. The phase portrait and the graph of solution with the initial condition $x(0) \in (0, 1)$ is showed in fig. 3.8. The singular point $x^* = 1$ corresponds to full lake (the number 1 stands, for example for million fishes).

The same equation (3.10.1) might describe the level $x(t)$ of love between λ and μ. The singular point $x = 0$ corresponds to the constant solution "no love all the time" and the singular point $x = 1$ is the level of love of Romeo and Julietta. The inflection point in the graph of the love (after which the love still increases, but not as fast as before) might correspond to a certain event in their relations (for example the marriage). Equation (3.10.1) might hold only if λ and μ do nothing against their love (μ does not leave λ to meet with his friends, λ does not speak with her mother by telephone more time than it is OK with μ, etc.). It will be so if they live in a desert island; for if not they do something against their love and it is like fishing in the lake.

Therefore let us consider the realistic case that people catch fishes from the lake and λ and μ do something against their life. Let us discuss a rigid plan: people fix that they will catch fishes with the velocity c fishes a year; λ and μ fix that they will do something against their love with velocity c a year. Then instead of the equation (3.10.1) we should consider the equation

$$x' = f(x) = x(1 - x) - c, \quad x = x(t) \tag{3.10.2}$$

The phase portrait of this equation depends on c. What happens if $c > 1/4$? In this case the function $f(x)$ is negative for all x, there are no singular points, and even if at the initial time $x(0) = 1$ (the lake was full of fish; λ and μ loved each other like Romeo and Julietta) the solution decreases to $-\infty$. See fig. 3.9. Therefore in some time there will be no fish in the lake; λ and μ will separate. Nobody want that. Therefore the rigid plan with $c > 1/4$ is not OK, people should catch less fish a year, λ and μ should do less against their love.

Let now $c < 1/4$, for example $c = 1/8$. In this case the phase portrait of equation (3.10.2) is different: there are two singular points, the points at which $x(1 - x) = 1/8$. They are approximately $x_1* \approx 0.14$ and $x_2^* \approx 0.86$. The phase portrait is showed in fig. 3.10 Let us assume, as above, that at time $t_0 = 0$ the lake is full, λ and μ love each other like Romeo and Julietta: $x(0) = 1$. The solution $x(t)$ is showed in fig. 3.10. We see that

(A) $c = 1/8$ people will catch all the time, till ∞ 1/8 fishes a year, the lake will contain not less than 0.86 fishes, λ and μ will do 1/8 a year something against their love, the level of love between λ and μ will be all time, till 120 not less than 86 percent of the level of love between Romeo and Julietta.

It is good! But now let us optimize the rigid plan. People want more fish, λ and μ want to do more against their love. The rigid plan $c > 1/4$ does not work, the rigid plan $c = 1/8$ works well. If one takes $c = 3/16$ then the same argument as above leads to the following:

(B) $c = 3/16$ people will catch all the time, till ∞ 1/8 fishes a year, the lake will contain not less than 0.75 fishes, λ and μ will do 1/8 a year something against their love, the level of love between λ and μ will be all time, till 120 not less than 75 percent of the level of love between Romeo and Julietta.

The variant (B) looks better than (A). Let us increase c more to get even better outcome. The optimization of rigid plan leads to $c = 1/4$. Let us see what happens

if $c = 1/4$. In this case the equation (3.10.2) has only one singular point $x = 1/2$ and the phase portrait is showed in fig. 3.11. Assuming, as above, that at time $t_0 = 0$ the lake is full, λ and μ love each other like Romeo and Julietta, i.e. $x(0) = 1$, the solution is showed in fig. 3.11. Therefore
(C) $c = 1/4$ people will catch all the time, till ∞ 1/4 fishes a year, the lake will contain not less than 1/2 of fish, λ and μ will do 1/4 a year something against their love, the level of love between λ and μ will be all time, till 120 not less than 1/2 the level of love between Romeo and Julietta.

The variant (C) is the best within optimization by rigid plan. It looks very good, but ... in this case there will be a catastrophe. In some time t_1 we will have $x(t_1) = 1/2 + \epsilon$, where ϵ is a small positive number. What happens if a little boy comes and catches few fishes? What happens if an egg fells down in the kitchen of the apartment λ and μ rent? Or some other small changes? Instead the condition $x(t_1) = 1/2 + \epsilon$ we will have $x(t_1) = \delta$ with a small positive δ. But with this initial condition at time t_1 the solution tends to $-\infty$ as $t \to \infty$.

Therefore in case (C), after some time t_1 and "due" to small changes, like a dropped egg in the kitchen, in some time $t_1 + t_2$ there will be no fishes in the lake, λ and μ will separate. The catastrophe holds not because of the egg, but because it was used optimization by rigid plan instead of feedback.

What is feedback in our examples? It is the following: people check all the time how many fishes are in the lake and decide how much to catch depending on that. The girl and the boy think all the time on the level of their relations and decide how much to do against their love depending on that.

The optimal feedback is showed in fig. 3.12: people catch fish with the velocity depending on the number of fishes, namely with the velocity $x/2$, so that

$$x' = f(x) = x(1-x) - \frac{1}{2}x, \quad x = x(t) \tag{3.10.3}$$

The phase portrait and the solution is showed in fig. 3.12. We obtain:

(D) feedback $x/2$:

the same as in (C): all the time till ∞ people catch not less than 1/4 a year; the lake is always at least half-full, but unlike (C) there will be no catastrophe: it follows from the phase portrait in fig. 3.12 that under a small change the worst that can happen is that people will to continue to catch all the time $1/4 - \delta$ a year where δ is a small number.

3.11. Exercises

1. . Let $f(x) = (x-1)^2(x-2)^3(x+1)^4(x+2)^5$ and let $x_1(t), ..., x_5(t)$ be the solutions of the equation $x'(t) = f(x(t))$ satisfying the initial condition
$x_1(1) = -1.5, \quad x_2(2) = 0, \quad x_3(5) = 1, \quad x_4(1) = 0.5, \quad x_5(0) = 1.5$
and defined for all t. Draw the 5 graphs, of $x_1(t), ..., x_7(t)$, in the same (t, x) plane.

2. Let $x_1(t), ..., x_5(t)$ be the solutions of the equation $x'(t) = sin(e^{x(t)})$ satisfying the initial condition $x_1(0) = 2, \quad x_2(1) = 2, \quad x_3(0) = 3, \quad x_4(1) = 3, \; x_5(-1) = 4$ and defined for all t. Draw the 5 graphs, of $x_1(t), ..., x_5(t)$ in the same (t, x) plane. For each of these graphs find the both coordinates (t_1, x_1) of all its inflection points (nikudat pitul). Integrals in the answers for t_1 are OK.

3. Find a formula for the solution $x(t)$ of the equation $x' = ax(b-x)$ where $a, b > 0$ are parameters. No inverse function and no integrals in the final answer.

4. Let $x(t)$ be the solution of the equation $x' = f(x)$ satisfying the initial condition $x(t_0) = x_0$ and defined on maximal possible interval (t^-, t^+), where the function $f(x)$ and t_0, x_0 are given below. Draw the graph of $x(t)$ and find t^- and t^+. Integral in the answers are OK only if they converge.

(a) $f(x) = (x-1)^2((x-2)(x-4)^4(x+2)^5, \ \ t_0 = 1, \ x_0 = -4$

(b) $f(x) = (1-x)\sqrt{x^2+1}, \ t_0 = 2, \ x_0 = 0$

(c) $f(x) = x \cdot ln(x^2+1), \ t_0 = 0, \ x_0 = -1$

(d) $f(x) = x \cdot ln(x^2+1), \ t_0 = -6, \ x_0 = 4$

5. Let $x(t)$ be the solution of the equation $x' = f(x)$ satisfying the initial condition $x(t_0) = x_0$ and defined on maximal possible interval (t^-, t^+). Give an example of a function $f(x) \in C^1(\mathbb{R})$ such that each of the requirements below holds.

(a) requirement 1: $t^+ = \infty$ and $t^- = -\infty$ for any x_0

requirement 2: $\lim_{t\to\infty} x(t) \neq +\infty$ and $\lim_{t\to-\infty} x(t) \neq +\infty$ for any x_0

requirement 3: $\lim_{t\to\infty} x(t) = -\infty$ if and only if $x_0 < 0$

(b) requirement 1: $t^+ = \infty$ and $t^- = -\infty$ for any x_0

requirement 2: $\lim_{t\to\infty} x(t) = +\infty$ if and only if $x_0 > 3$

requirement 3: $\lim_{t\to\infty} x(t) = -\infty$ if and only if $x_0 < -1$

requirement 4: $\lim_{t\to\infty} x(t) = 0$ if and only if $x_0 = 0$

(c) requirement 1: t^+ is a finite number if and only if $x\_0 > 1$

requirement 1: t^- is a finite number if and only if $x_0 < 1$

6. Find all inflection points (both t- and x-coordinates) in the graph of the solution of the equation $x' = 2 + sin^2 x$ satisfying the initial condition $x(0) = 0$ and defined on maximal possible interval.

7. Let $f(x)$ be a function whose graph is showed in fig. 3.13. Assume that the number of fishes in a lake changes by the equation $x'(t) = f(x(t))$ provided people do not fish. People start fishing when the lake is full: $x = 1$. Explain what will happen if people fish by the rigid plan (a), by the rigid plan (b), by feedback (c), by feedback (d), by feedback (e).

8. A girl λ and a boy μ optimize their life by the rigid plan

$$x'(t) = x(t)(1 - x(t)) - 1/4.$$

where t is time in weeks and $x(t)$ is the level of their love in comparison with Romeo and Julietta. Today the love each other as Romeo and Dzhulyetta ($x(0) = 1$). At time $t_1 = 50$ weeks an egg falls down in their kitchen, or something similar happens, which decreases the level of their love by 0.05. In how many weeks t^* after that their haverut will be over, i.e. $x(t_1 + t^*) = 0$? The answer t^* should be a number, no integrals.

CHAPTER 4

Equations of the form $x'' = f(x)$. Two body problem. Pendulum.

4.1. Introduction to the chapter

Equations of the form

$$x'' = f(x), \quad x = x(t) \tag{4.1.1}$$

mean, in physical language, that a body moves along the x-axes with acceleration $f(x)$ which depends only on the position of the body. By the second Newtom's law it means that the body moves under the action of the force $mf(x)$, where m is the mass of the body.

The initial conditions for the second order equations, in particular for equations (4.1.1) is the initial location and the initial velocity of the body. In math language:

$$x(t_0) = x_0, \quad x'(t_0) = v_0, \tag{4.1.2}$$

where t_0, x_0 and v_0 are given numbers.

4.2. Existence and uniqueness theorem. Theorem on prolongation of soluitions

The following theorem is a particular case of a much more general existence and uniqueness theorem which will be discussed in the end of the course.

Theorem 4.2.1. *Fix the initial conditions (4.1.2). Assume $f(x) \in C^1(U)$ where U is a neighborhood of the point x_0 in the x-axes. The equation (4.1.1) has a solution $x(t)$ satisfying (4.1.2). If $x(t)$ and $\widetilde{x}(t)$ are two such solutions, defined on intervals I and $\widetilde{I}$, then $x(t) = \widetilde{x}(t)$ for any $t \in I \cap \widetilde{I}$.*

Like for ODEs of order 1, the existence part of the theorem does not provide information on maximal possible interval of the definition of solution, even if $U = \mathbb{R}$. Such information can be obtained from the following prolongation theorem.

Theorem 4.2.2. *Let $x(t)$ be a solution of equation (4.1.1) defined for $t \in (a, b)$. Assume that there exist finite limits*

$$\lim_{t \to b} x(t) = B, \ \lim_{t \to b} x'(t) = B_1.$$

If $f(x) \in C^1(U)$ where U is a neighborhood of the point B in the x-axes then the solution $x(t)$ has a prolongation to the right: there exists a solution $\widetilde{x}(t)$ of the same equation defined for $t \in (a, b + \epsilon)$, $\epsilon > 0$, and coinciding with $x(t)$ for $t \in (a, b)$.

A theorem on prolongation to the left is similar. Unlike the case of first order ODE's there is no theorem stating that there are limits, finite or infinite, of $x(t)$ and $x'(t)$ as $t \to b$.

Theorem 4.2.2 follows from the existence part of Theorem 4.2.1 as follows. Consider the initial conditions $x(b) = B,\ x'(b) = B_1$. Let $\widehat{x}(t)$ be a solution satisfying these initial conditions and defined for $t \in (b-\epsilon, b+\epsilon)$. Construct the function $\widetilde{x}(t)$: $\widetilde{x}(t) = x(t)$ as $t \in (a,b)$ and $\widetilde{x}(t) = \widehat{x}(t)$ as $t \in [b, b+\epsilon)$. It is easy to prove that $\widetilde{x}(t)$ is a solution of the same equation.

The uniqueness part of Theorem 4.2.1 is illustrated in fig.4.1. If $f(x) \in C^1(\mathbb{R})$ then two solutions as in fig. 4.1.a are possible (unlike the case of equations of order 1), but two solutions as in fig. 4.1.b or 4.1.c are impossible.

Unlike the case of order 1 equations $x' = f(x)$, a non-constant solution $x(t)$ of (4.1.1) does not need to be monotonic function. But any critical point of $x(t)$ (point t_1 such that $x'(t_1) = 0$) must be a point of local maximum or local minimum, see fig. 4.2. It follows from he following statement.

Theorem 4.2.3. *Let $x(t)$ be a non-constant solution of (4.1.1) and let $x'(t_1) = 0$. Let $x(t_1) = x_1$. If $f(x) \in C^1(U)$ where U is a neighborhood of the point x_1 in the x-axes then $x''(t_1) \neq 0$ and consequently t_1 is the point of local maximum or local minimum of $x(t)$.*

Proof. Assume, to get contradiction, that $x''(t_1) = 0$. Then $f(x_1) = 0$ and then the constant function $\widehat{x}(t) \equiv x_1$ is a solution. The uniquiness theorem implies that $x(t) = \widehat{x}(t)$ for any t such that $x(t)$ is defined, i.e. $x(t)$ is a constant solution. □

4.3. Theorem on shift of time and inverse of time. Theorem on symmetries and on periodic solutions

Like for ODEs $x' = f(x)$ for (4.1.1) we have the theorem on shift of time, and unlike ODEs $x' = f(x)$ for (4.1.1) we also have a theorem on inverse of time.

Theorem 4.3.1. *Let $x(t)$ be a solution of (4.1.1) defined on the interval $t \in (a,b)$. Then the following functions are also solutions of the same equation:*

1. (shift of time): the function $\widetilde{x}(t) = x(t+t_1)$, for any $t_1 \in \mathbb{R}$, defined for $t \in (a-t_1, b-t_1)$;

2. (inverse of time): the function $\widehat{x}(t) = x(-t)$ defined for $t \in (-b,-a)$.

Proof. We have $\widetilde{x}'(t) = x'(t+t_1)$ and $\widehat{x}'(t) = -x'(t+t_1)$. It follows $\widetilde{x}''(t) = x''(t+t_1)$ and $\widehat{x}''(t) = x''(-t)$. These equations imply Theorem 4.3.1. □

Theorem 4.3.1 is illustrated in fig. 4.3. This Theorem implies the following theorem on the symmetry about any critical point.

Theorem 4.3.2. *Let $f(x) \in \mathbb{C}^1(\mathbb{R})$. Let $x(t)$ be a solution of (4.1.1) such that $x'(t_1) = 0$. Then the graph of $x(t)$ is symmetric about the vertical line $t = t_1$ in the (t,x)-plane, i.e. $x(t_1 - t) = x(t_1 + t)$ for any t such that $t_1 - t$ and $t_1 + t$ belong to the interval of definition of $x(t)$.*

Proof. By Theorem 4.3.1 the functions $\widetilde{x}(t) = x(t_1 + t)$ and $\widehat{x}(t) = x(t_1 - t)$ are solutions of the same equations. We have $\widetilde{x}(0) = \widehat{x}(0) = x(t_1)$. We also have $\widetilde{x}'(t_1) = x'(t_1)$ and $\widehat{x}'(t_1) = -x'(t_1)$. Since $x'(t_1) = 0$ we have $\widetilde{x}'(t_1) = \widehat{x}'(t_1) = 0$. Now Theorem 4.3.2 follows from the uniqueness theorem. □

What can be said about a function whose graph is symmetric about two vertical lines $t = t_1$ and $t = t_2$ in the (t, x)-plane? Intuition suggests that in this case the function is periodic.

THEOREM 4.3.3. *Let $f(x) \in \mathbb{C}^1(\mathbb{R})$. Let $x(t)$ be a solution of (4.1.1) such that $x'(t_1) = x'(t_2) = 0$ where $t_2 > t_1$. Then $x(t)$ is a periodic function:*

$$x(t+T) = x(t), \quad T = 2(t_2 - t_1)$$

for any t such that $x(t)$ and $x(t+T)$ are defined.

PROOF. By Theorem 4.3.2 we have

$$x(t_1 + t) = x(t_1 - t), \tag{4.3.1}$$

$$x(t_2 + t) = x(t_2 - t). \tag{4.3.2}$$

It follows

$$x(t) = x(t_1 + (t - t_1)) = \text{ (by 4.3.2) } = x(t_1 - (t - t_1)) = x(2t_1 - t) =$$
$$= x(t_2 + (2t_1 - t - t_2)) = \text{ (by 4.3.2) } = x(t_2 - (2t_1 - t - t_2)) = x(t + 2(t_2 - t_1)).$$

□

4.4. Theorem on energy. Energy equation

The following theorem in the math form of the energy preserving law
for equations (4.1.1).

DEFINITION. The kinematic and the potential energy of a solution $x(t)$ of (4.1.1) are the function of t defined as follows:

$$K(t) = \frac{(x'(t))^2}{2}, \quad P(t) = -\int_c^{x(t)} f(s)ds$$

where c is any constant (the potential energy is defined up to a constant).

THEOREM 4.4.1. *For any solution $x(t)$ of equation (4.1.1) with a continuous function $f(x)$ we have*

$$K(t) + P(t) \equiv const.$$

PROOF. We have

$$K'(t) = x'(t) \cdot x''(t), \quad P'(t) = -f(x(t)) \cdot x'(t).$$

Therefore

$$(K(t) + P(t))' = x'(t) \cdot (x''(t) - f(x(t))) \equiv 0.$$

□

Write the conclusion of Theorem 4.4.1 in the form

$$\frac{(x'(t))^2}{2} - \int_c^{x(t)} f(s)ds = C = const.$$

If $x(t)$ satisfies the initial conditions (4.1.2) then the constant C can be found from these initial condition by substituting $t = t_0$:

$$\frac{(x'(t))^2}{2} - \int_c^{x(t)} f(s)ds = \frac{v_0^2}{2} - \int_c^{x_0} f(x)dx.$$

It is convenient to take $c = x_0$. We obtain

$$\textbf{Energy equation}: \quad \frac{(x'(t))^2}{2} - \int_{x_0}^{x(t)} f(s)ds = \frac{v_0^2}{2}. \tag{4.4.1}$$

4.4.1. Example: energy equation for a spring. According to Hooke's law the energy equation for oscillations of a spring (see fig. 4.4) is $x'' = -kx$ with a certain coefficient k depending on the spring. We have $f(x) = -kx$ and the energy equation has the form

$$\frac{(x'(t))^2}{2} - \left(-k \cdot \left(\frac{x^2(t) - x_0^2}{2}\right)\right) = \frac{v_0^2}{2}$$

which can be simplified to

$$(x'(t))^2 + kx^2(t) = v_0^2 + kx_0^2.$$

4.4.2. Example: a rocket launched up from the surface of the Earth. According to the gravitation law, the equation is as follows: $x'' = -\frac{k}{x^2}$ where x is the distance to the center of the Earth, see fig. 4.5. The coefficient k can be found from the fact that for $x = R$ (the radius of the Earth) we have $x'' = -g$. It follows $k = gR^2$, so we deal with equation

$$x'' = -\frac{gR^2}{x^2}. \tag{4.4.2}$$

If the rocket is launched from the surface of the Earth with initial velocity v_0, we have the initial conditions

$$x(0) = R, \quad x'(0) = v_0.$$

The energy equation is as follows:

$$\frac{(x'(t))^2}{2} - \int_R^{x(t)} \frac{-gR^2}{s^2} ds = \frac{v_0^2}{2}.$$

It can be simplifies to

$$\frac{(x'(t))^2}{2} - \frac{2gR^2}{x(t)} = v_0^2 - 2gR. \tag{4.4.3}$$

4.5. Two body problem

4.5.1. The problem. Consider the following simple case of the two body problem, generalizing the case of a rocket launched up in section 4.4.2:

a big body, which is a ball of radius R, does not move and stands all the time at the point $x = 0$;

a small body of mass $m = 1$, which is assumed to be a point, moves along the x-axes under the force of attraction to the big body;

this force depends on the coordinate x of the small body, it is defined for all $x > R$ and it is strictly positive for all $x > R$.

At the initial time $t = 0$ the small body has the coordinate $x(0) = x_0 > R$ and the initial velocity $x'(0) = v_0$ directed opposite to the direction of the attraction force.

See fig. 4.6. We also will assume that the attraction force $F(x)$ is a differentiable function with continuous derivative. In math terms the problem is as follows:

$$x'' = -F(x) \in C^1(x > R), \quad F(x) > 0 \text{ for all } x > R$$
$$x(0) = x_0 > R, \quad x'(0) = v_0 > 0. \tag{4.5.1}$$

4.5.2. Qualitative analysis. Thinking about a rocket launched up, one can imagine two possibilities for solution $x(t)$ of (4.5.1), showed in fig. 4.7, a,b :

(a) the small body moves all the time away from the big body, and moves arbitrary far: $x(t) \to \infty$ as $t \to infty$.

(b) the small body moves away from the big body to some distance x_{max} in certain time t_1 and after that goes back and meets the big body in some finite time.

Maybe another type of solution is possible, for example solutions showed in figure 4.7, c_1, c_2, c_3? The answer is no.

THEOREM 4.5.1. *In the two body problem (4.5.1) there are not more than the following two possibilities (see fig. 4.7, a,b):*

(a) The solution $x(t)$ is defined for all $t > 0$. it increases for all $t > 0$ and tends to ∞ as $t \to \infty$;

(b) The solution $x(t)$ is defined for $t < T$ where $x(T) = R$. It increases as $t < t_1$ and decreases as $t > t_1$ so that t_1 is the point of maximum.

The time T in case (b) is the time when the two bodies meet.

PROOF.
Case 1. At first assume that $x'(t) \neq 0$ for all $t > 0$. Since $x'(0) = v_0 > 0$, the solution $x(t)$ is an increasing function. Its derivative $x'(t)$ is a decreasing function because $x''(t) < 0$ for all t. Since $x'(t) \neq 0$ we have $x'(t) < v_0$ for all $t > 0$.

It follows that for any finite $t^+ > 0$ there are finite limits of $x(t)$ and of $x'(t)$ as $t \to t^+$. Now the prolongation Theorem 4.2.2 implies that there is a solution defined for all $t > 0$. We have excluded the possibility c_2 in fig. 4.7. Now we have to exclude the possibility c_1 in fig. 4.7, i.e. to prove that $x(t) \to \infty$ as $t \to \infty$.

Assume, to get contradiction, that $x(t) \to B < \infty$ as $t \to \infty$. Consider the function $F(x)$ on the closed interval $[x_0, B]$. We have

$$F(x) > \epsilon > 0, \quad x \in [x_0, B]$$

and it follows

$$x''(t) < -\epsilon, \quad t \in (0, \infty).$$

But then $x'(t) \to -\infty$ as $t \to \infty$ which contradicts to our assumption that $x'(t) \neq 0$ for all $t > 0$.

We have proved that in Case 1 we have all statements in case (a) of Theorem 4.5.1.

Case 2 (alternative case). Assume now that $x'(t_1) = 0$ for some $t_1 > 0$ and $x'(t) \neq 0$ for $t < t_1$. Since $x''(t) < 0$ for all t, the solution $x(t)$ has a local maximum at the point t_1 and $x'(t) < 0$ for all $t > t_1$. Assume that $x(t)$ is defined for $0 < t < T$ and has no prolongation to the right. We have to show that $x(t) \to R$ as $t \to T$. Note that the case $x(T) \to \widetilde{R} < R$ is impossible because the function $f(x)$ is not defined for $x < R$. We have to exclude the case that $x(t) \to \widetilde{R} > R$

as $t \to T$. This possibility is excluded by the prolongation Theorem 4.2.2 provided $\lim_{t\to T} x'(t) \neq -\infty$ and consequently the derivative $x'(t)$ has a finite limit as $t \to T$.

Therefore the last item to be proved is as follows: if $x(t) \to \widetilde{R} > R$ as $t \to T$ then $\lim_{t\to T} x'(t) \neq -\infty$. To prove this claim consider the function $F(x)$ on the closed interval $[\widetilde{R}, x(t_1)]$. We have

$$F(x) < M, \quad x \in [\widetilde{R}, x(t_1)]$$

for some finite M, and it follows

$$0 > x''(t) > -M, \quad t \in (t_1, T) \quad \Longrightarrow \quad \lim_{t\to T} x'(t) \neq -\infty.$$

□

4.5.3. Distinguishing (a) and (b). Critical initial velocity.
The cases (a) and (b) in Theorem 4.5.1 can be distinguished using the energy equation. The energy equation in the two body problem is the energy equation (4.4.1) with $f(x) = -F(x)$:

$$\frac{(x'(t))^2}{2} + \int_{x_0}^{x(t)} F(s)ds = \frac{v_0^2}{2}. \tag{4.5.2}$$

Assume we have case (a). Taking the limit in (4.5.2) as $t \to \infty$ we obtain

$$v_0 \geq \sqrt{2\int_{x_0}^{\infty} F(x)dx}. \tag{4.5.3}$$

Assume now we have case (b). Substituting to (4.5.2) the time t_1 such that $x'(t_1) = 0$ we obtain

$$v_0 = \sqrt{2\int_{x_0}^{x(t_1)} F(x)dx}$$

and consequently

$$v_0 < \sqrt{2\int_{x_0}^{\infty} F(x)dx}. \tag{4.5.4}$$

Definition. The critical initial velocity in the two body problem is the velocity

$$v_{0,crit} = \sqrt{2\int_{x_0}^{\infty} F(x)dx}.$$

Note that the critical initial velocity $v_{0,crit}$ depends on the initial location x_0 of the small body.

Since v_0 satisfies either (4.5.3) or (4.5.4) we obtain the following theorem:

Theorem 4.5.2. *In the two body problem the case (a) in Theorem 4.5.1 holds if and only if $v_0 \geq v_{0,crit}$ and the case (b) holds if and only if $v_0 > v_{0,crit}$.*

Corollary 4.5.3. *If $\int_{x_0}^{\infty} F(x)dx = \infty$ then only case (b) is possible (the bodies will meet whatever is the initial velocity). If $\int_{x_0}^{\infty} F(x)dx = \infty$ (i.e. the integral converges) then both cases (a) and (b) are possible and which of them holds depends, when $F(x)$ is fixed, on the couple (v_0, x_0).*

4.5.4. Example: the second cosmic (escape) velocity. A rocket launched up from the surface of the Earth is a particular case of the two body problem with

$$F(x) = \frac{gR^2}{x^2}, \quad x_0 = R$$

where R is the radius of the Earth, see section 4.4.2. In this case

$$v_{0,crit} = \sqrt{2\int_R^\infty \frac{gR^2}{x^2}} = \sqrt{2gR} \approx 11200\ m/sec.$$

This velocity is also called the escape velocity or the second cosmic velocity.

4.5.5. Solving the two body problem. Let us show how to answer the following questions, see fig. 4.7, a,b:

1. Let $v_0 < v_{0,crit}$. Find the maximal distance between the two bodies, i.e. find $x_{max} = max_{t\geq 0}x(t)$.

2. Let $v_0 < v_{0,crit}$. In which time the distance between the bodies is maximal, i.e. find t_1 such that $x(t_1) = x_{max}$.

3. In which time t_2 the distance between the bodies is equal to a given number d such that $x_0 \leq d < x_{max}$?

4. Let $v_0 < v_{0,crit}$. In which time T the two bodies will meet, i.e. $x(T) = R$? (more precisely $lim_{t\to T}x(t) = R$).

SOLUTIONS

Question 1. We can find x_{max} by substituting to the energy equation (4.5.2) the time t_1 such that $x(t_1) = x_{max}$ and consequently $x'(t_1) = 0$. We obtain

$$x_{max}: \quad \int_{x_0}^{x_{max}} F(x)dx = \frac{v_0^2}{2} \tag{4.5.5}$$

which uniquely defines x_{max}.

Example. Let $F(x) = \frac{1}{x}$. Then $v_{0,crit} = \infty$, therefore $x_{max} < \infty$ for any x_0 and (4.5.5) gives

$$ln\, x_{max} - ln\, x_0 = \frac{v_0^2}{2} \quad \Longrightarrow \quad x_{max} = x_0 \cdot exp\left(\frac{v_0^2}{2}\right).$$

Example. Let $F(x) = \frac{1}{x^2}$. Then

$$v_{0,crit} = \sqrt{2\int_{x_0}^\infty \frac{dx}{x^2}} = \sqrt{\frac{2}{x_0}}$$

and if $v_0 < v_{0,crit}$ then x_{max} is a finite number defined by the equation

$$\int_{x_0}^{x_{max}} \frac{dx}{x^2} = \frac{v_0^2}{2}$$

and it follows

$$x_{max} = \frac{1}{\frac{1}{x_0} - \frac{v_0^2}{2}}.$$

Question 2. We can use the inverse function to the solution $x(t)$ restricted to $t \in [0, t_1]$. Since for $t \in [0, t_1)$ the derivative $x'(t)$ is non-negative, from the energy equation (4.5.2) we have

$$x'(t) = \sqrt{v_0^2 - 2\int_{x_0}^{x(t)} F(s)ds}, \quad t \in [0, t_1].$$

Let $t(x), x \in [x_0, x_{max}]$ be the inverse function. Then

$$t'(x) = \frac{1}{\sqrt{v_0^2 - 2\int_{x_0}^{x} F(s)ds}}, \quad x \in [x_0, x_{max}]$$

and consequently

$$t(x) = t(x_0) + \int_{x_0}^{x} \frac{du}{\sqrt{v_0^2 - 2\int_{x_0}^{u} F(s)ds}}.$$

Since $t(x_0) = 0$ and $t(x_{max}) = t_1$ we obtain

$$t_1 = \int_{x_0}^{x_{max}} \frac{dx}{\sqrt{v_0^2 - 2\int_{x_0}^{x} F(s)ds}}.$$

Example. If $F(x) = \frac{1}{x^2}$ then

$$t_1 = \int_{x_0}^{x_{max}} \frac{dx}{\sqrt{v_0^2 + 2\left(\frac{1}{x} - \frac{1}{x_0}\right)}}, \qquad x_{max} = \frac{1}{\frac{1}{x_0} - \frac{v_0^2}{2}}.$$

Question 3. If $v_0 \geq v_{0,crit}$ such time t_2 is unique and can be found in the same way as above using the inverse function $t(x)$. We obtain

$$t_2 = \int_{x_0}^{d} \frac{dx}{\sqrt{v_0^2 - 2\int_{x_0}^{x} F(s)ds}}.$$

If $v_0 < v_{0,crit}$ then there are two options for t_2, see fig. ****:

$$t_2 = t_{2,a} < t_1, \qquad t_2 = t_{2,b} > t_1.$$

By exactly the same argument as above we have

$$t_{2,a} = \int_{x_0}^{d} \frac{dx}{\sqrt{v_0^2 - 2\int_{x_0}^{x} F(s)ds}}.$$

To find $t_{2,b}$ we cannot use the inverse function to $x(t)$ for $t \in [0, t_{2,b}]$, but we can use the inverse function to $x(t)$ for $t \in [t_1, t_{2,b}]$. In this interval the derivative $x'(t)$ is non-positive, and from the energy equation (4.5.2) we have

$$x'(t) = -\sqrt{v_0^2 - 2\int_{x_0}^{x(t)} F(s)ds}, \quad t \in [t_1, t_{2,b}].$$

It follows

$$t_{2,b} - t_1 = \int_{x_{max}}^{d} \frac{dx}{-\sqrt{v_0^2 - 2\int_{x_0}^{x} F(s)ds}} = \int_{d}^{x_{max}} \frac{dx}{\sqrt{v_0^2 - 2\int_{x_0}^{x} F(s)ds}}$$

and consequently

$$t_{2,b} = \int_{x_0}^{x_{max}} \frac{dx}{\sqrt{v_0^2 - 2\int_{x_0}^{x} F(s)ds}} + \int_{d}^{x_{max}} \frac{dx}{\sqrt{v_0^2 - 2\int_{x_0}^{x} F(s)ds}}.$$

Remark. The same answer can be obtained using Theorem 4.3.2 on symmetries. By this theorem we have $t_{2,b} - t_1 = t_1 - t_{2,a}$ and therefore

$$t_{2,b} = 2t_1 - t_{2,a}.$$

Question 4. If $x_0 = R$ then by Theorem **** on symmetries we have $t^* = 2t_1$. If $x_0 > R$ we can compute x^* in the same way as above, see fig. ****. We obtain

$$t^* - t_1 = \int_{x_{max}}^{R} \frac{dx}{-\sqrt{v_0^2 - 2\int_{x_0}^{x} F(s)ds}} = \int_{R}^{x_{max}} \frac{dx}{\sqrt{v_0^2 - 2\int_{x_0}^{x} F(s)ds}}$$

and consequently

$$t^* = \int_{x_0}^{x_{max}} \frac{dx}{\sqrt{v_0^2 - 2\int_{x_0}^{x} F(s)ds}} + \int_{R}^{x_{max}} \frac{dx}{\sqrt{v_0^2 - 2\int_{x_0}^{x} F(s)ds}}.$$

4.6. The pendulum (or a swing)

4.6.1. Introduction. Consider a pendulum or a swing which might oscillate or turn over, see fig. 4.8. It is convenient to use the coordinate θ on the circle S^1 (the circle of the pendulum) such that $\theta = 0$ corresponds to the stable equilibrium and $\theta = \pi$ to the unstable equilibrium of the pendulum. The set of all possible positions of the pendulum is a circle S_1 (or real numbers modulo $2k\pi\pi$) with the coordinate θ measured in radians. The positive direction is the anticlockwise direction.

Our purpose is to understand the motion of the pendulum if at the initial time $t = 0$ we have $\theta = \theta_0 \in [0, \pi)$ and the initial angle velocity is $v_0 > 0$, see fig. 4.8,a. One can expect that there exists a certain critical velocity $v_{0,crit}$ (depending on θ_0) such that:

if $v_0 > v_{0,crit}$ the pendulum will turn over infinite number of times, see fig. 4.8,b.

if $v_0 < v_{0,crit}$ the pendulum will oscillate, see fig. 4.8,c.

We will find $v_{0,crit}$ and we will prove that

if $v_0 = v_{0,crit}$ then the pendulum will be approaching the position of unstable equilibrium, bit will never come to this position, see fig. 4.9.

4.6.2. Qualitative analysis. It is easy to obtain the following equation for the pendulum of length ℓ, where θ is measured in radians:

$$\theta'' = -\frac{g}{\ell} sin\,\theta, \quad \theta \in S^1. \tag{4.6.1}$$

It is important that here θ is not a real number, it is a point in the circle S^1, i.e. a real number modulo $2\pi k$. We will analyze solution $t \to \theta(t) \in S^1$ of this equation satisfying the initial condition

$$\theta(0) = \theta_0 \in [0, \pi), \quad \theta'(0) = v_0 > 0. \tag{4.6.2}$$

LEMMA 4.6.1. *Let $\theta(t)$ be the solution of (4.6.1) satisfying (4.6.2). If $\theta(t_1) = \pi$ for some time t_1 (i.e. the pendulum is in the position of the unstable equilibrium) then $\theta'(t_1) \neq 0$ (the velocity of the pendulum is not 0). If $\theta'(t_1) = 0$ then $\theta(t_1) \neq \pi$.*

PROOF. These statements follow from the uniqueness theorem and the fact that the equation (4.6.1) has the constant solution $\theta(t) \equiv \pi$ (corresponding to the case the the pendulum stays all the time in the position of unstable equilibrium). □

Now we need the energy equation (4.4.1) with $f(x) \to -\frac{g}{\ell} sin\, \theta$. The integral can be easily computed and we obtain

$$\frac{(\theta'(t))^2}{2} - \frac{g}{\ell} cos\, \theta(t) = \frac{v_0^2}{2} - \frac{g}{\ell} cos\, \theta_0. \tag{4.6.3}$$

LEMMA 4.6.2. *Let $\theta(t)$ be the solution of (4.6.1) satisfying (4.6.2).*

1. If $\theta(t_1) = \pi$ for some time t_1 then

$$v_0 > \sqrt{\frac{2g}{\ell}\,(1 + cos\, \theta_0)}. \tag{4.6.4}$$

2. If $\theta'(t_1) = 0$ for some time t_1 then

$$v_0 < \sqrt{\frac{2g}{\ell}\,(1 + cos\, \theta_0)}. \tag{4.6.5}$$

PROOF. Substitute $t = t_1$ to the energy equation (4.6.3). If $\theta(t_1) = \pi$ we obtain

$$\frac{v_0^2}{2} = \frac{(\theta'(t_1))^2}{2} + \frac{g}{\ell}(1 + cos\, \theta_0)$$

which implies (4.6.4) because by Lemma 4.6.1 $\theta(t_1) = \pi \implies \theta'(t_1) \neq 0$.

If $\theta'(t_1) = 0$ we obtain

$$\frac{v_0^2}{2} = \frac{g}{\ell}\,(cos\, \theta_0 - cos\, \theta(t_1))\,.$$

In this case by Lemma 4.6.1 we have $\theta(t_1) \neq 0$, therefore $cos\, \theta(t_1) > -1$ and (4.6.5) follows. □

Now it is clear that the critical initial angle velocity should be defined as follows:

$$\theta_{0,crit} = \sqrt{\frac{2g}{\ell}\,(1 + cos\, \theta_0)}. \tag{4.6.6}$$

THEOREM 4.6.3. *If $v_0 > v_{0,crit}$ then the pendulum will turn over infinitely many times. In this case the solution $t \to \theta(t)$ of (4.6.1) satisfying (4.6.2) is a <u>periodic</u> function defined for all $t \in \mathbb{R}$ such that $\theta'(t) \neq 0$ for all t and*

$$\{\theta(t),\ t \geq 0\} = S^1 \tag{4.6.7}$$

THEOREM 4.6.4. *If $v_0 < v_{0,crit}$ then the pendulum will oscillate. In this case the solution $t \to \theta(t)$ of (4.6.1) satisfying (4.6.2) is a <u>periodic</u> function defined for all $t \in \mathbb{R}$ such that*

$$\{\theta(t),\ t \geq 0\} = [-\theta_{max},\ \theta_{max}], \quad \theta_{max} \in (\theta_0, \pi). \tag{4.6.8}$$

THEOREM 4.6.5. *If $v_0 = v_{0,crit}$ the pendulum will approach the position of the unstable equilibrium, but will never be in this position. In this case the solution $t \to \theta(t)$ of (4.6.1) satisfying (4.6.2) is a function defined for all t such that $\theta'(t) \neq 0$ for all t and*

$$\{\theta(t),\ t \geq 0\} = [\theta_0, \pi), \quad \lim_{t\to\infty} \theta(t) = \pi. \tag{4.6.9}$$

It is important to note that the fact that the solution $\theta(t)$ is periodic for both cases that the pendulum turns over or oscillates is related to the fact that $t \to \theta(t)$ is a function from $\mathbb{R}$ to S^1 and not from $\mathbb{R}$ to $\mathbb{R}$. The two types of periodic solutions are distinguished by the image of $\theta(t)$ as $t \geq 0$, see fig. 4.10.

The proof of Theorems 4.6.3 and 4.6.4 is based on Lemmas 4.6.1 and 4.6.2 as well as on the energy equation (4.6.3) and Theorem 4.3.3 on periodic solutions. A precise proof requires rather big work similar to the proofs in sections 4.5.2, 4.5.3. I leave these proofs to a reader as a (non-simple) exercise.

PROOF OF THEOREM 4.6.5. Let $v_0 = v_{0,crit}$. If $\theta(t_1) = \pi$ for some t_1 then the energy equation (4.6.3) implies $\theta'(t_1) = 0$ which contradicts to Lemma 4.6.1. If $\theta'(t_1) = 0$ for some t_1 then the energy equation (4.6.3) implies $\theta(t_1) = \pi$ which again contradicts to Lemma 4.6.1. Therefore $\theta(t) \neq \pi$ and $\theta'(t) \neq 0$ for all t.

It follows that for $t \geq 0$ we can deal with $\theta(t)$ as with a function $t \to \mathbb{R}$. It is an increasing function and it is bounded from above by π. Therefore it has a limit $B \leq \pi$. We have to prove that $B = \pi$.

Assume, to get contradiction, that $B < \pi$. The function $\frac{g}{\ell} sin\,\theta$ takes positive values on the closed interval $[\theta_0, B]$, therefore it is bounded from below by some positive number ϵ. It follows $\theta''(t) < -\epsilon, \quad t \in [0, \infty)$. But then $\theta'(t) \to -\infty$ which contradicts to the fact that $\theta(t)$ is an increasing function.

4.6.3. Solving the equation of the pendulum (example). .

Let us solve the following problem:

Problem. The initial position of the pendulum is showed in fig. 4.11, a: $\theta_0 = \pi/2$, the initial velocity is 3 rad/sec. The problem is to find the first time t_1 such that at this time the pendulum will be in the position showed in fig. 4.11, b, when $\theta = -\pi/4$.

Certainly t_1 depends on the length ℓ of the pendulum. The required time t_1 exists if and only if $v_0 \neq v_{0,crit}$. If $v_0 > v_{0,crit}$ the pendulum will go through the position of the unstable equilibrium, then will go down to the required position. If $v_0 < v_{0,crit}$ the pendulum will rich the maximal angle $\theta_{max} \in (\pi/2, \pi)$, after that it will go clockwise to the required position. These two cases are principally different.

Therefore we have to start with finding the critical initial velocity. Since $\theta_0 = \pi/2$ we have $v_{0,crit} = \sqrt{\frac{2g}{\ell}}$ and it follows:

$$3 = v_0 > v_{0,crit} \ \Leftrightarrow \ell < \frac{2g}{9}, \quad \text{the pendulum will turn over}$$

$$3 = v_0 < v_{0,crit} \ \Leftrightarrow \ell > \frac{2g}{9}, \quad \text{the pendulum will oscillate}$$

$$3 = v_0 = v_{0,crit} \ \Leftrightarrow \ell = \frac{2g}{9}$$

Therefore if $\ell = \frac{2g}{9}$ then the required time t_1 does not exist. It exists for any $\ell \neq \frac{2g}{9}$.

Case 1: $\ell < \frac{2g}{9}$.

In this case the pendulum will turn over. For $t \in [0, t_1]$ we can deal with $\theta(t)$ as with an increasing function $t \to \mathbb{R}$ taking the value $\pi/2$ as $t = 0$ and the value $\frac{7\pi}{4}$ (and not $-\pi/4$!) as $t = t_1$, see fig. 4.12,a.

The energy equation (4.6.3 with $\theta_0 = \pi/2,\ v_0 = 3$ takes the form

$$\frac{(\theta'(t))^2}{2} - \frac{g}{\ell} \cos\theta(t) = \frac{9}{2}. \tag{4.6.10}$$

Since $\theta'(t) > 0$ for $t \in [0, t_1]$ we have

$$\theta'(t) = \sqrt{9 + \frac{2g \cos\theta(t)}{\ell}}, \quad t \in [0, t_1].$$

Using the inverse function $t(x), x \in [\pi/2, 7\pi/4]$ in the same way as many times above we obtain

$$t_1 = \int_{\pi/2}^{7\pi/4} \frac{d\theta}{\sqrt{9 + \frac{2g \cos\theta}{\ell}}}.$$

Case 2: $\ell > \frac{2g}{9}$.

In this case the pendulum will oscillate. We can deal with $\theta(t)$ as with a function $t \to \mathbb{R}$, but in this case $7\pi/4$ must be replaced by $-\pi/4$, see fig. 4.12,b. The required t_1 can be found in two steps, using the inverse function for $t \in [0, t^*]$ and the inverse function for $t \in [t^*, t_1]$, where t^* is the time such that $\theta(t^*) = \theta_{max} \in (\pi/2, \pi)$, see fig. 4.12,b. In the same way as in section 4.5.5 we obtain

$$t^* = \int_{\pi/2}^{\theta_{max}} \frac{d\theta}{\sqrt{9 + \frac{2g \cos\theta}{\ell}}}$$

$$t_1 - t^* = \int_{\theta_{max}}^{-\pi/4} \frac{d\theta}{-\sqrt{9 + \frac{2g \cos\theta}{\ell}}} == \int_{-\pi/4}^{\theta_{max}} \frac{d\theta}{\sqrt{9 + \frac{2g \cos\theta}{\ell}}}$$

and consequently

$$t_1 = \int_{\pi/2}^{\theta_{max}} Q(\theta) d\theta + \int_{-\pi/4}^{\theta_{max}} Q(\theta) d\theta, \quad \text{where } \ Q(\theta) = \frac{1}{\sqrt{9 + \frac{2g \cos\theta}{\ell}}}$$

It remains to find $\theta_{max} \in (\pi/2, \pi)$. It is simple: just substitute t^* to the energy equation (4.6.10). We obtain

$$\theta_{max} = arccos\left(-\frac{9\ell}{2g}\right),$$

with the value of $arccos$ in the interval $(\pi/2, \pi)$.

4.7. Exercises

1. Let $x(t)$ be solution of an equation of the form $x'' = f(x) \in C^1(\mathbb{R})$ satisfying the initial conditions $x(1) = 2, x'(1) = 3$ and defined for all t. Let $\widetilde{x}(t)$ be solution of the same equation satisfying the initial conditions $\widetilde{x}(0) = 2, \widetilde{x}'(0) = -3$ and defined for all t. Find t_1 such that $\widetilde{x}(13) = x(t_1)$. (Use Theorem 4.3.1).

2. Prove that in the two body problem of section 4.5 the critical initial velocity can be defined as follows: it is the only initial velocity such that $x(t) \to \infty$ and $x'(t) \to 0$ as $t \to \infty$.

3. A rocket has been launched up from the Earth surface with the initial velocity $v_0 = (1-\epsilon)v_{0,crit}$, where $v_{0,crit} = \sqrt{2gR}$ is the "escape velocity" (or second cosmic velocity), $\epsilon > 0$, and R is the radius of the Earth. Let $t^* = t^*(\epsilon)$ be the time in which the rocket will reach its maximal height. Obtain explicit (without integrals) formula for the function $t^*(\epsilon)$ and characterize the behavior of $t^*(\epsilon)$ ar $\epsilon \to 0$. For which ϵ (approximately) the rocket will reach its maximal height in a month after it has been launched?

Solving the problem use the equation $x'' = -\frac{gR^2}{x^2}$ where $x = x(t)$ is the distance between the rocket and the center of the Earth. You will have to compute the integral of the form $\int \frac{dx}{\sqrt{\frac{a}{x}+b}}$. Enjoy integration techniques from hedva or infi.

4. Assume that a big body and a small body are points (balls of radius 0), the big body does not move and attracts the small body of mass 1 kg with the force

$$a)\ \ F = \frac{1}{xlnx}, \qquad b)\ \ F = \frac{1}{\sqrt{x}} \qquad c)\ \ F = \frac{1}{x\sqrt{x}}$$

$kg \cdot m/sec^2$, where x is the distance between the bodies (in meters). At the initial time ($t = 0$) the distance between the bodies is 10 meters and the initial velocity of the small body is v_0 m/sec directed in such a way that the distance between the bodies starts to increase. In which time t^* the distance between the bodies will be 20 meters? The answer depends on v_0. If the answer is not unique you should find all answers. Integrals in the answers are OK.

5. Develop a complete theory for another version of two body problem on a line: a big body repels the small one with the force $F(x)$ and the initial velocity of the small body is directed towards the big body (fig. 4.13):

$$x'' = F(x) \in C^1(x > R),\ \ F(x) > 0 \text{ for all } x > R,$$

$$x(0) = x_0, \quad x'(0) = v_0 < 0.$$

Here "complete theory" means analogous of the theorems in sections 4.5.2, 4.5.3 describing two possible cases and distinguishing them in terms of critical velocity.

6. A big body which does not move repels a small body of mass 3 kg with the force $1/x$ (in $kg \cdot m/sec^2$) where $x = x(t)$ is the distance between the bodies (in meters). At the initial time-moment the distance between the bodies is 4 meters and the small body has velocity 2 m/sec towards the big body. Find (a) the minimal distance x_{min} between the bodies (without integrals in the answer) and (b) time t_1 such that $x(t_1) = x_{min}$ (integrals in the answer OK) and (c) time t_2 such that the distance between the bodies is 10 m (integrals in the answer OK).

7. Find the period of the pendulum for case 1 and for case 2 of section 4.6.3, i.e. the minimal number T such that $\theta(t+T) = \theta(t) \bmod 2k\pi$.
Integrals in the answers OK.

8. The initial position of the pendulum is showed in fig. 4.14
($\theta_0 = 3\pi/4,\ v_0 = 1\, rad/sec$). Find the first time t_1 such that

(a) the pendulum has velocity 2 rad /sec directed anticlockwise

(b) the pendulum has velocity 2 rad /sec directed clockwise

(c) the pendulum has velocity 0.5 rad /sec directed anticlockwise

(d) the pendulum has velocity 0.5 rad /sec directed clockwise.

The length ℓ of the pendulum is a parameter of the answer. Integrals in the answer are OK. It is very important to distinguish the case that the pendulum turns over from the case of oscillations. In some cases (that you have to find) t_1 does not exist.

CHAPTER 5

Systems of ODEs $x' = Ax$

5.1. Existence and uniqueness theorem

In this chapter we consider systems of ODEs of the form

$$x' = Ax, \quad x = x(t) = \begin{pmatrix} x_1(t) \\ \cdots \\ x_n(t) \end{pmatrix}, \quad A \text{ is an } n \times n \text{ matrix.} \tag{5.1.1}$$

For example, the system

$$x_1'(t) = 2x_1(t) + 5x_2(t), \quad x_2'(t) = -3x_1(t) + 7x_2(t)$$

can be writen in form (5.1.1) with $A = \begin{pmatrix} 2 & 5 \\ -3 & 7 \end{pmatrix}$.

System of form (5.1.1) are called linear homogeneous systems of first order ODEs with constant coefficients.

The initial condition in the case of systems of first order ODEs, in particular systems (5.1.1), is the condition

$$x(t_0) = x_0 \in \mathbb{R}^n \tag{5.1.2}$$

with a fixed time t_0 and fixed vector $x_0 \in \mathbb{R}^n$.

It is clear that if $x(t)$ is a solution of (5.1.1) then $\widetilde{x}(t) = x(t + \alpha)$ is also a solution, for any real α (shift of time). It allows to reduce the general initial condition (5.1.2) to the initial condition

$$x(0) = x_0 \in \mathbb{R}^n \tag{5.1.3}$$

Theorem 5.1.1. *For any $n \times n$ matrix A and any $x_0 \in \mathbb{R}^n$ system (5.1.1) has a solution defined for all $t \in \mathbb{R}$ and satisfying (5.1.3). Such a solution is unique.*

We postpone to the end of the course the proof of the uniqueness. The proof of the existence of a solution defined for all t is based on the exponent of a matrix, see section 5.3

5.2. The structure of the set of all solutions

Theorem 5.2.1. *Fix an $n \times n$ matrix A. The set of all solutions of (5.1.1) defined for all $t \in \mathbb{R}$ is a vector space over $\mathbb{R}$ of dimension n.*

Proof. The set of all differentiable vector-functions defined for all t is an infinite-dimensional vector space over $\mathbb{R}$. To show that the set of all solutions of (5.1.1) is a subspace of this space we have to check that the sum of two solutions is also a solution and that multiplying a solution by a real number we get another solution. Each of these claims is very simple and we leave the proof to a reader.

To prove that the dimension of the set of all solutions of (5.1.1) is equal to n we have to give an example of a basis consisting of n solutions. Take the standard basis of $\mathbb{R}^n$:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \cdots \\ 0 \end{pmatrix}, \ e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \cdots \\ 0 \end{pmatrix}, \ \cdots, \ e_n = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix}. \tag{5.2.1}$$

By the existence part of Theorem 5.1.1 there are solutions $x^{(1)}(t), ..., x^{(n)}(t)$ satisfying the initial conditions

$$x^{(1)}(0) = e_1, \ x^{(2)}(0) = e_2, \ \cdots, \ x^{(n)}(0) = e_n.$$

We claim that these n solutions is a basis of the vector space of all solutions. To prove it we have to show that these n solutions are linearly independent and that any solution is there linear combination with numerical coefficients.

The linear independence is obvious. In fact, if a linear combination of the vector-functions $x^{(1)}(t), ..., x^{(n)}(t)$ with coefficients $r_1, ...r_n \in \mathbb{R}$ is the zero vector, i.e. the zero vector-function, then $r_1 x^{(1)}(t) + \cdots + r_n x^{(n)}(t) \equiv 0$ and substituting $t = 0$ we obtain $r_1 e_1 + \cdots + r_n e_n = 0$ whence $r_1 = \cdots = r_n = 0$.

The proof that any solution is a linear combination of the solutions $x^{(1)}(t), ..., x^{(n)}(t)$ requires the uniqueness part of Theorem 5.1.1. Take any solution $x(t)$. Consider the vector $x(0) = (q_1, ..., q_n) \in \mathbb{R}^n$ (sometimes a vector will be written as a row). Consider the solution $\widehat{x}(t) = q_1 x^{(1)}(t) + \cdots + q_n x^{(n)}(t)$. We have $\widehat{x}(0) = x(0) = (q_1, ..., q_n)$ and by the uniqueness part of Theorem 5.1.1 $x(t) = \widehat{x}(t)$, i.e. $x(t)$ is a linear combination of the solutions $x^{(1)}(t), ..., x^{(n)}(t)$. □

5.3. Solutions in the form of the exponent of a matrix

The exponent of an $n \times n$ matrix A is defined by the same series as the exponent of a number:

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots \tag{5.3.1}$$

Here I is the identity $m \times n$ matrix. Consider also the series

$$e^{tA} = I + tA + \frac{t^2}{2!} A^2 + \frac{t^3}{3!} A^3 + \cdots = B(t) = \begin{pmatrix} b_{11}(t) & \cdots & b_{1n}(t) \\ \cdots & & \\ b_{n1}(t) & \cdots & b_{nn}(t) \end{pmatrix}, \ t \in \mathbb{R}. \tag{5.3.2}$$

Given a matrix $Q = Q(t)$ whose entries $Q_{ij}(t)$ are functions of t, the derivative $Q'(t)$ of this matrix is defined to be a matrix whose entries $Q'_{ij}(t)$. We will use the following theorem (proved in some infi courses).

Theorem 5.3.1. *The series (5.3.1) converges to a certain matrix, for any square matrix A. Consequently the series (5.3.2) converges to a certain matrix $B(t)$ whose entries $b_{ij}(t)$ are functions of t, defined for all $t \in \mathbb{R}$. These functions are infinitely-differentiable. The derivative of the sum of the series (5.3.2) is the sum of the derivatives of the matrices in this series, i.e.*

$$B'(t) = \left(e^{tA}\right)' = (I)' + (tA)' + \left(\frac{t^2}{2!} A^2\right)' + \left(\frac{t^3}{3!} A^3\right)' + \cdots.$$

Obviously

$$(I)' = 0,\ (tA)' = A,\ \left(\frac{t^2}{2!}A^2\right)' = tA^2,\ \left(\frac{t^3}{3!}A^3\right)' = \frac{t^2}{2!}A^3,\ \cdots .$$

Therefore

$$\left(e^{tA}\right)' = A + tA^2 + \frac{t^2}{2!}A^3 + \cdots = A\left(I + tA + \frac{t^2}{2!}A^2 + \cdots\right)$$

and we obtain, as a corollary, the following statement:

$$\left(e^{tA}\right)' = Ae^{tA}.$$

Take now any vector $x_0 \in \mathbb{R}$ and consider the vector-function $e^{tA}v_0$. It is clear that $\left(e^{tA}v_0\right)' = \left(e^{tA}\right)' v_0$ and consequently

$$\left(e^{tA}v_0\right)' = Ae^{tA}v_0.$$

Note that the exponent of the zero matrix is the matrix I, therefore the vector-function $e^{tA}x_0$ takes the value x_0 as $t = 0$. We obtain the following theorem.

THEOREM 5.3.2. *The vector-function $e^{tA}x_0$ is the solution of system (5.1.1) satisfying (5.1.3).*

The proof of Theorem 5.2.1 implies the following corollary.

THEOREM 5.3.3. *Let $e_1, ..., e_n$ is the standard basis of $\mathbb{R}^n$ (see 5.2.1). The n vector functions $e^{tA}e_1, ..., e^{tA}e_n$ is a basis of the vector space of all solutions of system (5.1.1).*

REMARK 5.3.4. Tracing the proof of Theorem 5.2.1 it is easy to see that in Theorem 5.3.3 the standard basis of $\mathbb{R}^n$ can be replaced by any basis of $\mathbb{R}^n$.

The applications of Theorems 5.3.2 and 5.3.3 for computation of solutions are rather restricted since these theorems involve an infinite series of matrices which might converge very slowly, especially when t is big. In sections 5.4- 5.8 we will give a way to find a basis of the vector space of all solutions of (5.1.1) in a much more explicit form.

Theorem 5.3.2 gives a good for computation formula for solution only in the case that A is a nilpotent matrix which means that $A^{k+1} = 0$ for some $k \geq 0$. The following statement is a direct corollary of Theorem 5.3.2.

THEOREM 5.3.5. *Assume that A is a nilpotent matrix, i.e. $A^{k+1} = 0$ for some $k \geq 0$. Then the solution of system (5.1.1) satisfying (5.1.3) is the polynomial vector function*

$$x(t) = \sum_{i=0}^{k} \frac{t^i}{i!}A^i, \quad \textit{where } A^0 = I.$$

REMARK 5.3.6. It is known that an $n \times n$ matrix is nilpotent if and only if it has only one eigenvalue 0 (of algebraic multiplicity n). For example the matrix $\begin{pmatrix} 3 & a \\ b & c \end{pmatrix}$ is nilpotent if and only if $c = -3$ and $ab = 3c$. One can see it immediately using the fact that a 2×2 matrix has only one zero eigenvalue if and only if both trace and determinant of this matrix are equal to 0.

Another application of Theorem 5.3.2 is computation of the solution of (5.1.1) satisfying (5.1.3) up to terms of order $\ell + 1$ as $t \to 0$. The following statement is also a direct corollary of Theorem 5.3.2.

THEOREM 5.3.7. *Let $\ell \geq 1$. The the solution of system (5.1.1) satisfying (5.1.3) has the form*

$$x(t) = \sum_{i=0}^{\ell} \frac{t^i}{i!} A^i + o\left(t^\ell\right) \quad \text{as } t \to 0.$$

5.4. Computation of a basis in the case that A is diagonalizable over $\mathbb{R}$

Recall from basic algebra course that the following conditions for a square real matrix A are equivalent:

1. there is a basis of $\mathbb{R}^n$ consisting of eigenvectors of A;

2. A is similar, over $\mathbb{R}$, to the matrix $D = diag(\lambda_1, ..., \lambda_n)$: the diagonal matrix with the eigenvalues of A on the diagonal. It means that $T^{-1}AT = D$ for some non-singular real matrix T (non-singular means invertible);

3. all eigenvalues of A are real and each of them has the same algebraic and geometric multiplicity.

If one (and then any other) of these conditions holds the matrix A is called diagonalizable over $\mathbb{R}$.

THEOREM 5.4.1. *Assume that A is diagonalizable over $\mathbb{R}$. Let $v_1, ..., v_n \in \mathbb{R}^n$ be linearly independent eigenvectors of A corresponding to the eigenvalues $\lambda_1, ..., \lambda_n \in \mathbb{R}$ respectively (the case $\lambda_i = \lambda_j$ for some $j \neq i$ is not excluded). Then the n vector-functions*

$$e^{\lambda_1 t} v_1, \ \cdots, \ e^{\lambda_n t} v_n \tag{5.4.1}$$

is a basis of the vector space of all solutions of (5.1.1).

PROOF. We have

$$\left(e^{\lambda_i t} v_i\right)' = \lambda_i e^{\lambda_i t} v_i = e^{\lambda_i t} \left(\lambda_i v_i\right) = e^{\lambda_i t} A v_i = A\left(e^{\lambda_i t} v_i\right)$$

and it follows that each of the vector-functions (5.4.1) is a solution of (5.1.1). To prove that these n solutions are linearly independent we assume that there linear combination with numerical coefficients is the zero function, we substitute $t = 0$ and obtain that the linear combination of the eigenvectors $v_1, ..., v_n$ with the same coefficients is equal to 0. Since the eigenvector are linearly independent it follows that each of the coefficients is equal to 0. □

5.5. System $x' = Ax$ for complex-valued vector-functions

To cover the case that some of the eigenvalues of A are not real we should consider the complex-valued solutions of system (5.1.1), i.e. vector functions

$$x(t) = (x_1(t), ..., x_n(t)), \quad x_k(t) = a_k(t) + ib_k(t), \ i = \sqrt{-1}, \quad k = 1, ..., n.$$

The variable t remains real. The derivative of complex-valued vector function is defined by

$$x'(t) = (x_1'(t), ..., x_n'(t)), \quad x_k'(t) = a_k'(t) + ib_k'(t), \quad i = \sqrt{-1}$$

DEFINITION 5.5.1. The exponent of a complex (including real) $n \times n$ matrix A (including the exponent of a complex number which is the case $n = 1$) is defined in the same way as the exponent of a real matrix, i.e. by the series (5.3.1).

All the results of sections 5.1 - 5.4 also hold for complex-valued solutions of system (5.1.1).

THEOREM 5.5.2. *Let A be any complex (including real) $n \times n$ matrix. For any $x_0 \in \mathbb{C}^n$ there exists one and only one complex-valued solution of system (5.1.1) satisfying the condition $x(0) = x_0$ and defined for all $t \in \mathbb{R}$. It can be given by the series $x(t) = e^{tA}x_0$. The set of all complex-valued solutions of system (5.1.1) is a vector space over $\mathbb{C}$ of dimension n.*

Recall from basic algebra course that the following conditions for a square comples matrix A are equivalent:

1. there is a basis of $\mathbb{C}^n$ consisting of eigenvectors of A;

2. A is similar, over $\mathbb{C}$, to the matrix $D = diag(\lambda_1, ..., \lambda_n)$: the diagonal matrix with the eigenvalues of A on the diagonal. It means that $T^{-1}AT = D$ for some non-singular complex matrix T (non-singular means invertible);

3. each of the eigenvalues of A has the same algebraic and geometric multiplicity.

If one (and then any other) of these conditions holds the matrix A is called diagonalizable over $\mathbb{C}$.

THEOREM 5.5.3. *If A is diagonalizable over $\mathbb{C}$ then one of the basis of the vector space of all complex-valued solutions of (5.1.1) is the n functions (5.4.1) where $v_1, ..., v_n \in \mathbb{C}^n$ are linearly independent eigenvectors of A corresponding to the eigenvalues $\lambda_1, ..., \lambda_n \in \mathbb{C}$ respectively (the case $\lambda_i = \lambda_j$ for some $j \neq i$ is not excluded).*

5.6. The Euler formula and the formula for $e^{z_1+z_2}$

One of the most valuable for applications in the following beautiful theorem.

THEOREM 5.6.1 (Euler).

$$e^{it} = cos\, t + i \cdot sin\, t, \quad i = \sqrt{-1}, \quad t \in \mathbb{R}.$$

One of the proofs is to distinguish the real part and the imaginary part in the series $e^{it} = 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \cdots$: we obtain the well-known series for $cos\, t$ and $sin\, t$.

An alternative proof, using ODEs, is as follows. Consider the equation $z'(t) = iz(t)$ for a complex valued function $z(t) = x(t) + iy(t)$. One of its solutions is $z_1(t) = e^{it}$. It satisfies the condition $z(0) = 1$. On the other hand, the equation $z'(t) = iz(t)$ can be expressed as the system of two equations for $x(t)$ and $y(t)$, namely $x'(t) = -y(t)$, $y'(t) = x(t)$. This system has a solution $x(t) = cos\, t$, $y(t) = sin\, t$. Therefore the equation $z' = iz$ has a solution $z_2(t) = cos\, t + i\, sin\, t$. One has $z_2(0) = 1$. By the uniqueness theorem $z_1(t) \equiv z_2(t)$ and the Euler formula follows.

We need one more result (without proof).

THEOREM 5.6.2. *$e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$ for any complex numbers z_1, z_2.*

Combining Theorems 5.6.1 and 5.6.2 we obtain the formula which will be used in the sequel:

$$e^{(a+bi)t} = e^{at}\left(cos(bt) + i \cdot sin(bt)\right), \quad i = \sqrt{-1} \tag{5.6.1}$$

5.7. Computation of a basis in the case that A is diagonalizable over $\mathbb{C}$, but not over $\mathbb{R}$

The matrix A is real in any case. Therefore for each non-real eigenvalue λ there is an eigenvalue $\bar{\lambda}$ and the eigenvector corresponding to $\bar{\lambda}$ can be chosen to be complexly-conjugate to the eigenvector corresponding to λ. Let the eigenvalues and the eigenvectors of A be as follows:

$$\begin{aligned} &\lambda_1 = a_1 + b_1 i \to v_1,\ \bar{\lambda}_1 = a_1 - b_1 i \to \bar{v}_1, \quad b_1 \neq 0, \quad v_1 \in \mathbb{C}^n - \mathbb{R}^n \\ &\cdots \\ &\lambda_s = a_s + b_s i \to v_s,\ \bar{\lambda}_s = a_s - b_s i \to \bar{v}_s, \quad b_s \neq 0, \quad v_s \in \mathbb{C}^n - \mathbb{R}^n \\ &\theta_1 \to w_1,\ \cdots,\ \theta_k \to w_k, \quad \theta_1, ..., \theta_k \in \mathbb{R}, \qquad w_1, ..., w_n \in \mathbb{R}^n. \end{aligned} \tag{5.7.1}$$

Then the tuple of n vector-functions

$$\begin{aligned} &f_1 = e^{\lambda_1 t} v_1,\ \bar{f}_1 = e^{\bar{\lambda}_1 t}\bar{v}_1, \quad \cdots, \\ &f_s = e^{\lambda_s t} v_s,\ \bar{f}_s = e^{\bar{\lambda}_s t}\bar{v}_s, \\ &g_1 = e^{\theta_1 t} w_1, \quad \cdots \quad g_k = e^{\theta_k t} w_k \end{aligned} \tag{5.7.2}$$

consisting of $2s$ complex-valued functions $f_1, \bar{f}_1, ..., f_s, \bar{f}_s$ and k real-valued functions $g_1, ..., g_k$ is a basis of the vector space of complex-valued solutions of system (5.1.1). Replace this tuple by another tuple of n functions as follows:

$$\begin{aligned} &h_1 = \frac{f_1 + \bar{f}_1}{2} = Re(f_1),\ h_2 = \frac{f_1 - \bar{f}_1}{2i} = Im(f_1), \\ &\cdots, h_s = \frac{f_s + \bar{f}_s}{2} = Re(f_s),\ h_2 = \frac{f_s - \bar{f}_s}{2i} = Im(f_s), \\ &g_1 = e^{\theta_1 t} w_1, \quad \cdots \quad g_k = e^{\theta_k t} w_k \end{aligned} \tag{5.7.3}$$

Since the functions (5.7.2) are linearly independent over $\mathbb{C}$, the functions (5.7.3) are also linearly independent over $\mathbb{C}$ and consequently over $\mathbb{R}$. Each of these functions is a real-valued solution of (5.1.1). Therefore the tuple of n functions (5.7.3) is a basis of the vector space of real-valued solutions of (5.1.1). Using the Euler formula we obtain the following theorem.

THEOREM 5.7.1. *Assume that the eigenvalues and the eigenvector of a real $n \times n$ matrix A have form (5.7.1). Then one of the basis of the vector space of real-valued solutions of system (5.1.1) is the tuple of vector functions*

$$\begin{aligned} &e^{a_1 t} Re\left((cos(b_1 t) + i \cdot sin(b_1 t)) \cdot v_1\right),\ e^{a_1 t} Im\left((cos(b_1 t) + i \cdot sin(b_1 t)) \cdot v_1\right), \\ &\cdots, \\ &e^{a_s t} Re\left((cos(b_s t) + i \cdot sin(b_s t)) \cdot v_1\right),\ e^{a_s t} Im\left((cos(b_s t) + i \cdot sin(b_s t)) \cdot v_1\right), \\ &e^{\theta_1 t} w_1, \quad \cdots \quad g_k = e^{\theta_k t} w_k. \end{aligned}$$

5.8. Computation of a basis in the case that A is not diagonalizable over $\mathbb{C}$ (only the case $n \leq 3$)

The way is as follows. Given a system $x' = Ax$ with a non-diagonalizable over $\mathbb{C}$ real $n \times n$ matrix A we introduce a new vector-function $y = (y_1(t), ..., y_n(t))$ related to $x = (x_1(t), ..., x_n(t))$ by a non-singular (=invertible) matrix T:

$$x = Ty, \quad x = \begin{pmatrix} x_1(t) \\ \cdots \\ x_n(t) \end{pmatrix}, \; y = \begin{pmatrix} y_1(t) \\ \cdots \\ y_n(t) \end{pmatrix}.$$

The system $x' = Ax$ takes the form

$$(Ty)' = ATy$$

and consequently

$$y' = Jy, \quad J = T^{-1}AT.$$

Here the matrix J is similar to A and T is a transition matrix from A to J (when A and are J fixed, the transition matrix T is not unique). By the choice of T we can make J to be <u>any</u> matrix similar to T. It is worth to chose T such that J is as simple as possible. Now we need the following statements from linear algebra.

PROPOSITION 5.8.1. *Any real 2×2 matrix A which is not diagonalizable over $\mathbb{C}$ is similar to the matrix*

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \tag{5.8.1}$$

where λ is the eigenvalue of A (it must be real and have algebraic multiplicity 2).

PROPOSITION 5.8.2. *Any real 3×3 matrix A which is not diagonalizable over $\mathbb{C}$ is similar to one of the matrices*

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \tag{5.8.2}$$

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix} \quad \text{here the case } \; \lambda = \mu \text{ is not excluded.} \tag{5.8.3}$$

The case (5.8.2) holds if and only if A has only one eigenvalue λ of algebraic multiplicity 3 and geometric multiplicity 1.

The case (5.8.3) with $\lambda = \mu$ holds if and only if A has two distinct eigenvalues $\lambda \neq \mu$, the eigenvalue λ has algebraic multiplicity 2 and geometric multiplicity 1, the eigenvalue μ has algebraic and geometric multiplicity 1;

The case (5.8.3) with $\lambda \neq \mu$ holds if and only if A has has only one eigenvalue λ of algebraic multiplicity 3 and geometric multiplicity 2.

The matrices (5.8.1), (5.8.2), (5.8.3) are all possible <u>Jordan normal forms</u> of a non-diagonalizable 2×2 or 3×3 matrix A. Propositions 5.8.1 and 5.8.2 give a simple way to find the Jordan normal form for any such A (for 2×2 matrices it is unique).

If J is one of the matrices (5.8.1), (5.8.2), (5.8.3), a basis of the vector space of solutions of system $y' = Jy$ is as follows.

PROPOSITION 5.8.3. *If J is one of the matrices (5.8.1), (5.8.2), (5.8.3) then the vector space of solutions of the system $y' = Jy$ has the following basis:*

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}: \quad \begin{pmatrix} e^{\lambda t} \\ 0 \end{pmatrix}, \begin{pmatrix} te^{\lambda t} \\ e^{\lambda t} \end{pmatrix};$$

$$J = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}: \quad \begin{pmatrix} e^{\lambda t} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} te^{\lambda t} \\ e^{\lambda t} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{t^2}{2}e^{\lambda t} \\ te^{\lambda t} \\ e^{\lambda t} \end{pmatrix};$$

$$J = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}: \quad \begin{pmatrix} e^{\lambda t} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} te^{\lambda t} \\ e^{\lambda t} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ e^{\mu t} \end{pmatrix}.$$

PROOF. It is easy to check that each of these vector-functions is a solution of the system $y' = Jy$ and that these solutions are linearly independent. □

This proposition implies the following corollary.

THEOREM 5.8.4. *Let A be a 2×2 or 3×3 matrix which is not diagonalizable over $\mathbb{C}$. Let J be the Jordan normal form of A and let T be the transition matrix: $T^{-1}AT = J$. The the vector space of solutions of the system $x' = Ax$ has the following basis:*

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}: \quad T\begin{pmatrix} e^{\lambda t} \\ 0 \end{pmatrix}, T\begin{pmatrix} te^{\lambda t} \\ e^{\lambda t} \end{pmatrix};$$

$$J = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}: \quad T\begin{pmatrix} e^{\lambda t} \\ 0 \\ 0 \end{pmatrix}, T\begin{pmatrix} te^{\lambda t} \\ e^{\lambda t} \\ 0 \end{pmatrix}, T\begin{pmatrix} \frac{t^2}{2}e^{\lambda t} \\ te^{\lambda t} \\ e^{\lambda t} \end{pmatrix};$$

$$J = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}: \quad T\begin{pmatrix} e^{\lambda t} \\ 0 \\ 0 \end{pmatrix}, T\begin{pmatrix} te^{\lambda t} \\ e^{\lambda t} \\ 0 \end{pmatrix}, T\begin{pmatrix} 0 \\ 0 \\ e^{\mu t} \end{pmatrix}.$$

Finding the transition matrix T

In order to use Theorem 5.8.4 one should know how to find the transition matrix T. It can be found as follows.

The case $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. Write the equation $T^{-1}AT = J$ in the form $AT = TJ$. Let $T_1, T_2 \in \mathbb{R}^2$ be the columns of T. The equation $AT = TJ$ is the same as the system

$$AT_1 = \lambda T_1, \quad AT_2 = \lambda T_2 + T_1$$

or equivalently

$$(A - \lambda I)T_1 = 0, \quad (A - \lambda I)T_2 = T_1.$$

Each of these equations is a linear system. The first linear system means that T_1 is one of the eigenvectors and we can chose any eigenvector. The second linear system has a solution T_2 (not unique) with any eigenvector T_1. Whatever T_1 and T_2 are chosen the vectors T_1, T_2 are linearly independent and consequently T is invertible matrix.

The case $J = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$. Write the equation $T^{-1}AT = J$ in the form $AT = TJ$. Let $T_1, T_2, T_3 \in \mathbb{R}^3$ be the columns of T. The equation $AT = TJ$ is the same as the system

$$AT_1 = \lambda T_1, \quad AT_2 = \lambda T_2 + T_1, \; AT_3 = \lambda T_3 + T_2$$

or equivalently

$$(A - \lambda I)T_1 = 0, \quad (A - \lambda I)T_2 = T_1, \; (A - \lambda I)T_3 = T_2.$$

Each of these equations is a linear system. The first linear system means that T_1 is one of the eigenvectors and we can chose any eigenvector. The second linear system has a solution T_2 (not unique) with any eigenvector T_1. The third linear system has a solution T_3 (not unique) whatever the eigenvector T_1 and whatever the solution T_2 of the second system are chosen. Whatever is our choice for the eigenvector T_1 and solutions T_2, T_3 of the second and the third linear system, the vectors T_1, T_2, T_3 are linearly independent and consequently T is invertible matrix.

The case $J = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}$. Write the equation $T^{-1}AT = J$ in the form $AT = TJ$. Let $T_1, T_2, T_3 \in \mathbb{R}^3$ be the columns of T. The equation $AT = TJ$ is the same as the system

$$AT_1 = \lambda T_1, \quad AT_2 = \lambda T_2 + T_1, \; AT_3 = \mu T_3.$$

or equivalently

$$(A - \lambda I)T_1 = 0, \quad (A - \lambda I)T_2 = T_1, \; (A - \mu I)T_3 = 0.$$

Each of these equations is a linear system. The first and the third linear systems mean that T_1 and T_3 are eigenvectors corresponding to the eigenvalues λ and μ. They are always linearly independent if $\lambda \neq \mu$. If $\lambda = \mu$ they can be chosen to be linearly independent. If $\lambda \neq \mu$ then the second linear system always has a solution T_2. It is not so if $\lambda = \mu$: in this case the second system has a solution $T_2 \in \mathbb{R}^2$ for a certain, but not any choice of T_1. If T_1 and T_3 are linearly independent eigenvectors and T_1 is chosen such that the second system has a solution T_2 then the vectors T_1, T_2, T_3 are linearly independent and consequently T is invertible matrix.

5.9. Example

Let us find the solution of the system

$$x' = Ax, \quad A = \begin{pmatrix} 2 & 1 & 3 \\ 2 & 3 & 6 \\ a & 0 & 0 \end{pmatrix}$$

with parameters $a, b \in \mathbb{R}$ satisfying the initial condition $x(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. The eigenvalues of A are as follows:

$$\lambda_1 = 1, \quad \lambda_{2,3} = 2 \pm \sqrt{4 + 3a}.$$

Case 1: $a > -4/3$ and $a \neq -1$.
In this case A has three distinct real eigenvalues and consequently A is diagonalizable over $\mathbb{R}$.

Case 2: $a < -4/3$.
In this case A has a real eigenvalue $\lambda_1 = 1$ and two non-real conjugate eigenvalues $\lambda_3 = \bar{\lambda}_2$, therefor A is diagonalizable over $\mathbb{C}$, but not over $\mathbb{R}$.

Case 3: $a = -4/3$. In this case A has two eigenvalues $\lambda_1 = 1$ of algebraic multiplicity 1 and $\lambda_2 = 2$ of algebraic multiplicity 2. The geometric multiplicity of $\lambda_2 = 2$ is equal to 1.

Case 4: $a = -1$. In this case A has two eigenvalues $\lambda_1 = 1$ of algebraic multiplicity 2 and $\lambda_2 = 3$ of algebraic multiplicity 1. The geometric multiplicity of $\lambda_1 = 1$ is equal to 1.

Solution in case 1. Let, for example $a = 0$ so that $A = \begin{pmatrix} 2 & 1 & 3 \\ 2 & 3 & 6 \\ 0 & 0 & 0 \end{pmatrix}$. The eigenvectors corresponding to the eigenvalues $1, 4, 0$ can be chosen as follows:

$$\lambda_1 = 1 \to \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}; \quad \lambda_2 = 4 \to \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}; \quad \lambda_3 = 0 \to \begin{pmatrix} 3 \\ 6 \\ -4 \end{pmatrix}.$$

Therefore any solution has the form

$$C_1 e^t \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} 3 \\ 6 \\ -4 \end{pmatrix}.$$

To find the solution satisfying the initial condition $x(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ we substitute $t = 0$ and obtain a linear system for C_1, C_2, C_3:

$$\begin{pmatrix} 1 & 1 & 3 \\ -1 & 2 & 6 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We obtain $C_1 = 0, C_2 = 3/4, C_3 = -1/4$. Therefore

$$x(t) = \frac{3}{4} e^{4t} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} 3 \\ 6 \\ -4 \end{pmatrix}$$

or equivalently

$$x_1(t) = \frac{3}{4}\left(e^{4t} - 1\right), \quad x_2(t) = \frac{3}{2}\left(e^{4t} - 1\right), \quad x_3(t) \equiv 1.$$

Solution in case 2. Let, for example $a = -3$ so that $A = \begin{pmatrix} 2 & 1 & 3 \\ 2 & 3 & 6 \\ -3 & 0 & 0 \end{pmatrix}$. The eigenvectors corresponding to the eigenvalues $1, 2 \pm i\sqrt{5}$ can be chosen as follows:

$$\lambda_1 = 1 \to v_1 = \begin{pmatrix} 1 \\ 8 \\ -3 \end{pmatrix}; \quad \lambda_2 = 2 + i\sqrt{5} \to v_2 = \begin{pmatrix} 2 + \sqrt{5}i \\ 4 + 2\sqrt{5}i \\ -3 \end{pmatrix}, \; \lambda_3 = \bar{\lambda}_2 \to v_3 = \bar{v}_2.$$

Therefore a basis of the vector space of all <u>real-valued</u> solutions can be chosen as follows:

$$e^t\begin{pmatrix}1\\8\\-3\end{pmatrix},\quad Re\left(e^{(2+i\sqrt{5})t}\begin{pmatrix}2+\sqrt{5}i\\4+2\sqrt{5}i\\-3\end{pmatrix}\right),\quad Im\left(e^{(2+i\sqrt{5})t}\begin{pmatrix}2+\sqrt{5}i\\4+2\sqrt{5}i\\-3\end{pmatrix}\right)$$

Using the Euler formula we can write down this basis without complex numbers:

$$e^t\begin{pmatrix}1\\8\\-3\end{pmatrix},\ e^{2t}\begin{pmatrix}2cos(\sqrt{5}t)-\sqrt{5}sin(\sqrt{5}t)\\4cos(\sqrt{5}t)-2\sqrt{5}sin(\sqrt{5}t)\\-3cos(\sqrt{5}t)\end{pmatrix},\ e^{2t}\begin{pmatrix}\sqrt{5}cos(\sqrt{5}t)+2sin(\sqrt{5}t)\\2\sqrt{5}cos(\sqrt{5}t)+4\sin(\sqrt{5}t)\\-3sin(\sqrt{5}t)\end{pmatrix}$$

Therefore any solution has the form

$$C_1e^t\begin{pmatrix}1\\8\\-3\end{pmatrix}+C_2e^{2t}\begin{pmatrix}2cos(\sqrt{5}t)-\sqrt{5}sin(\sqrt{5}t)\\4cos(\sqrt{5}t)-2\sqrt{5}sin(\sqrt{5}t)\\-3cos(\sqrt{5}t)\end{pmatrix}+C_3e^{2t}\begin{pmatrix}\sqrt{5}cos(\sqrt{5}t)+2sin(\sqrt{5}t)\\2\sqrt{5}cos(\sqrt{5}t)+4\sin(\sqrt{5}t)\\-3sin(\sqrt{5}t)\end{pmatrix}.$$

To find the solution satisfying the initial condition $x(0)=\begin{pmatrix}0\\0\\1\end{pmatrix}$ we substitute $t=0$ and obtain a system of linear equations for C_1, C_2, C_3:

$$\begin{pmatrix}1&2&\sqrt{5}\\8&4&2\sqrt{5}\\3&-3&0\end{pmatrix}\begin{pmatrix}C_1\\C_2\\C_3\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

and it remains to solve this system.

Solution in case 3: $a=-\frac{4}{3}$. In this case $A=\begin{pmatrix}2&1&3\\2&3&6\\-\frac{4}{3}&0&0\end{pmatrix}$. This matrix is similar to the Jordan normal form $J=\begin{pmatrix}2&1&0\\0&2&0\\0&0&1\end{pmatrix}$. We have to find a transition matrix: a non-singular matrix T such that $T^{-1}AT=J$. The columns T_1, T_2, T_3 of T satisfy the equations

$$(A-2I)T_1=0,\ (A-2I)T_2=T_1,\ (A-I)T_3=0.$$

The vectors (columns) T_1 and T_3 are the eigenvectors corresponding to the eigenvalues 2 and 1. We can chose them, for example, $T_1=\begin{pmatrix}3\\6\\-2\end{pmatrix}$, $T_3=\begin{pmatrix}3\\9\\-4\end{pmatrix}$. The vector (column) T_2 is a solution of the system $\begin{pmatrix}0&1&3\\2&1&6\\-\frac{4}{3}&0&-2\end{pmatrix}T_2=\begin{pmatrix}3\\6\\-2\end{pmatrix}$. This system, though its matrix is singular, must have infinitely many solutions. We may chose any of them, for example $T_2=\begin{pmatrix}3/2\\3\\0\end{pmatrix}$. Now we know the transition matrix

$T = \begin{pmatrix} 3 & 3/2 & 3 \\ 6 & 3 & 9 \\ -2 & 0 & -4 \end{pmatrix}$ and a basis of the vector space of all solutions:

$$T\begin{pmatrix} e^{2t} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3e^{2t} \\ 6e^{2t} \\ -2e^{2t} \end{pmatrix}, \ T\begin{pmatrix} te^{2t} \\ e^{2t} \\ 0 \end{pmatrix} = \begin{pmatrix} 3te^{2t} + \frac{3}{2}e^{2t} \\ 6te^{2t} + 3e^{2t} \\ -2te^{2t} \end{pmatrix}, \ T\begin{pmatrix} 0 \\ 0 \\ e^{t} \end{pmatrix} = \begin{pmatrix} 3e^{t} \\ 9e^{t} \\ -4e^{t} \end{pmatrix}.$$

Any solution has the form

$$x(t) = C_1\begin{pmatrix} 3e^{2t} \\ 6e^{2t} \\ -2e^{2t} \end{pmatrix} + C_2\begin{pmatrix} 3te^{2t} + \frac{3}{2}e^{2t} \\ 6te^{2t} + 3e^{2t} \\ -2te^{2t} \end{pmatrix} + C_3\begin{pmatrix} 3e^{t} \\ 9e^{t} \\ -4e^{t} \end{pmatrix}.$$

To find the solution satisfying the initial condition $x(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ we substitute $t = 0$ and obtain a system of linear equations for C_1, C_2, C_3:

$$\begin{pmatrix} 3 & 3/2 & 3 \\ 6 & 3 & -9 \\ -2 & 0 & -4 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and it remains to solve this system.

Solution in case 4: $a = -1$. In this case $A = \begin{pmatrix} 2 & 1 & 3 \\ 2 & 3 & 6 \\ -1 & 0 & 0 \end{pmatrix}$. This matrix is similar to the Jordan normal form $J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. We have to find a transition matrix: a non-singular matrix T such that $T^{-1}AT = J$. The columns T_1, T_2, T_3 of T satisfy the equations

$$(A - I)T_1 = 0, \ (A - I)T_2 = T_1, \ (A - 3I)T_3 = 0.$$

The vectors (columns) T_1 and T_3 are the eigenvectors corresponding to the eigenvalues 1 and 3. We can chose them, for example, $T_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $T_3 = \begin{pmatrix} 3 \\ 6 \\ -1 \end{pmatrix}$. The vector (column) T_2 is a solution of the system $\begin{pmatrix} 1 & 1 & 3 \\ 2 & 2 & 6 \\ -1 & 0 & -1 \end{pmatrix} T_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$. This system, though its matrix is singular, must have infinitely many solutions. We may chose any of them, for example $T_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Now we know the transition matrix $T = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 0 & 6 \\ -1 & 0 & -1 \end{pmatrix}$ and a basis of the vector space of all solutions:

$$T\begin{pmatrix} e^{t} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{t} \\ 2e^{t} \\ -e^{t} \end{pmatrix}, \ T\begin{pmatrix} te^{t} \\ e^{t} \\ 0 \end{pmatrix} = \begin{pmatrix} te^{t} + e^{t} \\ 2te^{t} \\ -te^{t} \end{pmatrix}, \ T\begin{pmatrix} 0 \\ 0 \\ e^{3t} \end{pmatrix} = \begin{pmatrix} 3e^{3t} \\ 6e^{3t} \\ -e^{3t} \end{pmatrix}.$$

Any solution has the form

$$C_1\begin{pmatrix}e^t\\2e^t\\-e^t\end{pmatrix}+C_2\begin{pmatrix}te^t+e^t\\2te^t\\-te^t\end{pmatrix}+C_3\begin{pmatrix}3e^{3t}\\6e^{3t}\\-e^{3t}\end{pmatrix}.$$

To find the solution satisfying the initial condition $x(0)=\begin{pmatrix}0\\0\\1\end{pmatrix}$ we substitute $t=0$ and obtain a system of linear equations for C_1, C_2, C_3:

$$\begin{pmatrix}1&1&3\\2&0&6\\-1&0&-1\end{pmatrix}\begin{pmatrix}C_1\\C_2\\C_3\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

and it remains to solve this system.

5.10. One more example

Let

$$Q=\begin{pmatrix}1&2&b\\0&1&c\\0&0&1\end{pmatrix},\ T=\begin{pmatrix}1&0&0\\1&1&0\\0&1&2\end{pmatrix},\ A=TQT^{-1},\qquad a,c\in\mathbb{R}.$$

Let us solve the system $x'=Ax$.

Note that $T^{-1}AT=Q$, therefore if we know a basis y_1, y_2, y_3 of the vector space of solutions of the system $y'=Qy$, a basis of the vector space of solutions of the system $x'=Ax$ is $x_1=Ty_1, x_2=Ty_2, x_3=Ty_3$. Therefore let us solve the system $y'=Qy$.

The matrix Q has the only eigenvalue 1 of algebraic multiplicity 3. Its geometric multiplicity is equal to 1 if $c\neq 0$ and to 2 if $c=0$.

Case 1: $c\neq 0$. In this case Q is similar to Jordan normal form $J=\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}$.

Let U be the transition matrix: $U^{-1}QU=J$ and let U_1, U_2, U_3 be the columns of U. They satisfy the equations

$$(Q-I)U_1=0,\ (Q-I)U_2=U_1,\ (Q-I)U_3=U_2$$

One of solutions is

$$U_1=\begin{pmatrix}1\\0\\0\end{pmatrix},\ U_2=\begin{pmatrix}0\\\frac{1}{2}\\0\end{pmatrix},\ U_3=\begin{pmatrix}0\\-\frac{b}{4c}\\\frac{1}{2c}\end{pmatrix}.$$

Therefore

$$U=\begin{pmatrix}1&0&0\\0&\frac{1}{2}&-\frac{b}{4c}\\0&0&\frac{1}{2c}\end{pmatrix}$$

and the vector space of solutions of the system $y'=Qy$ has a basis

$$y_1=U\begin{pmatrix}e^t\\0\\0\end{pmatrix}=\begin{pmatrix}e^t\\0\\0\end{pmatrix},\ y_2=U\begin{pmatrix}te^t\\e^t\\0\end{pmatrix}=\begin{pmatrix}te^t\\\frac{1}{2}e^t\\0\end{pmatrix},\ y_3=U\begin{pmatrix}\frac{t^2}{2}e^t\\te^t\\e^t\end{pmatrix}=\begin{pmatrix}\frac{t^2}{2}e^t\\\frac{1}{2}te^t-\frac{b}{4c}e^t\\\frac{1}{2c}e^t\end{pmatrix}.$$

Consequently a basis of the vector space of solutions of the system $x' = Ax$ is

$$x_1 = Ty_1 = \begin{pmatrix} e^t \\ 0 \\ 0 \end{pmatrix}, \quad x_2 = Ty_2 = \begin{pmatrix} te^t \\ \\ te^t + \frac{1}{2}e^t \\ \\ \frac{1}{2}e^t \end{pmatrix}, \quad x_3 = Ty_3 = \begin{pmatrix} \frac{t^2}{2}e^t \\ \\ \frac{t^2}{2}e^t + \frac{1}{2}te^t - \frac{b}{4c}e^t \\ \\ \frac{1}{2}te^t - \frac{b}{4c}e^t + \frac{1}{c}e^t \end{pmatrix}.$$

Case 2: $c = 0$. In this case Q is similar to Jordan normal form $J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
Let U be the transition matrix: $U^{-1}QU = J$ and let U_1, U_2, U_3 be the columns of U. They satisfy the equations

$$\begin{pmatrix} 0 & 2 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U_1 = 0, \quad \begin{pmatrix} 0 & 2 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U_2 = U_1, \quad \begin{pmatrix} 0 & 2 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U_3 = 0.$$

It means that U_1 and U_3 are linearly independent eigenvectors. In this case we <u>cannot</u> take arbitrary eigenvector U_1: the second and the third coordinate of U_1 must be equal to 0, otherwise the equation for U_2 has no solutions. We can take

$$U_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad U_3 = \begin{pmatrix} 0 \\ b \\ -2 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

, then

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & b \\ 0 & 0 & -2 \end{pmatrix}.$$

It follows that the vector space of solutions of the system $y' = Qy$ has a basis

$$y_1 = U\begin{pmatrix} e^t \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} e^t \\ 0 \\ 0 \end{pmatrix}, \quad y_2 = U\begin{pmatrix} te^t \\ e^t \\ 0 \end{pmatrix} = \begin{pmatrix} te^t \\ \frac{1}{2}e^t \\ 0 \end{pmatrix}, \quad y_3 = U\begin{pmatrix} 0 \\ 0 \\ e^t \end{pmatrix} = \begin{pmatrix} 0 \\ be^t \\ -2e^t \end{pmatrix}.$$

and the vector space of solutions of the system $x' = Ax$ has a basis

$$x_1 = Ty_1 = \begin{pmatrix} e^t \\ 0 \\ 0 \end{pmatrix}, \quad x_2 = Ty_2 = \begin{pmatrix} te^t \\ te^t + \frac{1}{2}e^t \\ \frac{1}{2}e^t \end{pmatrix} \quad x_3 = Ty_3 = \begin{pmatrix} 0 \\ be^t \\ (b-4)e^t \end{pmatrix}.$$

5.11. Exercises

1. Let $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ be solution of the system

$$x_1' = 2x_1 + 5x_2 - 7x_3 + 10x_4, \quad x_2' = -x_2 + 9x_3 + x_4,$$
$$x_3' = x_1 + 3x_4, \quad x_4' = x_2 - x_3 + x_4$$

satisfying the initial condition $x_1(0) = 1,\ x_2(0) = 3, x_3(0) = 0,\ x_4(0) = 2$. Find $r_1, r_2, r_3 \in \mathbb{R}$ such that $x_3(t) = r_1 + r_2 t + r_3 t^2 + o(t^2)$ as $t \to 0$.

2. Find the solution of the system $x_1' = x_1 + x_2,\ x_2' = ax_1 + bx_2$ satisfying the initial condition $x_1(0) = 1, x_2(0) = 0$. Here $a, b \in \mathbb{R}$ are parameters. The final answer must be without the exponent of a matrix and without any complex numbers.

3. Find the solution of the system

$$x_1' = x_1 + x_2 + x_3,\ x_2' = 2x_1 + 2x_2 + 2x_3,\ x_3' = ax_1 + bx_2 + 3x_3$$

satisfying the initial condition $x_1(0) = 0, x_2(0) = 0, x_3(0) = 1$. Here $a, b \in \mathbb{R}$ are parameters. The final answer must be without the exponent of a matrix and without any complex numbers.

4.a. Find a basis of all complex-valued solutions of the system $z_1' = iz_2,\ z_2' = -iz_1$, $i = \sqrt{-1}$.

4.b Express this system in the real form, i.e. express it as a system of ODEs for functions $x_1(t) = Re(z_1(t)), x_2(t) = Re(z_2(t)), y_1(t) = Im(z_1(t)), y_2(t) = Im(z_2(t))$.

5. Let

$$T = \begin{pmatrix} 2 & 0 & 4 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 5 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},\ J = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Find a basis of the vector space of all solutions of the system $x' = \left(TJT^{-1}\right)x$. The final answer should be without the exponent of a matrix.

There will be more exercises.

CHAPTER 6

Phase portraits

6.1. Phase portrait for autonomous systems of ODEs

Consider the system of ODEs

$$x_1' = f_1(x_1, ..., x_n), \ \dots, x_n' = f_n(x_1, ..., x_n) \tag{6.1.1}$$

with respect to unknown functions $x_1 = x_1(t), ..., x_n = x_n(t)$. It is the general form of autonomous system of first order ODEs. The word autonomous corresponds to the fact that the functions $f_1, ..., f_n$ do not depend on t. For any autonomous equation or system of equations we have the lemma on shift of time:
if $x(t) = (x_1(t), ..., x_n(t))$ is a solution then
$\widetilde{x}(t) = x(t + a)$ is also a solution for any a.

DEFINITION 6.1.1. Let $x(t) = (x_1(t), ..., x_n(t))$ be a solution of (6.1.1) defined on maximal possible interval (t^-, t^+). The set of points

$$\gamma = \{x(t), \ t \in (t^-, t^+)\} \subset \mathbb{R}^n$$

is called the phase curve corresponding to the solution $x(t)$. An oriented phase curve is a phase curve endowed with an arrow corresponding to the increase of time. The collection of all oriented phase curves of (6.1.1) (i.e. oriented phase curves corresponding to all possible solutions) is called the phase portrait of (6.1.1).

If for system (6.1.1) we have existence and uniqueness theorem, i.e. for any initial condition $x(0) = x_0 \in \mathbb{R}^n$ there exists a solution satisfying this initial condition, and any two such solutions are the same on the intersection of the intervals of their definition, then the phase portrait is a foliation of $\mathbb{R}^n$ which means that there is a phase curve passing through any fixed point of $\mathbb{R}^n$ and two different phase curves do not intersect. The general existence and uniqueness theorem (which is postponed to the end of the course) implies that for system (6.1.1) a sufficient condition for the existence and uniqueness property is $f_i(x_1, ..., x_n) \in C^1(\mathbb{R}^n)$, i.e. each of the functions $f_1, ..., f_n$ has a derivative with respect to any of the variables $x_1, ..., x_n$ and each of these derivatives is a continuous function.

PROPOSITION 6.1.2. *Assume that $f_i(x_1, ..., x_n) \in C^1(\mathbb{R}^n)$, $i = 1, ..., n$. Then for any point of $\mathbb{R}^n$ there is a phase curve passing through this point and two different phase curves do not intersect.*

PROOF. The existence of a phase curve passing through any fixed point follows from the existence theorem. To prove that two different phase curves do not intersect we need the uniqueness theorem and the shift of time property.

The proof is as follows. Let γ_1 and γ_2 be phase curves corresponding to solutions $x_1(t)$ and $x_2(t)$ (both $x_1(t)$ and $x_2(t)$ are vector-functions) defined on intervals

(a_1^-, a_1^+) and (a_2^-, a_2^+). Assume that γ and $\widetilde{\gamma}$ intersect at a point $A \in \mathbb{R}^n$. It means that $x_1(t_1) = A$ and $x_2(t_2) = A$ for some t_1, t_2. We have to prove that γ_1 and γ_2 are the same phase curve.

The vector functions

$$\widetilde{x}_1(t) = x_1(t + t_1) \text{ defined on the interval} = t \in I_1 = (a_1^- - t_1, a_1^+ - t_1)$$
$$\widetilde{x}_2(t) = x_2(t + t_2) \text{ defined on the interval} = t \in I_2 = (a_2^- - t_2, a_2^+ - t_2)$$

are solutions of the same system satisfying the conditions $\widetilde{x}_1(0) = \widetilde{x}_2(0) = A$. Since (a_1^-, a_1^+) and (a_2^-, a_2^+) are the maximal possible intervals of definition of the solutions $x_1(t)$ and $x_2(t)$ satisfying $x_1(t_1) = A$ and $x_2(t_2) = A$, the intervals I_1 and I_2 are maximal possible intervals for the solutions $\widetilde{x}_1(t)$ and $\widetilde{x}_2(t)$ satisfying $\widetilde{x}_1(0) = \widetilde{x}_2(0) = A$. Therefore I_1 and I_2 are the same time-interval. Let $I_1 = I_2 = I$. Note that the phase curves γ_1 and γ_2 can be expressed in the form

$$\gamma_1 = \{x_1(t),\ t \in (a_1^-, a_1^+)\} = \{\widetilde{x}_1(t),\ t \in I\}$$
$$\gamma_2 = \{x_2(t),\ t \in (a_2^-, a_2^+)\} = \{\widetilde{x}_2(t),\ t \in I\}$$

Since $\widetilde{x}_1(0) = \widetilde{x}_2(0) = A$, by the uniqueness theorem $\widetilde{x}_1(t) = \widetilde{x}_2(t)$ for any $t \in I$. Therefore γ_1 and γ_2 are the same phase curve: $\gamma_1 = \gamma_2$. □

6.2. The phase portrait saddle

6.2.1. Standard saddle. Consider the system

$$x' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} x : \quad x_1' = \lambda_1 x_1,\ x_2' = \lambda_2 x_2, \quad \lambda_1 < 0,\ \lambda_2 > 0 \tag{6.2.1}$$

It has solutions $x_1(t) = C_1 e^{\lambda_1 t},\ x_2(t) = C_2 e^{\lambda_2 t}$. The phase portrait of this system is showed in fig. 6.1 and it is called the standard saddle. The phase curve γ corresponding to the solution is as follows:

If $C_1, C_2 > 0$ then γ is contained in the domain $x_1, x_2 > 0$.

If $C_1 > 0, C_2 < 0$ then γ is contained in the domain $x_1 > 0, x_2 < 0$.

If $C_1 < 0, C_2 > 0$ then γ is contained in the domain $x_1 < 0, x_2 > 0$.

If $C_1, C_2 < 0$ then γ is contained in the domain $x_1, x_2 < 0$.

Any phase curve above approaches the x_2-axes as $t \to \infty$ and the x_1-axes as $t \to -\infty$.

If $C_1 > 0$ and $C_2 = 0$ then γ is the ray $\{x_1 > 0, x_2 = 0\}$ oriented towards 0.

If $C_1 < 0$ and $C_2 = 0$ then γ is the ray $\{x_1 < 0, x_2 = 0\}$ oriented towards 0.

If $C_1 = 0$ and $C_2 > 0$ then γ is the ray $\{x_1 = 0, x_2 > 0\}$ oriented towards ∞.

If $C_1 = 0$ and $C_2 < 0$ then γ is the ray $\{x_1 = 0, x_2 < 0\}$ oriented towards $-\infty$.

Finally, if $C_1 = C_2 = 0$ then γ is the point $x_1 = x_2 = 0$.

DEFINITION 6.2.1. A straight line $\ell \subset \mathbb{R}^2$ containing 0 (i.e. 1-dimensional subspace of $\mathbb{R}^2$) is called invariant if any phase curve γ containing a point in ℓ entirely belongs to ℓ. An invariant line ℓ is called stable if any phase curve in ℓ is oriented towards 0 and unstable if any phase curve in ℓ is oriented towards 0 after inverting the orientation (i.e. as $t \to -\infty$).

In the case of the standard saddle there are exactly two invariant lines: the x_1-axes is a stable invariant line and the x_2-axes is an unstable invariant line.

6.2.2. General saddle. In general, the saddle is the name for the phase portrait of a system $x' = Ax$ where A is a 2×2 matrix two real eigenvalues λ_1, λ_2, one positive and one negative.

To obtain the phase portrait we introduce a vector-function $y = (y_1(t), y_2(t))$ related to $x(t)$ by a linear transformation $x = Ty$ where T is the transition matrix from A to $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, i.e. $T^{-1}AT = D$. Then $y(t)$ satisfies the equation $y' = Dy$. i.e. the system (6.3.2) with x replaced by y.

It follows that the phase portrait can be obtained from the phase portrait of the standard saddle in the (y_1, y_2)-plane by the linear transformation $x = Ty$, see fig. 6.2. Any linear transformation brings a straight line to a straight line. Therefore the invariant lines in the (x_1, x_2)-plane are the images of the y_1-axes and the y_2-axes under the transformation $x = Ty$. The y_1-axes is spanned by the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, therefore its image is the line spanned by the vector $T\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. i.e. by the first column of the matrix T. Similarly, the image of the y_2-axes is the line spanned by the second column of the matrix T.

The columns of T are the eigenvectors of A corresponding to λ_1 and λ_2, therefore the invariant lines in the (x_1, x_2)-plane are the lines spanned by these eigenvectors. We should distinguish the stable and the unstable invariant line:

The stable invariant line is spanned by the eigenvector of A corresponding to the negative eigenvalue, and the unstable invariant line is spanned by the eigenvector of A corresponding to the positive eigenvalue.

Knowing the stable and the unstable invariant lines we can draw the whole phase portrait, see fig. 6.2: any phase curve beyond the invariant lines approaches the unstable invariant line as $t \to \infty$ and the stable invariant line as $t \to -\infty$.

6.2.3. Example. Consider the system $x_1' = -x_2,\ x_2' = -8x_1 + 2x_2$. The matrix $\begin{pmatrix} 0 & -1 \\ -8 & 2 \end{pmatrix}$ has eigenvalues -2 and 4. The corresponding eigenvectors are

$$-2 \to \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad 4 \to \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

Therefore the phase portrait is one showed in fig. 6.3.

6.2.4. Example: fight between two armies. Assume that during the fight between two armies the loss of soldiers in each of the armies is proportional to the product of the number of soldiers and the power of weapon in the army of the enemy. Assume also that the power of the weapon in each army does not change during the fight. Denote the power of the weapon in the first army by w_1 and in the second army by w_2 and assume that the first army has more powerful weapon: $w_1 > w_2$. Denote by s_1 and s_2 the number of soldiers in the first and the second army in the beginning of the fight and assume that $s_1 < s_2$. Thus the first army has more powerful weapon, but less soldiers. Which of the armies will win the fight?

The assumptions above lead to the system $x_1' = -kw_2x_2,\ x_2' = -kw_1x_1$, where $x_1 = x_1(t)$ and $x_2 = x_2(t)$ is the number of soldiers in the first and in the second army at time-moment t. Here k is some positive coefficient, the same in the two equations. We have the initial condition $x_1(0) = s_1,\ x_2(0) = s_2$.

The matrix of the system is $\begin{pmatrix} 0 & -kw_2 \\ kw_1 & 0 \end{pmatrix}$. Its eigenvalues are $\pm k\sqrt{w_1 w_2}$. The eigenvectors are

$$k\sqrt{w_1 w_2} \to \begin{pmatrix} \sqrt{w_2} \\ -\sqrt{w_1} \end{pmatrix}, \qquad k\sqrt{w_1 w_2} \to \begin{pmatrix} \sqrt{w_2} \\ \sqrt{w_1} \end{pmatrix}.$$

Therefore the phase portrait is one showed in fig. 6.4. In fact, we need only a part of this phase portrait, in the first quarter of the (x_1, x_2)-plane. From the phase portrait we see that the first army wins if and only if the phase curve corresponding to the solution of the system is below the line $x_2 = \sqrt{\frac{w_1}{w_2}} x_1$. The phase curve corresponding to the solution with the initial condition $(x_1(0), x_2(0)) = (s_1, s_2)$ is the phase curve in the phase portrait containing the point (s_1, s_2). Therefore the first army wins if and only if the point (s_1, s_2) is located below the line $x_2 = \sqrt{\frac{w_1}{w_2}} x_1$, i.e. $s_2 < \sqrt{\frac{w_1}{w_2}} s_1$ or equivalently:

the first army wins if and only if $w_1 > w_2 \cdot \left(\frac{s_2}{s_1}\right)^2$.

6.3. The phase portrait node

6.3.1. Standard stable node. Consider the system

$$x' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} x : \quad x_1' = \lambda_1 x_1, \ x_2' = \lambda_2 x_2, \quad \lambda_1, \lambda_2 < 0, \ |\lambda_1| < |\lambda_2|. \tag{6.3.1}$$

It has solutions $x_1(t) = C_1 e^{\lambda_1 t}, \ x_2(t) = C_2 e^{\lambda_2 t}$. The phase portrait of this system is showed in fig. 6.5, a and it is called the standard stable node. The phase curve γ corresponding to the solution is as follows:

If $C_1, C_2 > 0$ then γ is contained in the domain $x_1, x_2 > 0$.

If $C_1 > 0, C_2 < 0$ then γ is contained in the domain $x_1 > 0, x_2 < 0$.

If $C_1 < 0, C_2 > 0$ then γ is contained in the domain $x_1 < 0, x_2 > 0$.

If $C_1, C_2 < 0$ then γ is contained in the domain $x_1, x_2 < 0$

Any phase curve γ above approaches 0 as $t \to \infty$.

The closure of any phase curve γ above (i.e. $\gamma \cup \{0\}$) is tangent to the x_1-axes at 0.

If $C_1 > 0$ and $C_2 = 0$ then γ is the ray $\{x_1 > 0, x_2 = 0\}$ oriented towards 0.

If $C_1 < 0$ and $C_2 = 0$ then γ is the ray $\{x_1 < 0, x_2 = 0\}$ oriented towards 0.

If $C_1 = 0$ and $C_2 > 0$ then γ is the ray $\{x_1 = 0, x_2 > 0\}$ oriented towards 0.

If $C_1 = 0$ and $C_2 < 0$ then γ is the ray $\{x_1 = 0, x_2 < 0\}$ oriented towards 0.

Finally, if $C_1 = C_2 = 0$ then γ is the point $x_1 = x_2 = 0$.

Like in the case of the standard saddle there are exactly two invariant lines: the x_1-axes and the x_2-axes, but now each of them is a stable invariant line.

6.3.2. Standard unstable node. Consider the system

$$x' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} x : \; x_1' = \lambda_1 x_1, \; x_2' = \lambda_2 x_2, \quad \lambda_1, \lambda_2 > 0, \; \lambda_1 < \lambda_2. \tag{6.3.2}$$

It has solutions $x_1(t) = C_1 e^{\lambda_1 t}$, $x_2(t) = C_2 e^{\lambda_2 t}$. The phase portrait of this system is showed in fig. 6.5,b and it is called the standard unstable node. This phase portrait differs from the standard stable node only by orientation of the phase curves: each of the arrows is opposite. There are exactly two invariant lines: the x_1-axes and the x_2-axes, but now each of them is an unstable invariant line.

6.3.3. General stable and unstable node. It is the name for the phase portrait of a system $x' = Ax$ in the case that A is a 2×2 matrix with two distinct real eigenvalues, either both negative (stable node) or both positive (unstable node).

The phase portrait can be obtain from the phase portrait of the standard stable or unstable node in the same way as in section 6.2.2. See fig. 6.6. The invariant lines are spanned by the linearly independent eigenvectors of A. The closure of any phase curve beyond the invariant lines is tangent at 0 to the invariant line spanned by the eigenvector corresponding to the eigenvalue whose absolute value is smaller than the absolute value of the other eigenvalue.

6.3.4. Example. Consider the system $x_1' = -6x_1 + x_2$, $x_2' = -3x_1 - 2x_2$. The matrix $\begin{pmatrix} -6 & 1 \\ -3 & -2 \end{pmatrix}$ has eigenvalues -3 and -5. The corresponding eigenvectors are

$$-3 \to \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad -5 \to \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore the phase portrait is one showed in fig. 6.7.

6.4. The phase portrait focus

The focus is the name for the phase portrait of the system $x' = Ax$ where the real 2×2 matrix A has eigenvalues

$$\alpha \pm i\beta, \; \alpha \neq 0, \; \beta \neq 0, \quad i = \sqrt{-1}.$$

In this case any solution has the form

$$x_1(t) = e^{\alpha t} \left(C_1 sin(\beta t) + C_2 cos(\beta t) \right), \; x_2(t) = e^{\alpha t} \left(C_3 sin(\beta t) + C_4 cos(\beta t) \right)$$

and it is easy to see that if $(x_1(t), x_2(t)) \not\equiv (0,0)$ then $(C_1, C_2) \neq (0,0)$ and $(C_3, C_4) \neq (0,0)$. It follows that any phase curve except the point $(0,0)$ is a spiral. The spirals approach $(0,0)$ as $t \to \infty$ if $\alpha < 0$. In this case the focus is called stable. If $\alpha > 0$ the spirals approach $(0,0)$ as $t \to -\infty$. In this case the focus is called unstable. See fig. 6.8. In both case of stable or unstable focus the spirals can get twisted either clockwise or anticlockwise.

A simple way to understand if the spirals get twisted clockwise or anticlockwise is as follows. Take a point in the positive part of the x_1-axes, for example the point $A = (1,0)$. From the system we know if $x_2(t)$ increases or decreases at this point. If it increases at A then in the case of stable focus we have fig. 6.8,a2 and in the case of unstable focus we have fig. 6.8, b1. If $x_2(t)$ deccreases at the point A then we have fig. 6.8,a1 in the case of stable focus and fig. 6.8,b2 in the case of unstable focus.

6.4.1. Example. Consider the system

$$x' = Ax, \quad A = \begin{pmatrix} 0 & c \\ b & -2 \end{pmatrix},$$

where b and c are real parameters. The eigenvalues of A are $\lambda_{1,2} = -1 \pm \sqrt{1+bc}$. Therefore we have focus if $1+bc < 0$. If so, the focus is stable. Since $x_2' = bx_1 - 2x_2$, at the point $A = (1, 0)$ we have $x_2' = b$. Therefore if $1 + bc < 0$ and $b > 0$ we have the phase portrait in fig. 6.8,a2, and if $1 + bc < 0$ and $b < 0$ we have the phase portrait in fig. 6.8,a1.

6.5. The phase portrait center

The focus is the name for the phase portrait of the system $x' = Ax$ where the real 2×2 matrix A has eigenvalues

$$\pm i\beta, \ \beta \neq 0, \quad i = \sqrt{-1}.$$

In this case any solution has the form

$$x_1(t) = C_1 sin(\beta t) + C_2 cos(\beta t), \ x_2(t) = C_3 sin(\beta t) + C_4 cos(\beta t)$$

and it is easy to see that if $(x_1(t), x_2(t)) \not\equiv (0,0)$ then $(C_1, C_2) \neq (0,0)$ and $(C_3, C_4) \neq (0,0)$. It follows that any solution except $(x_1(t), x_2(t)) \equiv (0,0)$ is a periodic function.
Consequently any phase curve except the point $(0, 0)$ is a closed curve.

In fact, any phase curve except the point $(0, 0)$ is an ellipse (a circle is a particular case of an ellipse). To prove this fact we at first consider the standard center which is the phase portrait of the system

$$y' = Qy, \quad Q = Q = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \text{ or equivalently } y_1' = \beta y_2, \ y_2' = -\beta y_1,$$

The phase portrait of the system $y' = Qy$ is very simple, it is showed in fig. 6.9. The phase curves are the circles described by the equation $y_1^2 + y_2^2 = const$. To see it consider the function $F(t) = y_1^2(t) + y_2^2(t)$ where $(y_1(t), y_2(t))$ is a solution of the system $y' = Qy$. Differentiate this function by t: $F'(t) = 2y_1(t)y_1'(t) + 2y_2(t)y_2'(t)$. Since $y_1'(t) = \beta y_2(t)$ and $y_2'(t) = -\beta y_1(t)$ we obtain $F'(t) \equiv 0$ and consequently $F(t) \equiv const$.

Like in the case of a focus, in the case of any center, including the standard center in fig. 6.9, the orientation of phase curves might be clockwise (fig. 6.9,b) or anticlockwise (fig. 6.9.a) and it can be determined in exactly the same way as in section 6.4. Using this way we see that the phase portrait of the system $y' = Qy$ is one in fig. 6.9.b if $\beta > 0$ and in fig. 6.9,a if $\beta < 0$.

The matrix Q has eigenvalues $\pm\beta i$, the same as the matrix A. Therefore the matrices A and Q are similar and consequently the phase portrait of the system $x' = Ax$ can be obtained from the phase portrait of the system $y' = Qy$ by the linear transformation $x = Ty$ where T is a non-singular matrix such that $T^{-1}AT = Q$. It follows that the phase portrait of the system $x' = Ax$ consists of the curves which are obtained from the circles $y_1^2 + y_2^2 = const$ by some non-singular linear transformation $x = Ty$. It is well known that any linear transformation of the plane brings a circle to an ellipse. Therefore the phase portrait of the system $x' = Ax$ consists of ellipses (again circle is a particular case of an ellipse). See fig. 6.10.

Applying a linear transformation $x = Ty$ to the function $y_1^2 + y_2^2$ we obtain a function of the form $r_1x_1^2 + r_2x_1x_2 + r_3x_2^2$. Therefore in the case of any center the phase curves are described by the equation

$$r_1x_1^2 + r_2x_1x_2 + r_3x_2^2 = const$$

with certain r_1, r_2, r_3.

One of the ways to find r_1, r_2, r_3 is to compute the transition matrix T, a non-singular matrix such that $T^{-1}AT = Q$. There is a simpler way (from computational point of view) using the complex numbers. This way is as follows.

We know that A is similar to the matrix $D = \begin{pmatrix} \beta i & 0 \\ 0 & -\beta i \end{pmatrix}$, i.e. $R^{-1}AR = D$ for some non-singular matrix R. We know that R can be chosen to have the columns $v, \bar{v}$ where $v \in \mathbb{C}^2$ is an eigenvector corresponding to the eigenvalue βi. Consider the system $z' = Dz$ or equivalently $z_1' = \beta i z_1,\ z_2' = -\beta i z_2$. Let $z_1(t), z_2(t)$ be any complex-valued solution of this system. Consider the function $G(z) = z_1z_2$. Differentiate it by t: $G'(t) = z_2z_1' + z_1z_2' \equiv 0$. It follows that the phase portrait in $\mathbb{C}^2$ of the system $z' = Dz$ is described by the equations $z_1z_2 = const$. Let

$$R^{-1} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \implies z_1 = u_{11}x_1 + u_{12}x_2,\ z_2 = u_{21}x_1 + u_{22}x_2.$$

Then the phase portrait of the system $x' = Ax$ is described by the equations

$$z_1z_2 = const \ \rightarrow \ (u_{11}x_1 + u_{12}x_2)(u_{21}x_1 + u_{22}x_2) = const$$

and it remains to compute the entries u_{ij} of the inverse matrix R^{-1}.

6.5.1. Example. Consider the system $x' = Ax,\ A = \begin{pmatrix} 2 & -1 \\ 8 & -2 \end{pmatrix}$. The eigenvalues of A are $\pm 2i$. The eigenvector corresponding to the eigenvalue $2i$ can be chosen to be $\begin{pmatrix} 1 \\ 2(1-i) \end{pmatrix}$ so that the transition matrix from A to $D = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}$ is $R = \begin{pmatrix} 1 & 1 \\ 2(1-i) & 2(1+i) \end{pmatrix}$. Compute

$$R^{-1} = \frac{1}{4i}\begin{pmatrix} 2(1+i) & -1 \\ 2(-1+i) & 1 \end{pmatrix}. \tag{6.5.1}$$

The linear transformation $x = Rz$ brings the system $x' = Ax$ to the system $z' = Dz$ whose phase portrait is described by the equations $z_1z_2 = const$. We have $z = R^{-1}x$, and from (6.5.1) we have

$$z_1 = \frac{1}{4i}\left((2+2i)x_1 - x_2\right), \quad z_2 = \frac{1}{4i}\left((-2+2i)x_1 + x_2\right).$$

Compute

$$z_1z_2 = \frac{1}{16}\left(8x_1^2 - 4x_1x_2 + x_2^2\right).$$

Therefore the phase curves of the system $x' = Ax$ are described by the equations

$$8x_1^2 - 4x_1x_2 + x_2^2 = const.$$

The phase portrait is showed in fig. 6.11. The orientation is anticlockwise because at the point $A = (1, 0)$ we have $x_2' = 8$, therefore x_2 increases at the point A.

6.6. Exercises

1. Draw the phase portrait for the system

$$x_1' = x_2, \quad x_2' = ax_1 - 6x_2$$

for the following cases:

(a) $a = 7$, (b) $a = -5$, (c) $a = -10$

2. Draw the phase portrait for the system

$$x_1' = 3x_1 + x_2, \quad x_2' = bx_1 + ax_2$$

for the following cases:

(a) $a = -3,\ b = -10$, (b) $a = -3,\ b = 0$,

(c) $a = 0,\ b = -2$, (d) $a = 0,\ b = 4$, (e) $a = 0,\ b = -9$

3. Find the equations for the phase curves (no complex numbers in the final answer) and draw the phase portrait for the system

$$x_1' = x_1 + cx_2, \quad x_2' = -cx_1 - x_2$$

for the following cases: (a) $c = 2$, (b) $c = -2$.

4*. Draw the phase portrait for the system

$$x_1' = -x_1, \quad x_2' = 0$$

and use it give a way to draw the phase portrait for the system $x' = Ax$ where A is any 2×2 matrix with the zero eigenvalue and a negative eigenvalue.

5**. Draw the phase portrait for the system

$$x_1' = -x_1 + x_2, \quad x_2' = -x_2$$

and use it to give a way to draw the phase portrait for the system $x' = Ax$ where A is any non-diagonal 2×2 matrix with only one negative eigenvalue (of algebraic multiplicity 2).

CHAPTER 7

Linear homogeneous nth order ODEs with constant coefficients

7.1. Transferring to a system $y' = Ay$. Qualitative theorem

In this chapter we consider the equations of the form

$$x^{(n)} + a_{n-1}x^{(n-1)} + a_{n-2}x^{(n-2)} + \cdots + a_2x'' + a_1x' + a_0x = 0, \quad a_0, a_1, ...a_{n-1} \in \mathbb{R} \tag{7.1.1}$$

where $x^{(i)}$ denotes the ith derivative of the unknown function $x = x(t)$. Such equations are called linear homogeneous nth order ODE with constant coefficients. The initial conditions for this equation are as follows:

$$x(0) = c_0,\ x'(0) = c_1,\ x''(0) = c_2,\ \cdots,\ x^{(n-1)}(0) = c_{n-1} \tag{7.1.2}$$

where $c_0, ..., c_{n-1}$ are given real numbers.

Any equation (7.1.1) can be transferred to a system of equations of the form $y' = Ay$ with a certain $n \times n$ matrix A as follows. Introduce

$$y_1(t) = x(t),\quad y_2(t) = x'(t),\ y_3(t) = x''(t),\ \cdots, \quad y_{n-1}(t) = x^{(n-2)}(t),\quad y_n(t) = x^{(n-1)}(t). \tag{7.1.3}$$

Then

$$y_1' = y_2,\ y_2' = y_3,\ \cdots,\ y_{n-1}' = y_n$$
$$y_n' = -a_0y_1 - a_1y_2 - \cdots - a_{n-1}y_n.$$

It means that the vector-function $y(t) = (y_1(t), ..., y_n(t))$ satisfies the system

$$y' = Ay, \quad A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{pmatrix}. \tag{7.1.4}$$

The initial conditions (7.1.2) take the form

$$y(0) = (c_0, c_1,\ \ldots,\ c_{n-1}). \tag{7.1.5}$$

The function $x(t)$ is a solution of (7.1.1) satisfying (7.1.2) if and only if the vector function $y(t) = (y_1(t), ..., y_n(t)$ defined by (7.1.3) is a solution of (7.1.4) satisfying (7.1.5). Therefore the qualitative theorems in Chapter 5, sections 5.1, 5.2 imply the following theorem.

THEOREM 7.1.1. *Equation (7.1.1) has a solution $x(t)$ satisfying the initial condition (7.1.2) and defined for all $t \in \mathbb{R}$. Such a solution is unique. The set of all solutions of (7.1.1), defined for all $t \in \mathbb{R}$, is a vector space over $\mathbb{R}$ of dimension n.*

7.2. The characteristic polynomial. Compact form of equation (7.1.1)

The characteristic polynomial of equation (7.1.1) is the polynomial

$$P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0. \tag{7.2.1}$$

Given a polynomial (7.2.1) we associate to it the linear operator

$$P\left(\frac{d}{dt}\right) : \ C^\infty(\mathbb{R}) \ \to C^\infty(\mathbb{R})$$

$$P\left(\frac{d}{dt}\right)(f(t)) = f^{(n)}(t) + a_{n-1}f^{(n-1)}(t) + \cdots + a_2 f''(t) + a_1 f'(t) + a_0 f(t).$$

Equation (7.1.1) can be written in the form

$$P\left(\frac{d}{dt}\right)(x(t)) = 0$$

which means that $x(t)$ is a solution of (7.1.1) if and only if this function belongs to the kernel of the operator $P\left(\frac{d}{dt}\right)$.

7.3. A basis of the space of all solutions in the case that $P(\lambda)$ has n real roots

THEOREM 7.3.1. *Let λ_1 be the root of the polynomial (7.2.1). Then the equation $P\left(\frac{d}{dt}\right)(x(t)) = 0$ has a solution $x(t) = e^{\lambda_1 t}$.*

PROOF. We have

$$\left(e^{\lambda_1 t}\right)^{(i)} = \lambda_1^i e^{\lambda_1 t}$$

and it follows

$$P\left(\frac{d}{dt}\right)(e^{\lambda_1 t}) = \lambda_1^n e^{\lambda_1 t} + a_{n-1}\lambda_1^{n-1}e^{\lambda_1 t} + \cdots + a_2\lambda_1^2 e^{\lambda_1 t} + a_1\lambda_1 e^{\lambda_1 t} + a_0 e^{\lambda_1 t}.$$

Therefore

$$P\left(\frac{d}{dt}\right)(e^{\lambda_1 t}) = P(\lambda_1)e^{\lambda_1 t}. \tag{7.3.1}$$

If $P(\lambda_1) = 0$ we obtain $P\left(\frac{d}{dt}\right)(e^{\lambda_1 t}) = 0$. □

In the case that $P(\lambda)$ has n distinct real roots $\lambda_1, ..., \lambda_n$ the functions

$$e^{\lambda_1 t}, \ \ldots, e^{\lambda_n t} \tag{7.3.2}$$

are n solutions of equation $P\left(\frac{d}{dt}\right)(x(t)) = 0$.

PROPOSITION 7.3.2. *For any distinct numbers $\lambda_1, ..., \lambda_n$ the functions (7.3.2) are linearly independent over $\mathbb{R}$.*

PROOF. Assume that

$$r_1 e^{\lambda_1 t} + \cdots + r_n e^{\lambda_n t} \equiv 0. \tag{7.3.3}$$

We have to show that $r_1 = \cdots = r_n = 0$. One of the proofs is as follows. Substitute $t = 0$ to (7.3.3). We obtain $r_1 + \cdots + r_n = 0$. Now differentiate (7.3.3) and substitute $t = 0$. We obtain $\lambda_1 r_1 + \cdots + \lambda_n r_n = 0$. Differentiating (7.3.3) and substituting $t = 0$ we obtain $\lambda_1^2 r_1 + \cdots + \lambda_n^2 r_n = 0$. Continuing till the derivative of order $(n-1)$ of (7.3.3) we obtain the following linear system for $(r_1, ..., r_n)$:

$$Cr = 0,\ r = \begin{pmatrix} r_1 \\ \cdots \\ r_n \end{pmatrix},\ C = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \cdots & & & \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix}.$$

The matrix C can be met in many math areas. It is called the Vandermonde matrix. It is well-known that the determinant C is equal to $\prod_{1 \le i < j \le n} (\lambda_i - \lambda_j)$ ($\prod$ is the sign for the product). Since $\lambda_1, .., \lambda_n$ are distinct, $detC \neq 0$ and the system $Cr = 0$ has the only solution $r = 0$. □

Theorem 7.3.1 and Proposition 7.3.2 imply the following result.

THEOREM 7.3.3. *If a degree n polynomial $P(\lambda)$ has n distinct real roots $\lambda_1, ...\lambda_n$ then the set of functions*

$$e^{\lambda_1 t},\ \ldots,\ e^{\lambda_n t}$$

is a basis of the space of all solutions of the equation $P\left(\frac{d}{dt}\right)(x(t)) = 0$.

EXAMPLE. Let us find the solution of the equation $x'' + x' - 2x = 0$ satisfying the initial condition $x(0) = 1,\ x'(0) = 0$.

The characteristic polynomial $P(\lambda) = \lambda^2 + \lambda - 2$ has two real roots $\lambda_1 = 1, \lambda_2 = -2$. Therefore the set of all solutions of the given equation is $x(t) = C_1 e^t + C_2 e^{-2t}$. Substituting $t = 0$ to $x(t)$ and to $x'(t)$ we obtain

$$C_1 + C_2 = 1,\quad C_1 - 2c_2 = 0.$$

It follows $C_1 = 2/3,\ C_2 = 1/3$ and consequently $x(t) = \frac{1}{3}\left(2e^t + e^{-t}\right)$.

7.4. A basis of the space of all solutions in the case that $P(\lambda)$ has n complex roots

Assume now that the characteristic polynomial $P(\lambda)$ has n distinct roots, but some of them are not real. Since the coefficients of $P(\lambda)$ are real, the roots are as follows:

$$\begin{gathered} \alpha_1 \pm i\beta_1,\ \ldots,\ \alpha_s \pm i\beta_s,\ \ \theta_1,\ \ldots, \theta_p \\ \alpha 1, ..., \alpha_s, \beta_i, \ldots,\ \beta_s,\ \theta_1, ..., \theta_p \in \mathbb{R},\ \ \beta_1, \ldots, \beta_s \neq 0,\ \ 2s + p = n. \end{gathered} \tag{7.4.1}$$

Let us present a basis of all complex-valued solutions of equation $P\left(\frac{d}{dt}\right)(x(t)) = 0$.

Like in Chapter 5, Theorem 7.1.1 remains true for complex-valued solutions, with the only difference that they form an n-dimensional vector space over $\mathbb{C}$. Theorem 7.3.1 holds for complex-valued solutions without any change, and Proposition 7.3.2 holds for complex-valued functions 7.3.2 with any complex λ_i with the only change that these functions are linearly independent over $\mathbb{C}$. Therefore one of the basis of the vector space of complex-valued solutions is exactly the same as in section 7.3:

THEOREM 7.4.1. *If a degree n polynomial $P(\lambda)$ has n distinct complex roots (7.4.1) then the set of complex-valued functions*

$$e^{(\alpha_1+\beta_1 i)t},\ e^{(\alpha_1-\beta_1 i)t},\ \dots,\ e^{(\alpha_s+\beta_s i)t},\ e^{(\alpha_s-\beta_s i)t},\ \ e^{\theta_1 t},\ ...,\ e^{\theta_p t} \tag{7.4.2}$$

is a basis of the space of all complex-valued solutions of equation $P\left(\frac{d}{dt}\right)(x(t)) = 0$.

This basis can be transferred to a basis of the space of all real-valued solutions in the same way as in Chapter 5: we replace every couple of functions $f_j = e^{(\alpha_1+\beta_1 i)t}$, $g_j = e^{(\alpha_1-\beta_1 i)t}$ by the couple $\frac{f_j+g_j}{2},\ \frac{f_j-g_j}{2i},\ \ j = 1,..,s$. Since $g_j = \bar{f}_j$ we obtain n real-valued functions

$$Re\left(e^{(\alpha_1+\beta_1 i)t}\right),\ Im\left(e^{(\alpha_1+\beta_1 i)t}\right)\ \dots,\ Re\left(e^{(\alpha_s+\beta_s i)t}\right),\ Im\left(e^{(\alpha_s+\beta_s i)t}\right),$$
$$e^{\theta_1 t},\ ...,\ e^{\theta_p t}.$$

or equivalently

$$\begin{aligned} &e^{\alpha_1 t}\left(cos\left(\beta_1 t\right)\right),\ e^{\alpha_1 t}\left(sin\left(\beta_1 t\right)\right),\ \dots,\ e^{\alpha_s t}\left(cos\left(\beta_s t\right)\right),\ e^{\alpha_s t}\left(sin\left(\beta_s t\right)\right),\\ &e^{\theta_1 t},\ ...,\ e^{\theta_p t}.\end{aligned} \tag{7.4.3}$$

Each of this function is a solution of the same equation $P\left(\frac{d}{dt}\right)(x(t)) = 0$, and these functions are linearly independent over $\mathbb{C}$ (and consequently over $\mathbb{R}$) since the functions (7.4.2) are linearly independent over $\mathbb{C}$. Therefore the following statement holds:

THEOREM 7.4.2. *If a degree n polynomial $P(\lambda)$ has n distinct complex roots (7.4.1) then the set of n functions (7.4.3) is a basis of the space of all real-valued solutions of equation $P\left(\frac{d}{dt}\right)(x(t)) = 0$.*

EXAMPLE. Let us find the solution of the equation $x'''(t) = x(t)$ satisfying the initial conditions $x(0) = 0, x'()) = 1, x''(0) = 0$.

Write the equation in the form $x''' - x = 0$. The characteristic polynomial is $P(\lambda) = \lambda^3 - 1$. It has three distinct complex roots $1,\ \frac{1}{2}(-1 \pm \sqrt{3}i)$. Therefore the set of all solutions is as follows:

$$x(t) = e^{-\frac{1}{2}t}\left(C_1 cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 sin\left(\frac{\sqrt{3}}{2}t\right)\right) + C_3 e^t.$$

Substituting $t = 0$ to $x(t)$, to $x'(t)$, and to $x''(t)$ we obtain, form the initial conditions, the linear system

$$C_1 + C_3 = 0,\ -\frac{1}{2}C_1 + \frac{\sqrt{3}}{2}C_2 + C_3 = 1,\ \ -\frac{1}{2}C_1 - \frac{\sqrt{3}}{2}C_2 + C_3 = 0.$$

Solving this system we obtain $C_1 = -\frac{1}{3},\ C_2 = \frac{1}{\sqrt{3}},\ C_3 = \frac{1}{3}$. Therefore the required solution is as follows:

$$x(t) = \frac{1}{3}\left(e^{-\frac{1}{2}t}\left(-cos\left(\frac{\sqrt{3}}{2}t\right) + \sqrt{3}sin\left(\frac{\sqrt{3}}{2}t\right)\right) + e^t\right).$$

7.5. A basis of the space of all solutions in the case that $P(\lambda)$ has less than n complex roots

To find a basis of the vector space of all solutions in this case, several auxiliary propositions.

PROPOSITION 7.5.1. *For any two polynomials $P(\lambda)$ and $Q(\lambda)$ (not necessarily of the same degree) one has*

$$(PQ)\left(\frac{d}{dt}\right) = P\left(\frac{d}{dt}\right) \circ Q\left(\frac{d}{dt}\right)$$

where $\circ$ denotes the composition of two operators.

PROOF. Denote by D_i the linear operator sending a function $f(t)$ to its ith derivative:

$$D_i : \ C^\infty(\mathbb{R}) \ \to C^\infty(\mathbb{R}), \quad D_i(f(t)) = f^{(i)}(t), \qquad i \geq 0.$$

By $f^{(0)}$ (the zero order derivative of $f(t)$) we mean the function $f(t)$ itself. Let

$$P(\lambda) = a_0 + a_1\lambda + \cdots + a_k\lambda^k, \qquad Q(\lambda) = b_0 + b_1\lambda + \cdots + b_s\lambda^s.$$

Then

$$P\left(\frac{d}{dt}\right) = a_0D_0 + a_1D_1 + \cdots + a_kD_k, \qquad Q\left(\frac{d}{dt}\right) = b_0D_0 + b_1D_1 + \cdots + b_sD_s.$$

Since the operators D_i are linear we have

$$P\left(\frac{d}{dt}\right) \circ Q\left(\frac{d}{dt}\right) = \sum (a_ib_j) \cdot D_i \circ D_j,$$

where the sum is taken over all $i = 0, ..., k$ and all $j = 0, ..., s$.

The product $P(\lambda)Q(\lambda)$ can be written in the form

$$P(\lambda)Q(\lambda) = \sum a_ib_j\lambda^{i+j},$$

where as above the sum is taken over all $i = 0, ..., k$ and all $j = 0, ..., s$. Therefore

$$(PQ)\left(\frac{d}{dt}\right) = \sum (a_ib_j)D_{i+j}.$$

To prove the proposition it remains to note the obvious equation

$$D_i \circ D_j = D_{i+j}$$

which actually means that the jth derivative of the ith derivative is the derivative of order $i + j$. □

REMARK. Proposition 7.5.1 implies that the ring of all polynomials (of non-fixed degree) with the usual operations sum and multiplication is isomorphic to the ring of linear operators of the form $f(t) \to a_0f(t)+a_1f'(t)+\cdots+a_kf^{(k)}(t)$ (with non-fixed k) with the usual sum of two linear operators and multiplication = composition. The isomorphism is the map $P(\lambda) \to P\left(\frac{d}{dt}\right)$.

To formulate the second proposition we need the following notation.

NOTATION. Given a real or complex number λ_1, denote by L_{λ_1} the linear operator

$$L_{\lambda_1} : \ C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \quad L_{\lambda_1}(f(t)) = f'(t) - \lambda_1 f(t).$$

Denote by $L^r_{\lambda_1}$ the composition

$$L^r_{\lambda_1} = L_{\lambda_1} \circ L_{\lambda_1} \cdots \circ L_{\lambda_1} \quad (\ r \text{ times }).$$

LEMMA 7.5.2. *One has*

$$L_{\lambda_1}\left((t^i e^{\lambda_1 t}\right) = i \cdot t^{i-1} e^{\lambda_1 t}.$$

In fact,

$$L_{\lambda_1}\left((t^i e^{\lambda_1 t}\right) = \left((t^i e^{\lambda_1 t}\right)' - \lambda_1 t^i e^{\lambda_1 t} = i \cdot t^{i-1} e^{\lambda_1 t} + \lambda_1 t^i e^{\lambda_1 t} - \lambda_1 t^i e^{\lambda_1 t} = i \cdot t^{i-1} e^{\lambda_1 t}.$$

PROPOSITION 7.5.3. *For any $i \leq r-1$ one has*

$$L^r_{\lambda_1}\left(t^i e^{\lambda_1 t}\right) = 0.$$

PROOF. By Proposition 7.5.1 we have

$$L^r_{\lambda_1}\left(t^i e^{\lambda_1 t}\right) = L^{r-1}_{\lambda_1}\left(L_{\lambda_1}\left(t^i e^{\lambda_1 t}\right)\right).$$

Using Lemma 7.5.2 we obtain

$$L^r_{\lambda_1}\left(t^i e^{\lambda_1 t}\right) = i \cdot L^{r-1}_{\lambda_1}\left(t^{i-1} e^{\lambda_1 t}\right).$$

Using again Proposition 7.5.1 and Lemma 7.5.2 we obtain

$$L^r_{\lambda_1}\left(t^i e^{\lambda_1 t}\right) = i(i-1) \cdot L^{r-2}_{\lambda_1}\left(t^{i-2} e^{\lambda_1 t}\right).$$

Continuing to use Proposition 7.5.1 and Lemma 7.5.2 in the same way we obtain

$$L^r_{\lambda_1}\left(t^i e^{\lambda_1 t}\right) = i! \cdot L^{r-i}_{\lambda_1}\left(e^{\lambda_1 t}\right).$$

Since $\underline{i \leq r-1}$ we can use Proposition 7.5.1 one more time:

$$L^r_{\lambda_1}\left(t^i e^{\lambda_1 t}\right) = i! \cdot L^{r-i-1}_{\lambda_1}\left(L_{\lambda_1}\left(e^{\lambda_1 t}\right)\right)$$

and it remains to note that $L_{\lambda_1}\left(e^{\lambda_1 t}\right) = 0$. □

Propositions 7.5.1 and 7.5.3 allow to present n solutions of equation $P\left(\frac{d}{dt}\right)(x(t)) = 0$ where P is a polynomial of degree n. Let

$$\begin{aligned} &deg\ P = n, \\ &\lambda_1, ..., \lambda_s \in \mathbb{C} \text{ are } \underline{\text{distinct}} \text{ roots of } P \\ &r_1, ..., r_s \text{ are their multiplicities} \\ &(\text{by the basic theorem of algebra: } r_1 + \cdots + r_s = n\ ). \end{aligned} \tag{7.5.1}$$

Consider the complex-valued functions

$$\begin{aligned} &e^{\lambda_1 t},\ te^{\lambda_1 t},\ \ldots, t^{r_1 - 1} e^{\lambda_1 t} \ (\ r_1 \text{ functions}) \\ &e^{\lambda_2 t},\ te^{\lambda_2 t},\ \ldots, t^{r_2 - 1} e^{\lambda_2 t} \ (\ r_2 \text{ functions}) \\ &\cdots \\ &e^{\lambda_s t},\ te^{\lambda_s t},\ \ldots, t^{r_s - 1} e^{\lambda_s t} \ (\ r_s \text{ functions}) \end{aligned} \tag{7.5.2}$$

PROPOSITION 7.5.4. *Each of the n functions (7.5.2) is a complex-valued solution of equation $P\left(\frac{d}{dt}\right)(x(t)) = 0$.*

PROOF. Let $j \in \{1, ..., s\}$. It is known that if λ_j is a root of $P(\lambda)$ of multiplicity r_j then $P(\lambda) = (\lambda - \lambda_j)^{r_j} \cdot Q(\lambda)$ for some polynomial $Q(\lambda)$. By Proposition 7.5.1 we have

$$P\left(\frac{d}{dt}\right)\left(t^i e^{\lambda_j t}\right) = Q\left(\frac{d}{dt}\right)\left(L_{\lambda_j}^{r_j}\left(t^i e^{\lambda_j t}\right)\right).$$

If $i \leq r_j - 1$ then by Proposition 7.5.3 we have $L_{\lambda_j}^{r_j}\left(t^i e^{\lambda_j t}\right) = 0$ and consequently $P\left(\frac{d}{dt}\right)\left(t^i e^{\lambda_j t}\right) = 0$. □

PROPOSITION 7.5.5. *The n functions (7.5.2) are linearly independent over $\mathbb{C}$.*

The proof of this proposition in the general case is rather technical and we omit it. Propositions 7.5.4 and 7.5.5 imply:

THEOREM 7.5.6. *The n functions (7.5.2) is a basic of the vector space of all complex-valued solutions of equation $P\left(\frac{d}{dt}\right)(x(t)) = 0$.*

If all roots of the polynomial $P(\lambda)$ are real then each of the functions (7.5.2) is real-valued and these n functions are also a basis of the vector space of all real-valued solutions. If some of the roots are not real then these roots come in couples $\alpha \pm \beta i$ and these complexly-conjugate roots have the same multiplicity. In this case we can transfer (7.5.2) to a basis of the vector space of all real-valued solutions in the same way as in section 7.4.

EXAMPLE. Consider the equation

$$x^{15} - 2x^{(9)} + x''' = 0.$$

The characteristic polynomial is

$$P(\lambda) = \lambda^{15} - 2\lambda^9 + \lambda^3 = \lambda^3(\lambda^6 - 1)^2 = \lambda^2(\lambda^3 - 1)^2(\lambda^3 + 1)^2.$$

The equation $\lambda^3 - 1 = 0$ has complex solutions $1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$. The equation $\lambda^3 + 1 = 0$ has complex solutions $-1, \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$. Therefore the polynomial $P(\lambda)$ has the following roots:

root 0 of multiplicity 3;

roots ± 1, $-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$, $\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ each of multiplicity 2.

By Theorem 7.5.6 the vector space of all complex-valued solutions of equation $P\left(\frac{d}{dt}\right)(x(t)) = 0$ has a basis

$$\begin{gathered}
f_1 = 1,\ f_2 = t,\ f_3 = t^2,\ f_4 = e^t,\ f_5 = te^t,\ f_6 = e^{-t},\ f_7 = te^{-t}, \\
f_8 = e^{\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)t},\ f_9 = \bar{f}_8 = e^{\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)t}, \\
f_{10} = te^{\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)t},\ f_{11} = \bar{f}_{10} = te^{\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)t} \\
f_{12} = e^{\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)t},\ f_{13} = \bar{f}_{12} = e^{\left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)t}, \\
f_{14} = te^{\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)t},\ f_{15} = \bar{f}_{14} = te^{\left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)t}.
\end{gathered}$$

To transfer this basis to a basis of all real-valued solutions we replace the couple f_8, f_9 to $\frac{f_8 + f_9}{2} = Re\, f_8$, $\frac{f_8 - f_9}{2i} = Im\, f_8$ and we do the same with the couples (f_{10}, f_{11}), (f_{12}, f_{13}), (f_{14}, f_{15}). We obtain the following basis of the vector space of all real-valued solutions:

$$1,\ t,\ t^2,\ e^t,\ te^t,\ e^{-t},\ te^{-t},$$
$$e^{-\frac{1}{2}t}cos\left(\frac{\sqrt{3}}{2}t\right),\ e^{-\frac{1}{2}t}sin\left(\frac{\sqrt{3}}{2}t\right),\ e^{\frac{1}{2}t}cos\left(\frac{\sqrt{3}}{2}t\right),\ e^{\frac{1}{2}t}sin\left(\frac{\sqrt{3}}{2}t\right)$$
$$te^{-\frac{1}{2}t}cos\left(\frac{\sqrt{3}}{2}t\right),\ te^{-\frac{1}{2}t}sin\left(\frac{\sqrt{3}}{2}t\right),\ te^{\frac{1}{2}t}cos\left(\frac{\sqrt{3}}{2}t\right),\ te^{\frac{1}{2}t}sin\left(\frac{\sqrt{3}}{2}t\right)$$

7.6. Exercises

1. Find the solution of the equation $x''-2x'+ax=0$ satisfying the initial conditions $x(0)=1, x'(0)=0$. Here a is a real parameter. No complex numbers in the final answer.

2. Find the solution of the equation $x''''(t)=ax(t)$ satisfying the initial conditions $x(0)=1, x'(0)=0, x''(0)=0,\ x'''(0)=0$ for the cases: (a) $a=1$, (b) $a=-1$. No complex numbers in the final answer.

3. Find a basis of the vector space of all real-valued solutions of the equation $P\left(\frac{d}{dt}\right)(x(t))=0$ where $P(\lambda)=(\lambda^2-\lambda+a)^3(\lambda^2+b)^2$. Here a and b are real parameters. No complex numbers in the final answer.

4. Consider the equation $x''+ax'+bx=0$ with real parameters a,b. Give a necessary and sufficient condition on the pair (a,b) under which:

(a) any non-constant solution is periodic

(b) any solution tends to 0 as $t\to\infty$

(c) any solution tends to 0 as $t\to-\infty$.

5. Give a necessary and sufficient condition on the roots of a polynomial $P(\lambda)$ (of any degree) and their multiplicities under which the equation $P\left(\frac{d}{dt}\right)(x(t))=0$ has the following property:

(a) any non-constant solution is periodic

(b) any solution tends to 0 as $t\to\infty$

(c) any solution tends to 0 as $t\to-\infty$.

6. Prove that the functions $e^t,\ e^{-t},\ te^t,\ te^{-t}$ are linearly independent over $\mathbb{R}$.

CHAPTER 8

Linear homogeneous and non-homogeneous ODEs and system of ODES

8.1. Qualitative theorems

This chapter is devoted to the equations of the form

$$x^{(n)}+a_{n-1}(t)x^{(n-1)}+\cdots+a_2(t)x''(t)+a_1(t)x'(t)+a_0(t)x(t)=0,\quad x=x(t) \tag{8.1.1}$$

$$x^{(n)}+a_{n-1}(t)x^{(n-1)}+\cdots+a_2(t)x''(t)+a_1(t)x'(t)+a_0(t)x(t)=f(t) \tag{8.1.2}$$

which are called homogeneous (equation (8.1.1)) and non-homogeneous (equation (8.1.2)) linear nth order ODEs (in general with non-constant coefficients, the case $a_i(t)\equiv a_i = const$ is a particular case), and to the systems of the form

$$y'=A(t)y,\quad y=y(t)=\begin{pmatrix} y_1(t)\\ \cdots \\ y_n(t)\end{pmatrix},\quad A(t)=\begin{pmatrix} A_{11}(t) & \cdots & A_{1n}(t)\\ & \cdots & \\ A_{n1}(t) & \cdots & A_{nn}(t)\end{pmatrix} \tag{8.1.3}$$

$$y'=A(t)y+F(t)\quad F(t)=\begin{pmatrix} f_1(t)\\ \cdots \\ f_n(t)\end{pmatrix} \tag{8.1.4}$$

which are called homogeneous (system (8.1.3)) and non-homogeneous (system (8.1.4)) linear systems of ODEs of order 1 (in general with non-constant coefficients, the case that $A(t)$ is a constant matrix is a particular case).

The word "linear" corresponds to the following very simple property of the solutions:

Theorem 8.1.1. *The set of all solutions of equation (8.1.1) or system (8.1.3) defined on as fixed interval $t\in(\alpha,\beta)$ is a vector space over $\mathbb{R}$ (a subspace of differentiable functions or vector-functions). The set of all solutions of (8.1.2) has the form $x^*(t)+x_h(t)$, where $x^*(t)$ is any fixed (particular) solution of this equation and $x_h(t)$ is an arbitrary solution of the homogeneous equation (8.1.1) with the same coefficients. Similarly, the set of all solutions of system (8.1.4) has the form $y^*(t)+y_h(t)$, where $y^*(t)$ is any fixed (particular) solution of this system and $y_h(t)$ is an arbitrary solution of the homogeneous system (8.1.3) with the same matrix $A(t)$.*

Proof. The sum of two solutions of a homogeneous equation or system is also a solution of the same equation or system. Multiplying any solution of a homogeneous equation or system by a real number we obtain a function or vector-function which is also a solution. The difference of any two solutions of a non-homogeneous equation or system is a solution of the corresponding homogeneous equation or system. □

THEOREM 8.1.2. *Assume that the coefficients* $A_{ij}(t)$ *of the matrix* $A(t)$ *in (8.1.3) or (8.1.4) are continuous functions on the interval* $t \in (\alpha, \beta)$.

1. For any $t_0 \in (\alpha, \beta)$ *and any* $y_0 \in \mathbb{R}^n$ *system (8.1.3) has a unique solution defined for* $t \in (\alpha, \beta)$ *and satisfying the initial condition* $y(t_0) = y_0$.

2. The same holds for system (8.1.4) provided that $F(t)$ *in (8.1.4) is a continuous vector function on* (α, β).

3. The set of all solutions of (8.1.3) defined for $t \in (\alpha, \beta)$ *is a vector space over* $\mathbb{R}$ *of dimension* n.

I will not prove the first statement, but using it I will prove, in this chapter, the second and the third statement. Theorem 8.1.2 implies a similar theorem for linear nth order equations.

THEOREM 8.1.3. *Assume that the coefficients* $a_i(t)$ *in (8.1.1) or (8.1.2) are continuous functions on the interval* $t \in (\alpha, \beta)$.

1. For any $t_0 \in (\alpha, \beta)$ *and any tuple* $c_0, c_1, ..., c_{n-1} \in \mathbb{R}^n$ *equation (8.1.1) has a unique solution defined for* $t \in (\alpha, \beta)$ *and satisfying the initial condition*

$$x(t_0) = c_0, x'(t_0) = c_1, ..., x^{(n-1)}(t_0) = c_{n-1}.$$

2. The same holds for equation (8.1.2) provided that $f(t)$ *in (8.1.2) is a continuous vector function on* (α, β).

3. The set of all solutions of (8.1.1) defined for $t \in (\alpha, \beta)$ *is a vector space over* $\mathbb{R}$ *of dimension* n.

THEOREM 8.1.3 FROM THEOREM 8.1.2. Theorem 8.1.2 implies Theorem 8.1.3 because equation (8.1.1) or (8.1.2) can be transferred to system (8.1.3) or (8.1.4) with

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0(t) & -a_1(t) & -a_2(t) & \cdots & -a_{n-2}(t) & -a_{n-1}(t) \end{pmatrix}, \quad F(t) = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ f(t) \end{pmatrix}$$

by introducing, exactly as in section 7.1, the vector function

$$y(t) = \begin{pmatrix} x(t) \\ x'(t) \\ x''(t) \\ \cdots \\ x^{(n-1)}(t) \end{pmatrix}$$

PROOF OF STATEMENT 3. OF THEOREM 8.1.2. The proof is exactly that same as the proof of Theorem 5.2.1 in section 5.2: in the proof of that theorem we did not use that matrix A is constant. The only difference is that instead of solutions defined for all t we work with solutions defined for $t \in (\alpha, \beta)$. In exactly the same way as in the proof of Theorem 5.2.1 we use the existence and uniqueness of solutions satisfying a fixed initial condition (the first statement of Theorem 8.1.2).

8.2. From a basis of solutions of homogeneous system to a solution of non-homogeneous system

Assume that the entries $A_{ij}(t)$ of $A(t)$ and the vector function $F(t)$ in (8.1.4) are continuous on (α,β) and assume that we know a basis of the vector space of all solutions of the homogeneous system (8.1.3) defined on (α,β):

basis of the vector space of solutions of (8.1.3):

$$y^{(1)}(t)=\begin{pmatrix}y_1^{(1)}(t)\\ \cdots\\ y_n^{(1)}(t)\end{pmatrix},\ \ldots,\ y^{(n)}(t)=\begin{pmatrix}y_1^{(n)}(t)\\ \cdots\\ y_n^{(n)}(t)\end{pmatrix} \tag{8.2.1}$$

We can find a particular solution $y^*(t)$ of system (8.1.3), and consequently the set of all solutions of this system (see Theorem 8.1.1) by the following simple way, called the method of variation of parameters.

The set of all solutions of (8.1.3), defined for $t\in(\alpha,\beta)$ is the set of arbitrary linear combinations of the vector functions (8.2.1):

$$C_1y^{(1)}(t)+C_2y^{(2)}(t)+\cdots+C_ny^{(n)}(t),\quad C_1,...,C_n\in\mathbb{R}.$$

We will show that the non-homogeneous system (8.1.4), with the same matrix $A(t)$ as in (8.1.3) has a particular solution $y^*(t)$ of the form

$$y^*(t)=C_1(t)y^{(1)}(t)+C_2(t)y^{(2)}(t)+\cdots+C_n(t)y^{(n)}(t), \tag{8.2.2}$$

with some functions $C_1(t),...,C_n(t)$ instead of the constants $C_1,...,C_n$ (which explains the name "variation of parameters"). Certainly these functions are not arbitrary, they must satisfy certain conditions. To give these conditions consider the $n\times n$ matrix with the columns (8.2.1), i.e. the matrix

$$W(t)=\begin{pmatrix}y_1^{(1)}(t) & y_1^{(2)}(t) & \cdots & y_1^{(n)}(t)\\ y_2^{(1)}(t) & y_2^{(2)}(t) & \cdots & y_2^{(n)}(t)\\ & & \cdots & \\ y_n^{(1)}(t) & y_n^{(2)}(t) & \cdots & y_n^{(n)}(t)\end{pmatrix} \tag{8.2.3}$$

It is called the matrix of Wronski corresponding to basis (8.2.1).

THEOREM 8.2.1. *Assume that the entries of $A(t)$ are continuous functions on (α,β) and (8.2.1) is a basis of the vector space of all solutions of the homogeneous system (8.1.3) defined for $t\in(\alpha,\beta)$. For any $t\in(\alpha,\beta)$ one has: $det\,W(t)\neq 0$.*

PROOF. Let $t_0\in(\alpha,\beta)$. To prove that $det\,W(t_0)\neq 0$ is the same as to prove that the vectors $y^{(1)}(t_0),...,y^{(n)}(t_0)\in\mathbb{R}^n$ are linearly dependent, i.e. the equation $r_1y^{(1)}(t_0)+\cdots+r_ny^{(n)}(t_0)=0$ implies $r_1,...,r_n=0$. The function $y(t)=r_1y^{(1)}(t)+\cdots+r_ny^{(n)}(t)$ is a solution of the homogeneous system (8.1.3) satisfying $y(t_0)=0$. The uniqueness theorem (the first statement of theorem 8.1.3) implies $y(t)\equiv 0$ which means that the linear combination of the vector functions (8.2.1) is zero (which is much stronger than the zero linear combination of the vectors in $\mathbb{R}^n$, the values of these functions at t_0). Since these vector-functions form a basis they are linearly independent over $\mathbb{R}$ and consequently $r_1=\cdots=r_n=0$. $\square$

THEOREM 8.2.2. *Let $C_1(t), ..., C_n(t)$ be differentiable functions whose derivatives satisfy the linear system*

$$W(t) \cdot \begin{pmatrix} C_1'(t) \\ \cdots \\ C_n'(t) \end{pmatrix} = F(t) = \begin{pmatrix} f_1(t) \\ \cdots \\ f_n(t) \end{pmatrix}, \quad t \in (\alpha, \beta) \tag{8.2.4}$$

Then the vector function (8.2.2) is a solution of the non-homogeneous equation (8.1.4).

REMARK. System (8.2.4) is a system of usual linear equations with respect to the tuple $(C_1'(t), ..., C_n'(t))$. Theorem 8.2.1 implies that this linear system has unique solution $(C_1'(t), ..., C_n'(t))$. Integrating these functions we obtain $C_1(t), ..., C_n(t)$. We need one particular solution, therefore we can take any anti-derivatives.

PROOF. We have

$$(y^*(t))' = C_1'(t)y^{(1)}(t) + C_2'(t)y^{(2)}(t) + \cdots + C_n'(t)y^{(n)}(t) +$$
$$+ C_1(t)(y^{(1)})'(t) + C_2(t)(y^{(2)})'(t) + \cdots + C_n(t)(y^{(n)})'(t).$$

Since $y^{(i)}(t)$ is a solution of the homogeneous system (8.1.3) we have $(y^{(i)})'(t) = A(t)y^{(i)}(t), \ i = 1, ..., n$ and consequently

$$(y^*(t))' = C_1'(t)y^{(1)}(t) + C_2'(t)y^{(2)}(t) + \cdots + C_n'(t)y^{(n)}(t) +$$
$$+ C_1(t)A(t)y^{(1)}(t) + \cdots + C_n(t)A(t)y^{(n)}(t) =$$
$$= C_1'(t)y^{(1)}(t) + C_2'(t)y^{(2)}(t) + \cdots + C_n'(t)y^{(n)}(t) + A(t)y^*(t).$$

It remains to note that system (8.2.4) implies $C_1'(t)y^{(1)}(t) + C_2'(t)y^{(2)}(t) + \cdots + C_n'(t)y^{(n)}(t) = F(t)$. Therefore $(y^*(t))' = Ay^*(t) + F(t)$. □

8.3. From a basis of solutions of homogeneous nth order equation to a solution of non-homogeneous equation

Assume that the coefficients $a_i(t)$ and the vector function $f(t)$ in (8.1.2) are continuous on (α, β) and assume that we know a basis of the vector space of all solutions of the homogeneous equation (8.1.1) defined on (α, β):

$$\text{basis of the vector space of solutions of (8.1.1):} \quad x_1(t), \ x_2(t), \ \ldots, x_n(t) \tag{8.3.1}$$

We can find a particular solution $x^*(t)$ of system (8.1.1), and consequently the set of all solutions of this system (see Theorem 8.1.1) by the following simple way, also called the method of variation of parameters.

The set of all solutions of (8.1.1), defined for $t \in (\alpha, \beta)$ is the set of arbitrary linear combinations of the vector functions (8.3.1):

$$C_1x_1(t) + C_2x_2(t) + \cdots + C_nx_n(t), \quad C_1, ..., C_n \in \mathbb{R}.$$

We will show that the non-homogeneous system (8.1.4), with the same matrix $A(t)$ as in (8.1.3) has a particular solution $y^*(t)$ of the form

$$x^*(t) = C_1(t)x_1(t) + C_2(t)x_2(t) + \cdots + C_n(t)x_n(t), \tag{8.3.2}$$

with some functions $C_1(t), ..., C_n(t)$ instead of the constants $C_1, ..., C_n$ (which explains the name "variation of parameters"). Certainly these functions are not arbitrary, they must satisfy certain conditions. To give these conditions consider the $n \times n$ matrix

$$W(t) = \begin{pmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \\ x_1'(t) & x_2'(t) & \cdots & x_n'(t) \\ x_1''(t) & x_2''(t) & \cdots & x_n''(t) \\ & & \cdots & \\ x_1^{(n-1)}(t) & x_2^{(n-1)}(t) & \cdots & x_n^{(n-1)}(t) \end{pmatrix} \tag{8.3.3}$$

It is called the matrix of Wronski corresponding to basis (8.3.1).

THEOREM 8.3.1. *Assume that the coefficients $a_i(t)$ of the homogeneous equation (8.1.1) are continuous functions on (α, β) and (8.3.1) is a basis of the vector space of all solutions of (8.1.1) defined for $t \in (\alpha, \beta)$. Consider the matrix (8.3.3). For any $t \in (\alpha, \beta)$ one has: $\det W(t) \neq 0$.*

PROOF. Transfer (8.1.1) to a homogeneous system (8.1.3), see section 8.1. The columns of matrix form a basis of the vector space of all solutions of this system defined on (α, β). Therefore Theorem 8.3.1 follows from Theorem 8.2.1.

□

THEOREM 8.3.2. *Let $C_1(t), ..., C_n(t)$ be differentiable functions whose derivatives satisfy the linear system*

$$W(t) \cdot \begin{pmatrix} C_1'(t) \\ C_2'(t) \\ \cdots \\ C_{n-1}'(t) \\ C_n'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ f(t) \end{pmatrix}, \quad t \in (\alpha, \beta) \tag{8.3.4}$$

where $W(t)$ is the matrix (8.3.3). Then the function (8.3.2) is a solution of the non-homogeneous equation (8.1.2).

PROOF. Transfer (8.1.2) to a system (8.1.4), see section 8.1. Any solution of (8.1.2) is the first coordinate of a solution of (8.1.4). Therefore Theorem 8.3.2 follows from Theorem 8.2.2. □

8.4. Solving equation $x' = a(t)x + b(t)$

Assume that the functions $a(t)$ and $b(t)$ are continuous on (α, β). Take any point $t_0 \in (\alpha, \beta)$ and consider the function

$$A(t) = e^{\int_{t_0}^{t} a(s)ds}.$$

Its derivative is equal to $A'(t) = a(t)A(t)$, therefore the single function $A(t)$ is a basis of the 1-dimensional vector space of all solution of the homogeneous equation $x' = a(t)x$, defined on (α, β):

<u>any solution of the equation $x' = a(t)x$ has the form $x(t) = C \cdot A(t)$ for some $C \in \mathbb{R}$</u>

(changing $t_0 \in (\alpha, \beta)$ leads to changing C).

By Theorem 8.3.2 (or Theorem 8.2.2) with $n = 1$ the equation $x' = a(t)x + b(t)$ has a solution $x^*(t) = C(t)A(t)$ where $C(t)$ satisfies the equation $C'(t)A(t) = b(t)$. One of the solutions of the equation for $C(t)$ is

$$C(t) = \int_{t_0}^{t} \frac{b(s)}{A(s)} ds.$$

Therefore the set of all solutions of the equation $x' = a(t)x + b(t)$ is as follows:

$$x(t) = A(t) \cdot \int_{t_0}^{t} \frac{b(s)}{A(s)} ds + C \cdot A(t) = A(t) \left(C + \int_{t_0}^{t} \frac{b(s)}{A(s)} ds \right)$$

$$A(t) = e^{\int_{t_0}^{t} a(s)ds}.$$

If we have initial condition $x(t_0) = x_0$ it is convenient (though not necessary) to take t_0 namely from this initial condition. Since $A(t_0) = 1$ and the integral from t_0 to t_0 vanishes we obtain, substituting t_0: $C = x_0$.

Example. Let us find solution of the equation

$$x' = \frac{x}{\sqrt{t}} + sin\, t$$

satisfying the initial condition $x(1) = 3$ and defined for $t > 0$. We have

$$A(t) = e^{\int_1^t \frac{ds}{\sqrt{s}}} = e^{2\sqrt{t}-2}.$$

Therefore

$$x(t) = e^{2\sqrt{t}-2} \left(3 + \int_1^t sin\, s \cdot e^{2-2\sqrt{s}} ds \right) = e^{2\sqrt{t}} \left(3e^{-2} + \int_1^t sin\, s \cdot e^{-2\sqrt{s}} ds \right).$$

8.5. Solving non-homogeneous system (8.1.4) with a constant matrix A (example)

We know how to find a basis of the space of all solutions of any system $y' = Ay$ (Chapter 5), therefore we can solve any system of the form $y' = Ay + F(t)$ with a constant matrix A.

Let us find solution of the system

$$y_1' = y_1 - y_2 + \sqrt{t^2+1}, \quad y_2' = 10y_1 + 3y_2 + t \tag{8.5.1}$$

defined for all t and satisfying the initial condition $y_1(0) = 1,\ y_2(0) = 0$.

Write the system in the form $y' = Ay + F(t)$, where $A = \begin{pmatrix} 1 & -1 \\ 10 & 3 \end{pmatrix}$ and $F(t) = \begin{pmatrix} \sqrt{t^2+1} \\ t \end{pmatrix}$. The eigenvalues of A are $2 \pm 3i$. Take the eigenvectors $\begin{pmatrix} -1 \\ 1 \pm 3i \end{pmatrix}$. The complexly conjugate functions

$$z_1(t) = e^{(2+3i)t} \cdot \begin{pmatrix} -1 \\ 1+3i \end{pmatrix}, \quad z_2(t) = \bar{z}^{(1)}(t) = e^{(2-3i)t} \cdot \begin{pmatrix} -1 \\ 1-3i \end{pmatrix}$$

is a basis of the vector space of all complex-valued solutions of the homogeneous system $y' = Ay$. The functions

$$y^{(1)}(t) = Re \left(e^{(2+3i)t} \cdot \begin{pmatrix} -1 \\ 1+3i \end{pmatrix} \right), \quad y^{(2)}(t) = Im \left(e^{(2+3i)t} \cdot \begin{pmatrix} -1 \\ 1+3i \end{pmatrix} \right),$$

i.e.

$$y^{(1)}(t) = e^{2t}\begin{pmatrix} -cos(3t) \\ cos(3t) - 3sin(3t) \end{pmatrix}, \quad y^{(2)}(t) = e^{2t}\begin{pmatrix} -sin(3t) \\ sin(3t) + 3cos(3t) \end{pmatrix}.$$

By Theorem 8.2.2 equation (8.5.1) has a solution $y^*(t) = C_1(t)y^{(1)}(t)+C_2(t)y^{(2)}(t)$ where $C_1(t)$ and $C_2(t)$ are any functions satisfying the system

$$W(t)\cdot\begin{pmatrix} C_1'(t) \\ C_2'(t) \end{pmatrix} = \begin{pmatrix} \sqrt{t^2+1} \\ t \end{pmatrix},$$

$$W(t) = \begin{pmatrix} -e^{2t}cos(3t) & -e^{2t}sin(3t) \\ e^{2t}\left(cos(3t) - 3sin(3t)\right) & e^{2t}\left(sin(3t) + 3cos(3t)\right) \end{pmatrix}$$

Solving this linear system (with unknowns $C_1'(t), C_2'(t)$) it is convenient to use Cramer's rule. We have $det(W(t)) = -3e^{4t}$ and by Cramer's rule

$$C_1'(t) = -\frac{1}{3}e^{-2t}\left(\sqrt{t^2+1}\left(sin(3t) + 3cos(3t)\right) + tsin(3t)\right),$$

$$C_2'(t) = -\frac{1}{3}e^{-2t}\left(\sqrt{t^2+1}\left(-cos(3t) + 3sin(3t)\right) - tcos(3t)\right).$$

Since the initial conditions are given at $t_0 = 0$, it is convenient to take the following anti-derivatives:

$$\begin{aligned} C_1(t) &= -\frac{1}{3}\int_0^t e^{-2s}\left(\sqrt{s^2+1}\left(sin(3s) + 3cos(3s)\right) + s\cdot sin(3s)\right)ds \\ C_2(t) &= -\frac{1}{3}\int_0^t e^{-2s}\left(\sqrt{s^2+1}\left(-cos(3s) + 3sin(3s)\right) - s\cdot cos(3s)\right). \end{aligned} \tag{8.5.2}$$

The general solution of the system (8.5.1) is

$$\begin{aligned} y(t) = (c_1 + C_1(t))\,e^{2t}&\begin{pmatrix} -cos(3t) \\ cos(3t) - 3sin(3t) \end{pmatrix} + \\ + (c_2 + C_2(t))\,e^{2t}&\begin{pmatrix} -sin(3t) \\ sin(3t) + 3cos(3t) \end{pmatrix} \end{aligned} \tag{8.5.3}$$

where c_1, c_2 are arbitrary real numbers and the functions $C_1(t)$ and $C_2(t)$ are given by (8.5.2). Substituting the initial condition $y_1(0) = 1,\ y_2(0) = 0$ we obtain a system of linear equations for c_1, c_2: $-c_1 = 1,\ c_1 + 3c_2 = 0$. Therefore the solution satisfying the given initial condition is given by the formula (8.5.3) with $c_1 = -1, c_2 = 1/3$ and $C_1(t), C_2(t)$ defined by (8.5.2).

8.6. Solving non-homogeneous equation (8.1.2) with constant coefficients a_i (example)

We know how to find a basis of the space of all solutions of any linear homogeneous nth order equation with constant coefficients (Chapter 7), therefore we can solve any equation of form (8.1.2) with constant a_i and any function $f(t)$.

Let us find solution of the equation

$$x'' - 2x' + 5x = \sqrt{t} \tag{8.6.1}$$

defined for $t > 0$ and satisfying the initial conditions $x(1) = 0, x'(1) = 1$.

The vector space of all solutions of the homogeneous equation

$$x'' - 2x' + 5x = 0$$

defined for $t > 0$ has a basis

$$x_1(t) = e^t cos(2t),\ x_2(t) = e^t sin(2t).$$

By Theorem 8.3.2 equation (8.6.1) has a solution $x^*(t) = C_1(t)x_1(t) + C_2(t)x_2(t))$ where $C_1(t)$ and $C_2(t)$ are any functions satisfying the system

$$W(t) \cdot \begin{pmatrix} C_1'(t) \\ C_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{t} \end{pmatrix},$$

$$W(t) = \begin{pmatrix} e^t cos(2t) & e^t sin(2t) \\ e^t\left(cos(2t) - 2sin(2t)\right) & e^t\left(sin(2t) + 2cos(2t)\right) \end{pmatrix}$$

Solving this linear system (with unknowns $C_1'(t), C_2'(t)$) it is convenient to use Cramer's rule. We have $det(W(t)) = 2e^{2t}$ and by Cramer's rule

$$C_1'(t) = -\frac{1}{2}e^{-t}\sqrt{t} \cdot sin(2t),\quad C_2'(t) = \frac{1}{2}e^{-t}\sqrt{t} \cdot cos(2t).$$

It follows that the set of all solutions of the equation (8.6.1) is as follows:

$$x(t) = \left(c_1 - \frac{1}{2}\int_1^t e^{-s}\sqrt{s} \cdot sin(2s)ds\right) e^t cos(2t) +$$
$$+ \left(c_2 + \frac{1}{2}\int_1^t e^{-s}\sqrt{s} \cdot cos(2s)ds\right) e^t sin(2t),$$

where c_1, c_2 are arbitrary real numbers. Substituting the initial conditions $x(1) = 0, x'(1) = 1$ we obtain the following linear system for c_1, c_2:

$$cos2 \cdot c_1 + sin2 \cdot x_2 = 0$$

$$(cos2 - 2sin2)c_1 + (sin2 + 2cos2)c_2 = e^{-1}.$$

Solving this system we obtain

$$c_1 = -\frac{sin2}{2e},\quad c_2 = \frac{cos2}{2e}.$$

8.7. The cases when a partial solution can be found directly without variation of parameters

In this section we present the most important cases that we do not need rather long computations realizing the method of variation of parameters.

8.7.1. Equation $x' = ax + b,\ a, b \in \mathbb{R},\ a \neq 0$. Obviously one of solutions is

$$x^*(t) \equiv -\frac{b}{a}$$

8.7.2. Systems $y' = Ay + b,\ detA \neq 0,\ b \in \mathbb{R}^n$. Obviously one of solutions is

$$y^*(t) \equiv -A^{-1}b.$$

8.7.3. Equations of the form $P\left(\frac{d}{dt}\right)(x(t)) = e^{\alpha t}$ where α is not a root of $P(\lambda)$. In chapter 7 we showed that $P\left(\frac{d}{dt}\right)(e^{\alpha t}) = P(\alpha)e^{\alpha t}$. Therefore the equation in the title of this subsection has a solution

$$x^*(t) = \frac{e^{\alpha t}}{P(\alpha)}.$$

8.7.4. Equations of the form $P\left(\frac{d}{dt}\right)(x(t)) = e^{\alpha t}$ where α is not root of $P(\lambda)$ of multiplicity r. In this case the polynomial $P(\lambda)$ can be expressed in the form

$$P(\lambda) = Q(\lambda)(\lambda-\alpha)^r, \quad Q(\lambda) \text{ polynomial}, \ Q(\alpha) \neq 0. \tag{8.7.1}$$

Consider the linear operator

$$L_\alpha: \ f(t) \to f'(t) - \alpha \cdot f(t).$$

By Proposition 7.5.1 we have

$$P\left(\frac{d}{dt}\right)(t^r e^{\alpha t}) = Q\left(\frac{d}{dt}\right)\left(L_\alpha^r(t^r e^{\alpha t})\right)$$

where L_α^r is $L_\alpha \circ L_\alpha \circ \cdots \circ L_\alpha$ (r times). Lemma 7.5.2 implies

$$L_\alpha^r(t^r e^{\alpha t}) = r! e^{\alpha t}.$$

Therefore

$$P\left(\frac{d}{dt}\right)(t^r e^{\alpha t}) = r! Q\left(\frac{d}{dt}\right)(e^{\alpha t}) = r! Q(\alpha) e^{\alpha t}.$$

It follows that the equation in the title of this subsection has a solution

$$x^*(t) = \frac{t^r e^{\alpha t}}{r! Q(\alpha)}.$$

Note that

$$r! Q(\alpha) = P^{(r)}(\alpha).$$

Therefore the same solution can be expressed in the form

$$x^*(t) = \frac{t^r e^{\alpha t}}{P^{(r)}(\alpha)}.$$

8.7.5. Equations of the form $P\left(\frac{d}{dt}\right)(x(t)) = e^{\alpha t} cos(\beta t)$ and $P\left(\frac{d}{dt}\right)(x(t)) = e^{\alpha t} sin(\beta t)$. Consider the equation

$$P\left(\frac{d}{dt}\right)(z(t)) = e^{(\alpha+\beta i)t} \tag{8.7.2}$$

with respect to a complex-valued function $z(t)$. Since

$$e^{\alpha t} cos(\beta t) = Re\left(e^{(\alpha+i\beta)t}\right), \ e^{\alpha t} sin(\beta t) = Im\left(e^{(\alpha+i\beta)t}\right)$$

the equations in the title of this subsection have solutions

$$x^*(t) = Re(z^*(t)), \quad x^*(t) = Im(z^*(t))$$

respectively, where $z^*(t)$ is one of solutions of equation (8.7.2). The solution $z^*(t)$ can be found in the same way as in section 8.7.4 or 8.7.5, with α in those sections replaced by $\alpha + \beta i$. If $\alpha + \beta i$ is not a root of the polynomial $P(\lambda)$ we have

$$z^*(t) = \frac{e^{(\alpha+\beta i)t}}{P(\alpha+\beta i)}$$

If $\alpha + \beta i$ is a root of the polynomial $P(\lambda)$ of multiplicity r we can express $P(\lambda)$ in the form $P(\lambda) = Q(\lambda)(\lambda - (\alpha+\beta i))^r$ and we have

$$z^*(t) = \frac{t^r e^{(\alpha+\beta i)t}}{r! Q(\alpha+\beta i)} = \frac{t^r e^{(\alpha+\beta i)t}}{P^{(r)}(\alpha+\beta i)}.$$

8.7.6. Equations of the form $P\left(\frac{d}{dt}\right)(x(t)) = r_1 f(t) + \cdots + rsf_s(t)$. Assume that we can find a solution $x_i^*(t)$ of the equations $P\left(\frac{d}{dt}\right)(x(t)) = f_i(t),\ i = 1, ..., s.$ It is clear that the function

$$x^*(t) = r_1 x_1(t) + \cdots + r_s x_s(t)$$

is a solution of the equation in the title of this subsection.

8.7.7. Example. Let us find a partial solution $x^*(t)$ of the linear order 6 ODE

$$P\left(\frac{d}{dt}\right)(x(t)) = 2e^{ct} sint + 5cost, \quad P(\lambda) = (\lambda^2 + 6\lambda + 10)(\lambda^2 + 1)^2.$$

Here $c \in \mathbb{R}$ is a parameter. We have

$$x^*(t) = x_1^*(t) + x_2^*(t),$$
$$x_1^*(t) = 2Im(z_1^*(t)), \quad x_2^*(t) = 5Re(z_2^*(t)),$$

where

$$z_1^*(t) \text{ is a solution of the equation } P\left(\frac{d}{dt}\right)(z(t)) = e^{(c+i)t},$$
$$z_2^*(t) \text{ is a solution of the equation } P\left(\frac{d}{dt}\right)(z(t)) = e^{it}.$$

The polynomial $P(\lambda)$ has roots $-3 \pm i$ of multiplicity 1 and $\pm i$ of multiplicity 2. Therefore

$$z_1^*(t) = \frac{e^{(c+i)t}}{P(c+i)} \quad \text{if } c \neq -3,\ c \neq 0$$
$$z_1^*(t) = \frac{te^{(-3+i)t}}{P'(-3+i)} \quad \text{if } c = -3$$
$$z_1^*(t) = \frac{t^2 e^{it}}{P''(i)} \quad \text{if } c = 0$$
$$z_2^*(t) = \frac{t^2 e^{it}}{P''(i)}.$$

Compute (which takes around 10 minutes)

$$P(c+i) = c^2(c+3)\left(c^3 + 3c^2 - 12c - 12 + \left(6c^2 + 12c - 8\right)i\right).$$

To compute $P''(i)$ takes less time:

$$P''(i) = 2!\left(\lambda^2 + 6\lambda + 10\right)_{\lambda=i} = 6 \cdot (3 + 2i).$$

We also need to compute $P'(-3+i)$. To compute it express $P(\lambda)$ in the form

$$P(\lambda) = (\lambda + 3 - i)Q(\lambda), \quad Q(\lambda) = (\lambda + 3 + i)(\lambda^2 + 1)^2.$$

It follows

$$P'(-3+i) = Q(-3+i) = 18(12 + 5i).$$

In the case $c \neq 0, c \neq -3$ we have

$$x_1^*(t) = 2Im\left(\frac{e^{ct}(cost + isint)}{c^2(c+3)((c^3 + 3c^2 - 12c - 12 + (6c^2 + 12c - 8)\,i)}\right) =$$
$$= \frac{2e^{ct}(-D_1 cost + D_2 sint)}{D_3}$$

where $D_1 = 6c^2 + 12c - 8, \quad D_2 = c^3 + 3c^2 + -12c - 12, \quad D_3 = c^2(c+3)(D_1^2 + D_2^2).$

If $c = -3$ we have

$$x_1^*(t) = 2Im\left(\frac{te^{-3t}(cost + isint)}{18(12+5i)}\right) = \frac{te^{-3t}}{9\cdot(12^2+5^2)}\left(-5cost + 12sint\right).$$

For $c = 0$ we have

$$x_1^*(t) = 2Im\left(\frac{t^2(cost + isint)}{6(3+2i)}\right) = \frac{t^2}{39}\left(-2cost + 3sint\right).$$

Finally, $x_2^*(t)$ does not depend on c and we have

$$x_2^*(t) = 5Re\left(\frac{t^2(cost + isint)}{6(3+2i)}\right) = \frac{5t^2}{6\cdot 13}\left(3cost + 2sint\right).$$

8.8. The Euler equation

The homogeneous linear equation of the form

$$t^2x'' + btx' + cx = 0, \quad x = x(t), \quad b, c \in \mathbb{R} \tag{8.8.1}$$

is called the Euler equation. This equation does not have the form (8.1.1), but can be brought to this form by dividing by t^2. We obtain an equation of form (8.1.1) whose coefficients are continuous function in any interval which does not contain the point $t = 0$. Therefore the qualitative theorems of section 8.1 hold for equation (8.8.1) in the time interval $t > 0$ or $t < 0$. We will consider the case $t > 0$.

The special form of equation (8.8.1) allows to find a solution in the simple form $x(t) = t^r$. Substituting $x = t^r$ to the equation we see that this function is a solution if and only if the number r satisfies the square equation

$$r(r-1) + br + c = 0. \tag{8.8.2}$$

It is very easy to prove (an exercise) that the functions t^{r_1} and t^{r_2} are linearly independent over $\mathbb{R}$ provided $r_1 \neq r_2$. Therefore the following statement holds.

THEOREM 8.8.1. *If equation (8.8.2) has two real distinct solutions r_1, r_2 then the couple of functions*

$$x_1(t) = t^{r_1}, \quad x_2(t) = t^{r_2}$$

is a basis of the vector space of all solutions of equation (8.8.1) defined for $t > 0$.

If equation (8.8.2) has two non-real solutions $r_{1,2} = \alpha \pm \beta i,\ \beta \neq 0$ then Theorem 8.8.1 holds for complex-valued solutions. To understand the function t^r for non-real r write it in the form

$$r = \alpha + \beta i \implies t^r = e^{rln\,t} = e^{(\alpha+\beta i)ln\,t} = e^{\alpha ln\,t}\cdot e^{i(\beta\ln\,t)} =$$

$$= t^{\alpha}\left(cos(\beta ln\,t) + isin(\beta ln\,t)\right).$$

Therefore in the case that equation (8.8.2) has solutions $r_{1,2} = \alpha \pm \beta i, \beta \neq 0$ the couple of complexly conjugated functions

$$z_1(t) = t^{\alpha}\left(cos(\beta ln\,t) + isin(\beta ln\,t)\right), \quad z_2(t) = \bar{z}_1(t) = t^{\alpha}\left(cos(\beta ln\,t) + isin(\beta ln\,t)\right)$$

is a basis of the vector space of all complex-valued solutions defined for $t > 0$. To obtain a basis of the vector space of real valued solutions we replace $z_1(t)$ and $z_2(t)$ b he real valued functions

$$\begin{aligned} x_1(t) = (z_1 + z_2)/2 = Re(z_1(t)) = t^{\alpha}\left(cos(\beta ln\, t)\right), \\ x_2(t) = (z_1 - z_2)/(2i) = Im(z_1(t)) = t^{\alpha}\left(sin(\beta ln\, t)\right). \end{aligned} \tag{8.8.3}$$

We obtain:

THEOREM 8.8.2. *If equation (8.8.2) has two solutions* $r_{1,2} = \alpha \pm \beta i$, $\beta \neq 0$ *then the couple of functions*

$$x_1(t) = t^{\alpha}\left(cos(\beta ln\, t)\right), \quad x_2(t) = t^{\alpha}\left(cos(\beta ln\, t)\right)$$

is a basis of the vector space of all solutions of equation (8.8.1) defined for $t > 0$.

It remains the case that equation 8.8.2 has only one solution, of multiplicity 2.

THEOREM 8.8.3. *If equation 8.8.2 has only one solutions* r_1 *(of multiplicity 2) then the couple of functions*

$$x_1(t) = t^{r_1}, \quad x_2(t) = t^{r_1} ln\, t$$

is a basis of the vector space of all solutions of equation (8.8.1) defined for $t > 0$.

This theorem is proved (and explained) in the next section because it requires a theorem on reduction of order of a linear equation.

Example. Let us find the set of all solutions of the equation

$$t^2 x'' + 3tx' + 5x = t^2. \tag{8.8.4}$$

defined for $t > 0$. At first we should find the set of all solutions of the equation

$$t^2 x'' + 3tx' + 5x = 0. \tag{8.8.5}$$

It is the Euler equation. Substituting $x = t^r$ we obtain that it is a solution if and only if $r(r-1) + 3r + 5 = 0$, i.e. $r = -1 \pm 2i$. It follows that the set of all solutions of (8.8.4) is

$$c_1 x_1(t) + c_2 x_2(t) + x^*(t), \quad c_1, c_2 \in \mathbb{R}$$

where

$$x_1(t) = \frac{1}{t} cos(2 ln\, t), \quad x_2(t) = \frac{1}{t} sin(2 ln\, t)$$

and $x^*(t)$ is any partial solution. To find $x^*(t)$ we use the method of variation of parameters. We write equation 8.8.4 in the form

$$x'' + \frac{3}{t} x' + \frac{5}{t^2} x = 1. \tag{8.8.6}$$

Since $t > 0$, the corresponding homogeneous equation $x'' + \frac{3}{t}x' + \frac{5}{t^2}x = 0$ is equivalent to (8.8.5), therefore $x_1(t), x_2(t)$ is a basis of the vector space of all solutions of this homogeneous equation. A particular solution $x^*(t)$ of equation (8.8.6), which is equivalent to (8.8.4) because $t > 0$, has the form

$$x^*(t) = C_1(t)x_1(t) + C_2(t)x_2(t),$$

where the derivatives of the functions $C_1(t)$ and $C_2(t)$ satisfy the linear system

$$W(t)\begin{pmatrix} C_1'(t) \\ C_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Here $W(t)$ is the matrix of Wronski defined by the basis $x_1(t), x_2(t)$ of the vector space of all solutions of the homogeneous equation (8.8.5):

$$W(t) = \begin{pmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{t}cos(2ln\, t) & \frac{1}{t}sin(2ln\, t) \\ -\frac{1}{t^2}(cos(2ln\, t) + 2sin(2ln\, t)) & \frac{1}{t^2}(-sin(2ln\, t) + 2cos(2ln\, t)) \end{pmatrix}$$

We have $detW(t) = \frac{2}{t^3}$. Using Cramer's rule we obtain

$$C_1'(t) = -\frac{t^2}{2}sin(2\ln\, t), \quad C_2'(t) = \frac{t^2}{2}cos(2\ln\, t).$$

Therefore the set of all solutions of equation (8.8.4) can be expressed as follows:

$$x(t) = \frac{c_1\dot{c}os(2ln\,(t))}{t} + \frac{c_2\dot{s}in(2ln\,(t))}{t} -$$

$$-\frac{cos(2ln\, t)}{2t} \cdot \int_1^t s^2 sin(2\ln\, s)ds + \frac{sin(2ln\, t)}{2t} \cdot \int_1^t s^2 cos(2\ln\, s)ds, \qquad c_1, c_2 \in \mathbb{R}.$$

8.9. Reduction of order

Assume that we know a non-zero solution $x_1(t)$ of a linear homogeneous equation

$$x'' + a(t)x' + b(t)x = 0. \tag{8.9.1}$$

The following theorem and its proof allow to find another solution $x_2(t)$ such that $x_1(t)$ and $x_2(t)$ are linearly independent and consequently the couple $x_1(t), x_2(t)$ is a basis of the vector space of all solutions.

THEOREM 8.9.1. *Assume that the coefficients $a(t)$ and $b(t)$ in (8.9.1) are continuous functions on an interval (α, β) and $x_1(t)$ is a solution such that $x_1(t) \neq 0$ for any $t \in (\alpha, \beta)$. Then the equation has another solution $x_2(t)$ of the form*

$$x_2(t) = C(t)x_1(t)$$

which is also defined on (α, β) such that $C(t) \not\equiv const$ and consequently $x_1(t)$ and $x_2(t)$ are linearly independent and form a basis of all solutions defined on (α, β).

To apply this theorem one should clearly understand its proof.

PROOF. Let us find condition on $C(t)$ under which $x_2(t) = C(t)x_1(t)$ is a solution. One has

$$x_2'(t) = C'(t)x_1(t) + C(t)x_1'(t),$$
$$x_2''(t) = C''(t)x_1(t) + 2C'(t)x_1'(t) + C(t)x_1''(t).$$

Therefore $x_2(t)$ is a solution of (8.9.1) if and only if

$$C''(t)x_1(t)+2C'(t)x_1'(t)+C(t)x_1''(t)+a(t)\left(C'(t)x_1(t) + C(t)x_1'(t)\right)+b(t)C(t)x_1(t) = 0.$$

Write this equation in the form

$$C''(t)x_1(t) + C'(t)\left(2x_1'(t) + a(t)x_1(t)\right) + C(t)\left(x_1''(t) + a(t)x_1'(t) + b(t)x_1(t)\right) = 0.$$

Since $x_1(t)$ is a solution, we have $x_1''(t) + a(t)x_1'(t) + b(t)x_1(t) = 0$. Therefore the obtained equation for $C(t)$ simplifies to

$$C''(t)x_1(t) + C'(t)\left(2x_1'(t) + a(t)x_1(t)\right) = 0.$$

Introduce $D(t) = C'(t)$. Since $x_1(t) \neq 0$ for $t \in (\alpha, \beta)$ we obtain the following equation for $D(t)$:

$$D'(t) + f(t)D(t) = 0, \quad f(t) = \frac{2x_1'(t)}{x_1(t)} + a(t).$$

This equation has a solution (see section 8.4) $D(t) = e^{-\int_{t_0}^{t} f(s)ds}$ where t_0 is any point in the interval (α, β). Therefore $x_2(t)$ is a solution of equation (8.9.1) if we take

$$C(t) = \int_{t_0}^{t} D(s)dx, \quad D(t) = e^{-\int_{t_0}^{t} f(s)ds}, \; f(t) = \frac{2x_1'(t)}{x_1(t)} + a(t)$$

where t_0 is any point in (α, β). Since $D(t) > 0$ the function $C(t)$ is not a constant.

□

Example. It is easy to check that the equation

$$tx'' - (t + a)x' + ax = 0$$

with a real parameter $a > 0$ has a solution $x_1(t) = e^t$. Let us find the solution of the equation satisfying the initial conditions

$$x(t_0) = 1, \; x'(t_0) = 0, \quad t_0 > 0.$$

At first we have to find a linearly independent solution $x_2(t)$ defined for $t > 0$. We know that there is a solution $x_2(t)$ of the form $x_2(t) = C(t)e^t$. We have $x_2'(t) = C'(t)e^t + C(t)e^t$, $x_2''(t) = C''(t)e^t + 2C'(t)e^t + c(t)e^t$. Substituting to the equation we obtain that the function $C(t)$ must satisfy

$$C''(t)t + C'(t)(t - a) = 0.$$

Let $D(t) = C'(t)$. Since $t > 0$ the obtained equation holds if and only if

$$D'(t) = \frac{(a - t)}{t} D(t).$$

Compute

$$\int \frac{(a - t)dt}{t} = a \cdot ln|t| - t.$$

Since $t > 0$ one of solutions of the equation for $D(t)$ is

$$D(t) = e^{a ln\, t - t} = t^a e^{-t}.$$

Now for $C(t)$ we have the equation

$$C'(t) = t^a e^{-t}$$

and one of solutions is $C(t) = \int_{t_0}^{t} s^a e^{-s} ds$. Therefore

$$x_2(t) = e^t \int_{t_0}^{t} s^a e^{-s} ds.$$

Now we know the set of all solutions, it is

$$e^t \left(c_1 + c_2 \int_{t_0}^{t} s^a e^{-s} ds \right).$$

Substituting the initial conditions we obtain

$$x(t_0) = e^{t_0} c_1 = 1, \quad x'(t_0) = c_1 e^{t_0} + c_2 t_0^a = 0.$$

It follows that $c_1 = e^{-t_0}$ and $c_2 = -t_0^{-a}$.

Proof of Theorem 8.8.3. We have the solution $x_1(t) = t^{r_1}$ and we know that there is a linearly independent solution of the form $x_2(t) = C(t)t^{r_1}$. Substituting $x_2(t)$ to equation (8.8.1) we obtain

$$t^{r_1+2}C''(t) + C'(t)t^{r_1+1}(2r_1 + b) = 0$$

or equivalently

$$tC''(t) + C'(t)(2r_1 + b) = 0.$$

Since r_1 is the root of the polynomial $P(r) = r(r-1) + br + c$ of multiplicity 2, one has $P'(r_1) = 0$ which means $2r_1 + b = 1$. The obtained equation for $C(t)$ takes the form

$$tC''(t) + C'(t) = 0.$$

Let $D(t) = C'(t)$. Since $t > 0$ the obtained equation for $C(t)$ is equivalent to the equation $D'(t) = -\frac{1}{t}D(t)$. One of solutions is $D(t) = \frac{1}{t}$. Then $C'(t) = \frac{1}{t}$ and one of solutions is $C(t) = ln\, t$.

8.10. Exercises

1. Let A be a real 2×2 matrix with eigenvector $(2, 5+i)$ corresponding to the eigenvalue $\lambda = -3 + 4i$. Find the set of all solutions of the system

$$y' = Ay + \begin{pmatrix} \sqrt{t^2+1} \\ t \end{pmatrix}$$

defined for all t and find the solution of this system satisfying the initial condition $y(2) = (3, 0)$. Integrals in the final answers OK. No complex numbers in the final answers.

2. Consider the linear system

$$y_1' = y_2 + cos(\omega_1 t), \quad y_2' = -2y_1 + sin(\omega_2 t)$$

with real parameters ω_1, ω_2. Find a necessary and sufficient condition on (ω_1, ω_2) under which any solution of this system defined for all t is a bounded vector-function (i.e. there exists C such that $|y_1(t)|, |y_2(t)| < C$ for all t). Give an example of (ω_1, ω_2) such that there is an unbounded solution (i.e. such C does not exist) and find one of such solutions. Is it true that in this case any solution is unbounded?

3. Find the set of all solutions of the system

$$y_1' = y_1 + y_2 + 5, \quad y_2' = -5y_1 - 3y_2 + 1$$

(without using the method of variation of parameters). No integrals and no complex numbers in the final answer.

4. Find the solution of the following equations satisfying the condition $x(0) = 0$.

(a) $x' = -2x + 7, \quad t \in \mathbb{R}$

(b) $x' = -tx + sin\, t, \quad t \in \mathbb{R}$

(c) $tx' = x + cos\, t, \quad t > 0$

(d) $(t^2+1)x' = x + cos\, t, \quad t \in \mathbb{R}$

The final answer should not contain $\int t dt, \int \frac{dt}{t}$, other integrals OK.

5. Find the set of all solutions of the equation $x'''(t) = x(t) + sin(t^2)$ defined for all t. Integrals in the final answer are OK.

6. Find a particular (i.e. any single) solution of the equation $x'''(t) = ax(t) + e^{bt} sin(\omega t)$ with real parameters a, b, ω.

7. Find a particular (i.e. any single) solution of the equation $P\left(\frac{d}{dt}\right)(x(t)) = cost$ where $P(\lambda) = (\lambda^2+1)^4(\lambda^3+5\lambda^2+10\lambda-4)$.

8. Find the set of all solutions of the equation $P\left(\frac{d}{dt}\right)(x(t)) = e^{-2t}(sint + cost) + sin(3t)$ where $P(\lambda) = (\lambda^2+4\lambda+5)(\lambda^2+a)^2$ and a is a real parameter.

9. Find a necessary and sufficient condition on the real numbers a, b, ω under which any solution of the equation $x'' + ax' + bx = cos(\omega t)$, defined for all t, is bounded, i.e. there exists C such that $|x(t)| < C$ for all t.

10. Find the set of all solutions of the system

$$y_1' = y_1 + y_2 + y_3 + y_4,\ y_2' = y_2 + y_3 + y_4,\ y_3' = y_3 + y_4,\ y_4' = y_4$$

without using the Jordan normal form (solve the last equation, after it the third equation, then the second, and finally the first equation).

11. Prove that if $x_1(t) = sint,\ x_2(t) = cost$ and $x_3(t) = f(t)$ are solutions of the equation of the form $(t^2-1)x''' + a(t)x'' + b(t)x' + c(t)x = 0$ defined for $t > 1$, where $a(t), b(t), c(t) \in C^0(1, \infty)$, and the function $f(t)$ satisfies $f''(5) + f(5) = 0$ then $f(t) = r_1 sint + r_2 cost$ for some $r_1, r_2 \in \mathbb{R}$.

12. Find the set of all solutions of the equation $t^2x'' + 7tx' + 13x = \sqrt{t},\ x = x(t)$ defined for $t > 0$. Integrals in the final answer OK.

13. The equation $(1-t^2)x'' + 2tx' - 2x = 0$ has a solution $x_1(t) = t$. Find the set of all solutions of this equations defined for $t > 1$.

CHAPTER 9

Stability of equilibrium points for autonomous systems

9.1. Equilibrium (= singular)points

By definition, a singular, or an equilibrium point of an autonomous system

$$y' = F(y), \quad y = \begin{pmatrix} y_1(t) \\ \cdots \\ y_n(t) \end{pmatrix}, \quad F(y) = \begin{pmatrix} f_1(y) \\ \cdots \\ f_n(y) \end{pmatrix} \tag{9.1.1}$$

is a point $y^* \in \mathbb{R}^n$ such that $F(y^*) = 0$, i.e. $f_1(y^*) = \cdots = f_n(y^*) = 0$.

Obviously y^* is an equilibrium point if and only if $y(t) \equiv y^*$ is a constant solution of the system.

For example the system

$$y_1' = y_2, \quad y_2' = y_1 sin(y_2) + y_1^3 cos(y_2)$$

has the only singular point $(0, 0)$, the system

$$y_1' = y_2, \quad y_2' = y_1 y_2 + sin(y_1)$$

has infinitely many singular points $(\pi k, 0),\ k \in \mathbb{Z}$, the system

$$y_1' = y_1^2 + y_2^2 - 1, \quad y_2' = a y_1^2 + y_2^2 - 2$$

with a real parameter $a > 2$ has 4 singular points

$$p_1 = \left(\frac{1}{\sqrt{a-1}}, \sqrt{\frac{a-2}{a-1}} \right), \; p_2 = \left(\frac{1}{\sqrt{a-1}}, -\sqrt{\frac{a-2}{a-1}} \right),$$

$$p_3 = \left(-\frac{1}{\sqrt{a-1}}, \sqrt{\frac{a-2}{a-1}} \right), \; p_4 = \left(-\frac{1}{\sqrt{a-1}}, -\sqrt{\frac{a-2}{a-1}} \right)$$

If $a = 2$ this system has two singular points $(1, 0)$ and $(-1, 0)$. If $a < 2$ this system has no singular points.

9.2. Definition of stability

Definition 9.2.1. An equilibrium point $y^* \in \mathbb{R}^n$ of system (9.1.1) is called asymptotically stable if for any neighborhood U of y^* contains a smaller neighbourhood W of y^* such that for any $y_0 \in W$ this system has a solution $y(t)$ satisfying the initial condition $y(0) = y_0$ and defined for all $t \geq 0$ and any such solution satisfies the following conditions: 1. $y(t) \in U$ for all $t \geq 0$; 2. $lim_{t \to \infty} y(t) = y^*$. If only the requirement 1. holds then y^* is called stable by Lyapunov (so that asymptotic stability is in general a stronger condition than stability by Lyapunov).

This definition is illustrated by phase portraits in fig. 9.1-9.3. It is possible that any solution of a system tends to an equilibrium point as $t \to \infty$, but this equilibrium point is not stable by Lyapunov, see fig. 9.3.

REMARK. Usually the words "stable equilibrium point" mean stability by Lyapunov without excluding the case od asymptotic stability, and the words "not stable equilibrium point" mean that that it is not stable by Lyapunov (and therefore not asymptotically stable).

9.3. Stabilty of the origin for the system $y' = Ay$

In this case we can give a complete answer. The results of Chapter 5 imply the following theorem.

THEOREM 9.3.1. *Let A be an $n \times n$ matrix. The stability of the equilibrium point $0 \in \mathbb{R}^n$ of the system $y' = Ay$ depends as follows on the eigenvalues of the matrix A:*

1. If the real part of any eigenvalue is strictly negative then the equilibrium point $0 \in \mathbb{R}^n$ is asymptotically stable.

2. If there exists at least one eigenvalue of A whose real part is strictly positive then the equilibrium point $0 \in \mathbb{R}^n$ is not stable by Lyapunov.

3. In the remaining case that the real part of any eigenvalue of A is ≤ 0 and there exists at least one eigenvalue whose real part is equal to 0, the equilibrium point $0 \in \mathbb{R}^n$ is stable by Lyapunov, but not asymptotically stable.

9.4. Lyapunov theorem

Let y^* be an equilibrium point of system (9.1.1). Introduce $x = y - y^*$. The system takes the form

$$x' = F(x + y^*) = \begin{pmatrix} f_1(x+y^*) \\ \cdots \\ f_n(x+y^*) \end{pmatrix} = \widetilde{F}(x) = \begin{pmatrix} \widetilde{f}_1(x) \\ \cdots \\ \widetilde{f}_n(x) \end{pmatrix}.$$

The equilibrium point $y = y^*$ is now $x = 0 \in \mathbb{R}^n$. We assume $F \in C^\infty$, then the vector function $\widetilde{F}(x)$ can be expressed in the form

$$\widetilde{F}(x) = Ax + o(||x||) \text{ as } x \to 0 \in \mathbb{R}^n,$$

where A is the Jacobi matrix

(9.4.1)
$$A = \begin{pmatrix} \frac{\partial \widetilde{f}_1}{\partial x_1}(0) & \frac{\partial \widetilde{f}_1}{\partial x_2}(0) & \cdots & \frac{\partial \widetilde{f}_1}{\partial x_n}(0) \\ & & \cdots & \\ \frac{\partial \widetilde{f}_n}{\partial x_1}(0) & \frac{\partial \widetilde{f}_n}{\partial x_2}(0) & \cdots & \frac{\partial \widetilde{f}_n}{\partial x_n}(0) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(y^*) & \frac{\partial f_1}{\partial y_2}(y^*) & \cdots & \frac{\partial f_1}{\partial y_n}(y^*) \\ & & \cdots & \\ \frac{\partial f_n}{\partial y_1}(y^*) & \frac{\partial f_n}{\partial y_2}(y^*) & \cdots & \frac{\partial f_n}{\partial y_n}(y^*) \end{pmatrix}$$

We have now a system of the form

(9.4.2)
$$x' = Ax + o(||x||) \text{ as } x \to 0 \in \mathbb{R}^n.$$

Obviously the equilibrium point y^* of system (9.1.1) is stable (by Lyaponov or asymptotically) if and only if the equilibrium point $0 \in \mathbb{R}^n$ of system (9.4.2) has the same property. Is it the same property as the stability of the system $x' = Ax$ obtained by taking away $o(||x||)$?

Remark. The system $x' = Ax$ is called the linearization of (9.1.1) at y^*.

The answer to this question was obtained by Lyapunov. In cases 1 and 2 of Theorem 9.3.1 the terms $o(||x||)$ in (9.4.2) play no role in the stability question. But they are important in case 3 of Theorem 9.3.1.

Theorem 9.4.1 (Lyapunov). *Let $F(y) \in C^\infty$ (this condition can be weakened). Let y^* be an equilibrium point of system (9.1.1). Let A be the matrix (9.4.1).*

1. If the real part of any eigenvalue of A is strictly negative then y^* is an asymptotically stable equilibrium point.

2. If there exists at least one eigenvalue of A whose real part is strictly positive then y^* is not stable by Lyapunov (and therefore not asymptotically stable).

3. In the remaining case that the real part of any eigenvalue of A is ≤ 0 and there exists at least one eigenvalue whose real part is equal to 0, the information on the matrix A is not enough to answer the question on stability: dependently on the non-linear terms $o(||x||)$ in (9.4.2) in point y^* can be asymptotically stable, or stable by Lyapunov and not asymptotically stable, or not stable by Lyapunov.

9.5. A lemma on eigenvalues of 2×2 matrices

Before illustrating Lyapunov theorem by examples, let us give a simple necessary and sufficient condition under which the eigenvalues of a 2×2 matrix satisfy the condition 1. or condition 2. in Theorem 9.4.1.

Since the trace of a matrix is the sum of the eigenvalues and the determinant is their product (with multiplicities) we have the following table relating all possible cases for the eigenvalues with the signs of the trace and the sign of the determinant:

Case for the eigenvalues λ_1, λ_2 of a 2×2 matrix A (including $\lambda_1 = \lambda_2$)	$traceA$	$detA$
$\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1, \lambda_2 < 0$	negative	positive
$\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1, \lambda_2 > 0$	positive	positive
$\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 > 0, \lambda_2 < 0$	?	negative
$\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 = 0,\ \lambda_2 < 0$	negative	zero
$\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 = 0,\ \lambda_2 > 0$	positive	zero
$\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 = 0,\ \lambda_2 = 0$	zero	zero
$\lambda_{1,2} = \alpha \pm \beta i$, $\beta \neq 0,\ \alpha < 0$	negative	positive
$\lambda_{1,2} = \alpha \pm \beta i$, $\beta \neq 0,\ \alpha > 0$	positive	positive
$\lambda_{1,2} = \pm\beta i, \beta \neq 0$	zero	positive

This table implies the following useful lemma.

LEMMA 9.5.1. *The eigenvalues of a 2×2 matrix A satisfy the condition 1. in Theorem 9.4.1 if and only if $traceA < 0$ and $detA > 0$. The eigenvalues satisfy the condition 2. in Theorem 9.4.1 if and only if either $traceA > 0$ or $detA < 0$.*

9.6. Example with $n = 2$

Consider the system

$$x_1' = x_1^2 + x_2^2 - 1, \quad x_2' = ax_1 + bx_2, \quad b \neq 0.$$

There are two singular points:

$$p_1 = \left(\frac{|b|}{\sqrt{a^2+b^2}}, \quad -\frac{a|b|}{b\sqrt{a^2+b^2}} \right)$$
$$p_2 = \left(-\frac{|b|}{\sqrt{a^2+b^2}}, \quad \frac{a|b|}{b\sqrt{a^2+b^2}} \right)$$

The Jacobi matrix is $A = \begin{pmatrix} 2x_1 & 2x_2 \\ a & b \end{pmatrix}$. We have

$$A(p_1) = \begin{pmatrix} \frac{2|b|}{\sqrt{a^2+b^2}} & -\frac{2a|b|}{b\sqrt{a^2+b^2}} \\ a & b \end{pmatrix}.$$

$$A(p_2) = \begin{pmatrix} -\frac{2|b|}{\sqrt{a^2+b^2}} & \frac{2a|b|}{b\sqrt{a^2+b^2}} \\ a & b \end{pmatrix}.$$

Compute

$$detA(p_1) = \frac{2|b|(a^2+b^2)}{b\sqrt{a^2+b^2}}, \quad traceA(p_1) = \frac{2|b|}{\sqrt{a^2+b^2}} + b$$
$$detA(p_2) = -\frac{2|b|(a^2+b^2)}{b\sqrt{a^2+b^2}}, \quad traceA(p_2) = -\frac{2|b|}{\sqrt{a^2+b^2}} + b$$

Note that

- if $b > 0$ then $traceA(p_1) > 0$, therefore p_1 is not stable;
- if $b < 0$ then $detA(p_1) < 0$, therefore like above p_1 is not stable;
- it follows that p_1 is not stable for any parameters $a, b \neq 0$;
- if $b > 0$ then $detA(p_2) < 0$, therefore p_2 is not stable;
- if $b < 0$ then $detA(p_2) > 0$ and $traceA(p_2) < 0$, therefore p_2 is asymptotically stable;
- thus p_2 is stable (asymptotically) if $b < 0$ and not stable (by Lyapunov) if $b > 0$.

9.7. Example with $n = 3$

Consider the system

$$x_1' = x_2^2 + x_3, \quad x_2' = x_3^2 + x_1, \; x_3' = x_1^2 + x_2.$$

It is easy to see that there are two equilibrium points

$$p_1 = (0, 0, 0), \quad p_2 = (-1, -1, -1).$$

The Jacobi matrix is $A = \begin{pmatrix} 0 & 2x_2 & 1 \\ 1 & 0 & 2x_3 \\ 2x_1 & 1 & 0 \end{pmatrix}$. We have

$$A(p_1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad A(p_2) = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 0 & 0 \\ -2 & 1 & 0 \end{pmatrix}.$$

The characteristic polynomial of the matrix $A(p_1)$ is $\lambda^3 - 1$. One of the eigenvalues is equal to 1. Therefore p_1 is not stable. The characteristic polynomial of the matrix $A(p_2)$ is $\lambda^3 + 6\lambda + 7$. Let $\lambda_1, \lambda_2, \lambda_3$ be its roots, i.e. eigenvalues of the matrix $A(p_2)$. By Vieta's theorem $\lambda_1 + \lambda_2 + \lambda_3 = 0$. Therefore $Re\lambda_1 + Re\lambda_2 + Re\lambda_3 = 0$. Sice 0 is not an eigenvalues, the case $Re\lambda_1 = Re\lambda_2 = Re\lambda_3 = 0$ is impossible. It follows that there is at least one eigenvalues whose real part is positive. Therefore like p_1 the equilibrium point p_2 is not stable.

9.8. Exercises

1. Find all singular points of the system

$$x_1' = \frac{x_1}{x_2} - 1, \quad x_2' = f(x_1)$$

$$f(x_1) = (x_1 + 3)(x_1 + 2)(x_1 + 1)(x_1 - 1)(x_1 - 2)(x_1 - 3)$$

and determine which of them are asymptotically stable.

2. Find all singular points of the system

$$x_1' = ln(x_1 x_2), \quad x_2' = ax_1 - b,$$

where a, b are real parameters, $a \neq 0, b \neq 0$. Determine which of them are asymptotically stable.

3. Find all singular points of the system

$x_1' = (x_2 - x_1) \cdot x_3^2$

$x_2' = x_1 x_3 - x_2$

$x_3' = (x_3 + \frac{7}{2})(x_3 + \frac{5}{2})(x_3 + \frac{3}{2})(x_3 + \frac{1}{2})(x_3 - \frac{1}{2})(x_3 - \frac{3}{2})(x_3 - \frac{5}{2})(x_3 - \frac{7}{2})$

and determine which of them are asymptotically stable.

4. Consider the system

$$x_1' = -x_1^N, \quad x_2' = x_3, \quad x_3' = ax_2, \quad a \in \mathbb{R}, \quad N \in \{1, 2, 3, ...\}.$$

Prove that the equilibrium point $(0, 0, 0)$ is stable by Lyapunov if and only if $a < 0$ and N is an odd number.

CHAPTER 10

Solvable first order ODEs

10.1. Introduction

We have a complete theory of autonomous first order ODEs $x' = f(x)$ (Chapter 3) and a complete theory of linear first order ODEs $x' = a(t)x + b(t)$ (section 8.4).

In this chapter we present several other types of first order ODEs $x' = f(t,x)$ which can be solved. Here the world "to solve" should be understood as follows: we can find a formula (possibly involving integrals that cannot be expressed by elementary functions) relating t and x for all solutions $x(t)$ of the equation.

More precisely, "to solve" means to find a function $H(t,x)$ such that

$$H(t, x(t)) = c = const \quad \text{for any solution } x(t). \tag{10.1.1}$$

The constant c depends on the initial conditions. In some cases such a formula implies a direct formula for $x(t)$, in many cases not. Many first order ODEs are not solvable, i.e. a function $H(t,x)$ cannot be found, even if integrals are allowed.

One should understand that even if an equation is solvable, i.e. we know a function $H(t,x)$ satisfying (10.1.1), it does not mean at all that we know the basic qualitative properties of solutions: on which intervals they are defined, for which t they increase or decrease, what are their limits as $t \to \pm\infty$, etc. Certainly if the function H is simple enough, it can help a lot to obtain the answers. But in many cases the function $H(t,x)$, for solvable equations, is as involved as

$$H(t,x) = \sqrt{\frac{t + ln\, x}{x^2 - t}} + sin\left(\frac{xt}{x^2 - t^2}\right)$$

or

$$H(t,x) = \int_0^{x/t} e^{-sin\sqrt{s^4 - \sqrt{s}}} ds + cos\left(\frac{1}{x^2 + t^2 - 1}\right).$$

In such cases it cannot help much to answer qualitative questions on solutions.

On the other hand, in many cases the qualitative properties of solutions can be understood without solving it, whether or not solving is possible. We already illustrated it in chapters 3 and 4.

10.2. Separable equations: $x' = a(t)b(x)$

Equations of the form

$$x' = a(t)b(x), \quad x = x(t), \quad a(t) \in C^0, \quad b(x) \in C^1 \tag{10.2.1}$$

so that the function $f(t,x)$ in the first order ODE of general form, $x' = f(t,x)$, is the product of two functions, one depends only on t, the other only on x, are called separable. We need the assumptions $a(t) \in C^0, b(x) \in C^1$ in order to use the existence and the uniqueness theorem, as well as the theorem on prolongation of solutions, see Chapter 2.

Equations of form (10.2.1) are present in any level text book on ODEs and usually they are solved as in the following example:

EXAMPLE. The standard way of solving the equation $x' = x^2 \cdot cos\, t$ is as follows. Write the equation in the form $\frac{dx}{dt} = x^2 \cdot cos\, t$, after that in the form $\frac{dx}{x^2} = cos\, tdt$. The next step is to add the sign of integral to the both sides of the equation: $\int \frac{dx}{x^2} = \int cos\, tdt$. Integrating we get $-\frac{1}{x} = sin\, t + c$ where c is a "free" constant. It follows $x(t) = -\frac{1}{c+sin\, t}$. The constant c depends on the initial condition. If for example we have $x(0) = 1$ then substituting $t = 0$ we obtain $c = -1$ and consequently $x(t) = \frac{1}{1-sin\, t}$.

Certainly the manipulations used in this example are not more than notations, a kind of "slang" which corresponds to a certain theorem. These notations (or slang if one wishes) are very convenient and very standard and it is worth to use them, but a student (especially of Math faculty) should clearly understand what theorem is covered by these notations.

This theorem is as follows.

THEOREM 10.2.1.
1. Let $x(t)$ be a non-constant solution of equation (10.2.1) satisfying the condition $x(t_0) = x_0$ and defined for $t \in (\alpha, \beta)$. Then $b(x(t)) \neq 0$ for any $t \in (\alpha, \beta)$, consequently $t \to \int_{x_0}^{x(t)} \frac{ds}{b(s)}$ is a differentiable function, and one has

$$\int_{x_0}^{x(t)} \frac{ds}{b(s)} = \int_{t_0}^{t} a(s)ds, \quad t \in (\alpha, \beta). \tag{10.2.2}$$

2. The converse also holds: if $x(t)$ is a differentiable function such that $b(x(t)) \neq 0$ for all $t \in (\alpha, \beta)$ and such that (10.2.2) holds then $x(t)$ is a solution of (10.2.1) for $t \in (\alpha, \beta)$.

PROOF. Let $x(t), t \in (\alpha, \beta)$ be a non-constant solution. The fact that $b(x(t)) \neq 0$ for all $t \in (\alpha, \beta)$ follows from the uniqueness theorem and the condition that $x(t)$ is a non-constant solution: if $b(x(t_1)) = 0$ for some $t_1 \in (\alpha, \beta)$ then the function $\widetilde{x}(t) \equiv x(t_1)$ is a solution of (10.2.1) and since $\widetilde{x}(t_1) = x(t_1)$ we have contradiction to the uniqueness theorem. To show that (10.2.2) holds it suffices to note that the functions

$$F(t) = \int_{x_0}^{x(t)} \frac{ds}{b(s)}, \quad G(t) = \int_{t_0}^{t} a(s)ds$$

have the same value at t_0: $F(t_0) = G(t_0) = 0$ and their derivatives are identically equal:

$$F'(t) = \frac{1}{b(x(t))} \cdot x'(t) = \frac{1}{b(x(t))} \cdot (b(x(t))a(t)) = a(t) = G'(t) = a(t).$$

Here we certainly used that $x(t)$ is a solution of (10.2.1). To prove the second statement it suffices to observe that if $F(t) = G(t)$ then $F'(t) = G'(t)$ and it follows

$$\frac{1}{b(x(t))} \cdot x'(t) = a(t) \quad \Longrightarrow \quad x'(t) = a(t)b(x(t)).$$

□

EXAMPLE 10.2.2. Consider the equation

$$x' = (x-1)(x-3)\frac{t^3}{t^4+1}.$$

Let us analyze the solution of this equation satisfying the initial condition

$$x(1) = 2$$

and defined on maximal possible time-interval (t^-, t^+).

We have constant solutions $x_1(t) \equiv 1$ and $x_2(t) \equiv 3$. The initial condition $x(1) = 2$ and the uniqueness theorem imply

$$x(t) \in (1,3), \quad t \in (t^-, t^+). \tag{10.2.3}$$

It follows:

$$x'(t) > 0 \text{ if } t \in (t^-, t^+),\ t < 0 \qquad x'(t) < 0 \text{ if } t \in (t^-, t^+),\ t > 0 \tag{10.2.4}$$

Therefore <u>if</u> $0 \in (t^-, t^+)$ then 0 is a point of maximum of $x(t)$.

From (10.2.3) and (10.2.4) it follows that there are limits

$$\lim_{t\to t^+} x(t) = A \in [1,3], \quad \lim_{t\to t^-} x(t) = B \in [1,3]. \tag{10.2.5}$$

Now Theorems on prolongation of solutions in section 2.4 imply

$$t^- = -\infty, \qquad t^+ = \infty.$$

To find A in B in (10.2.5) we use Theorem 10.2.1. According to this theorem

$$\int_2^{x(t)} \frac{ds}{(s-1)(s-3)} = \int_1^t \frac{s^3 ds}{s^4+1}.$$

The integrals

$$\int_1^{\infty} \frac{s^3 ds}{s^4+1}, \quad \int_1^{-\infty} \frac{s^3 ds}{s^4+1}$$

<u>diverge</u>. Therefore the integrals $\int_2^A \frac{ds}{(s-1)(s-3)}$ and $\int_2^B \frac{ds}{(s-1)(s-3)}$ diverge. It allows to conclude that each of the numbers A and B is either 1 or 3. Since $x(t)$ decreases as $t > 0$ and increases as $t > 0$ we obtain

$$A = B = 1.$$

Our analysis shows that the graph of $x(t)$ is one showed in fig. 10.1.

EXAMPLE 10.2.3. Consider the equation

$$x' = (x-1)(x-3)\frac{t^3}{(t^4+1)^2}.$$

Let us analyze the solution of this equation satisfying the initial condition

$$x(1) = 2$$

and defined on maximal possible time-interval (t^-, t^+).

Arguing in exactly the same way as in Example 10.2.2 we obtain

$$t^- = -\infty, \quad t^+ = \infty$$

$$x(t) \text{ increases as } t < 0, \qquad x(t) \text{ decreases as } t > 0$$

$$lim_{t\to\infty} x(t) = A \in [1,3], \quad lim_{t\to-\infty} x(t) = B \in [1,3],$$

$$\int_2^{x(t)} \frac{ds}{(s-1)(s-3)} = \int_1^t \frac{s^3 ds}{(s^4+1)^2}.$$

The latter equation implies

$$\int_2^A \frac{dx}{(x-1)(x-3)} = \int_1^\infty \frac{t^3 dt}{(t^4+1)^2},$$

$$\int_2^B \frac{dx}{(x-1)(x-3)} = \int_1^{-\infty} \frac{t^3 dt}{(t^4+1)^2}.$$

The integrals in the right hand side parts of these equations converge. It allows us to conclude that

$$A > 1, \quad B > 1$$

(explicit computation of A and B is possible, but requires a lot of work). The graph of the solution is showed in fig. 10.2.

EXAMPLE 10.2.4. Consider the equation

$$x' = (x-1)^3(x-3)^2 sin\, t.$$

Let us analyze the solution of this equation satisfying the initial condition

$$x(1) = 2$$

and defined on maximal possible time-interval (t^-, t^+).

Arguing as in the previous examples we obtain that $t^+ = \infty,\ t^- = -\infty$ and $x(t)$ increases as $sin\, t > 0$ and decreases as $sin\, t < 0$. We also have

$$\int_2^{x(t)} \frac{ds}{(s-1)^3(s-3)^2} = \int_1^t sin\, s\, ds$$

and it follows that the function $x(t)$ has no limit as $t \to \pm\infty$. The graph of $x(t)$ is showed in fig. 10.3.

EXAMPLE 10.2.5. Consider the equation

$$x' = (x^4 - 1) \cdot t$$

Let us analyze the solution of this equation satisfying the initial condition

$$x(0) = 2$$

and defined on maximal possible time-interval (t^-, t^+).

The uniqueness theorem implies that $x(t) > 1$ for any $t \in (t^-, t^+)$. We also see that $x(t)$ increases if $t \in (t^-, t^+),\ t > 0$ and decreases if $t \in (t^-, t^+),\ t > 0$. In this case the theorem on prolongation of solutions does not alow us to decide if t^+ and t^- are $\pm\infty$ or finite numbers. But we can answer this question using the equation

$$\int_2^{x(t)} \frac{ds}{s^4 - 1} = \int_0^t s ds = \frac{t^2}{2}. \tag{10.2.6}$$

Assume that $t^+ = \infty$. In this case it is easy to prove (without solving the equation) that $x(t) \to \infty$ as $t \to \infty$ and (10.2.6) implies $\int_2^\infty \frac{ds}{s^4-1} = \infty$ which is a contradiction: the integral $\int_2^\infty \frac{ds}{s^4-1}$ converges. Therefore $t^+ < \infty$. In the same way we prove that $t^- = -\infty$. The prolongation theorem implies $x(t) \to \infty$ as $t \to \pm\infty$. Using again (10.2.6) we obtain

$$t^\pm = \pm\sqrt{2\int_2^\infty \frac{dx}{x^4 - 1}}.$$

The graph of $x(t)$ is showed in fig. 10.4.

EXAMPLE 10.2.6. ni Consider the equation

$$x' = (x^4 - 1) \cdot t^2.$$

Let us analyze the solution of this equation satisfying the initial condition

$$x(0) = 2$$

and defined on maximal possible time-interval (t^-, t^+).

Like in the previous example we have $x(t) > 1$ for any $t \in (t^-, t^+)$, but now the function $x(t)$ increases for all $t \in (t^-, t^+)$. We can use the theorem on prolongation of solutions to conclude that $t^- = -\infty$. We cannot use theorem on prolongation of solutions to see if $t^+ = \infty$ or t^+ is a finite number, but we can answer this question arguing in the same way as in Example 10.2.5. We have

$$\int_2^{x(t)} \frac{ds}{s^4 - 1} = \int_0^t s^2 ds = \frac{t^3}{3}. \tag{10.2.7}$$

If $t^+ = \infty$ then it is easy to see (without solving the equation) that $x(t) \to \infty$ as $t \to t^+$ and (10.2.7) gives $\int_2^{x(t)} \frac{ds}{s^4-1} = \infty$ whereas we know that this integral converges. The contradiction shows that t^+ is a finite number. Now we use the prolongation theorem to conclude that $x(t) \to \infty$ as $t \to t^+$. Using again (10.2.7) we obtain

$$t^+ = \left(3\int_2^\infty \frac{dx}{x^4 - 1}\right)^{1/3}.$$

Since $x(t)$ is an increasing function and $x(t) > 1$ there exists $lim_{t\to-\infty} x(t) = B \geq 1$. Taking the limit as $t \to -\infty$ in (10.2.7) we obtain $\int_2^B \frac{ds}{s^4-1} = -\infty$ and it follows $B = 1$. The graph of the solution is showed in fig. 10.5.

EXAMPLE 10.2.7. Consider the equation

$$x' = (x^2 - 1)(t - 1).$$

Let us analyze the solution of this equation satisfying the initial condition

$$x(0) = a < -1.$$

and defined on maximal possible time-interval (t^-, t^+).

The uniqueness theorem implies $x(t) < -1$ for all $t \in (t^-, t^+)$. It follows that $x(t)$ increases as $t \in (t^-, t^+), t > 1$ and decreases as $t \in (t^-, t^+), t < 1$. The theorem on prolongation of solutions implies $t^- = -\infty$. It follows that there is a limit $lim_{t\to-\infty} x(t) = B \geq -1$. Taking the limit as $t \to -\infty$ in the equation

$$\int_a^{x(t)} \frac{ds}{s^2 - 1} = \int_0^t (s - 1)dx = \frac{t^2}{2} - t \tag{10.2.8}$$

we obtain $\int_2^B \frac{dx}{x^2-1} = \infty$ and it follows $B = -1$.

This part of the analysis is simple, but the part concerning t^+ is not. **Assume** $t^+ > 1$. In this case $x(t)$ increases as $t > 1$ and since $x(t) < -1$ we can use the prolongation theorem to conclude that $t^+ = \infty$. Equation (10.2.8) implies that $lim_{t\to\infty} x(t) = -1$. Therefore inder assumption $t^+ > 1$ we obtain that $t^+ = \infty$ and the graph of the solution is showed in fig. 10.6,a.

If it is not true that $t^+ > 1$, i.e. $t^+ \leq 1$ then $x(t)$ is a decreasing function for all $t \in (-\infty, t^+)$ and the prolongation theorem implies that $lim_{t\to t^+} x(t) = -\infty$. The graph of $x(t)$ is showed in fig. 10.6.b.

Thus we have tow principally different possibilities: a. and b. in fig. 10.6 and we know that one of them holds. Which one?

To answer we have to make computation of the integral in (10.2.8). One can compute

$$a < -1,\ x < -1 \implies \int_a^x \frac{ds}{s^2 - 1} = \frac{1}{2} ln\left(\frac{x-1}{x+1}\right) - \frac{1}{2} ln\left(\frac{a-1}{a+1}\right)$$

After this computation we obtain from (10.2.8):

$$\frac{x-1}{x+1} = \frac{a-1}{a+1} e^{t^2 - 2t}.$$

It is easy to solve this equation with respect to x. Doing it we see that $x(t)$ is defined for all t such that

$$e^{t^2-2t} \neq \frac{a+1}{a-1}.$$

Now we have to find t, id any, such that $e^{t^2-2t} = \frac{a+1}{a-1}$ or equivalently

$$t^2 - 2t - ln\left(\frac{a+1}{a-1}\right) = 0. \tag{10.2.9}$$

The roots of this square equation are

$$t_1 = 1 + \sqrt{1 + ln\left(\frac{a+1}{a-1}\right)},\quad t_2 = 1 - \sqrt{1 + ln\left(\frac{a+1}{a-1}\right)} \tag{10.2.10}$$

It is easy to compute

$$a < -1,\ 1 + ln\left(\frac{a+1}{a-1}\right) = 0\ \Leftrightarrow a = \frac{1+e}{1-e}.$$

Therefore if $a < \frac{1+e}{1-e}$ the equation (10.2.9) has no real roots and consequently $x(t)$ is defined for all $t \in \mathbb{R}$, i.e. $t^+ = \infty$ and we have case a. in fig. 10.6. If $a \in \left(-1,\ \frac{1+e}{1-e}\right)$ the equation (10.2.9) has two real roots defined by (10.2.10), and if $a = \frac{1+e}{1-e}$ it has one real root. In any of these cases the roots are positive and the smaller root is t_1. Therefore in the case $a \in \left(-1,\ \frac{1+e}{1-e}\right]$ the solution is defined for $t \in (-\infty, t^+)$ with $t^+ = t_1 = 1 - \sqrt{1 + ln\left(\frac{a+1}{a-1}\right)}$.

The final conclusion is as follows:

Denote

$$\mu = \frac{1+e}{1-e} \approx -2.16.$$

If $a \in (-1, \mu)$ we have case a. in fig. 10.6, and if $a \leq \mu$ we have case b. in fig. 10.6.

10.3. Equations of the form $x' = f\left(\frac{x}{t}\right),\ t > 0$

Such equations can be solve by introducing the function $y(t) = \frac{x(t)}{t}$. We have

$$y' = \left(\frac{x}{t}\right)' = \frac{x't - x}{t^2} = \frac{1}{t}\left(f\left(\frac{x}{t}\right) - \frac{x}{t}\right) = \frac{1}{t}(f(y) - y).$$

Thus $y(t)$ satisfies the equation

$$y' = \frac{f(y) - y}{t}$$

which is a separable equation. We know how to solve it. We obtain a formula relating y and t. Substituting $y = x/t$ we obtain a formula relating x and t.

EXAMPLE 10.3.1. Let $x(t)$ be the solution of the equation $x' = \frac{x^2}{t^2} + 1,\ t > 0$ satisfying the initial condition $x(3) = 2$. Then the function $y = y(t) = \frac{x(t)}{t}$ satisfies the equation $y' = \frac{y^2+1-y}{t}$ and the initial condition $y(3) = \frac{x(3)}{3} = \frac{2}{3}$. We obtain

$$\int_{2/3}^{y(t)} \frac{ds}{s^2 + 1 - s} = \int_3^t \frac{ds}{s} = ln\left(\frac{t}{3}\right)$$

and it follows

$$t = 3 \cdot exp\left(\int_{2/3}^{y(t)} \frac{ds}{s^2 + 1 - s}\right).$$

Therefore $x(t)$ and t are related by the formula

$$t = 3 \cdot exp\left(\int_{2/3}^{\frac{x(t)}{t}} \frac{ds}{s^2 + 1 - s}\right).$$

EXAMPLE 10.3.2. Let $x(t)$ be the solution of the equation

$$x' = \frac{x^3 + 2x^2t + t^3}{xt^2 + 3t^3},\ t > 0$$

satisfying the initial condition $x(1) = 0$. This equation has the form $x' = f(\frac{x}{t})$ because dividing the nominator and the denominator by t^3 we obtain

$$x' = f(y) = \frac{y^3 + 2y^2 + 1}{y + 3}, \quad y = \frac{x}{t}.$$

The function $y(t)$ satisfies the equation

$$y' = \frac{1}{t}\left(\frac{y^3 + 2y^2 + 1}{y + 3} - y\right) = \frac{y^3 + y^2 - 3y + 1}{t \cdot (y + 3)}$$

and the initial condition $y(1) = \frac{x(1)}{1} = 0$. Solving it we obtain

$$\int_0^{y(t)} \frac{(s + 3)ds}{s^3 + s^2 - 3s + 1} = \int_1^t \frac{ds}{s} = lnt$$

which gives us the forllowing formula relating $x(t)$ and t:

$$t = exp\left(\int_0^{\frac{x(t)}{t}} \frac{(s + 3)ds}{s^3 + s^2 - 3s + 1}\right).$$

10.4. Equations of the form $x' = f(ax + bt + c),\ a \neq 0$

Such equations can be solved by introducing $y(t) = ax(t) + bt + c$. The function $y(t)$ satisfies the equation

$$y'(t) = ax'(t) + b = af(y) + b$$

which can be solved as a separable equation. Substituting $y = ax + bt + c$ to the formula relating $y(t)$ and t we obtain a formula relating $x(t)$ and t.

Example 10.4.1. Let $x(t)$ be the solution of the equation $x' = (2x+5t+1)^3+1$ satisfying the initial condition $x(3) = 1$. The function $y(t) = 2x(t)+5t+1$ satisfies the equation

$$y' = 2x' + 5 = 2(y^3 + 1) + 5 = 2y^3 + 7$$

and the initial condition $y(3) = 2x(3) + 5 \cdot 3 + 1 = 18$. Therefore

$$\int_{18}^{y(t)} \frac{ds}{2s^3 + 7} = \int_3^t ds = t - 3$$

and we obtain the following formula relating $x(t)$ and t:

$$t = 3 + \int_{18}^{2x(t)+5t+1} \frac{ds}{2s^3 + 7}.$$

10.5. Bernoulli equation $x' = a(t)x + b(t)x^\beta,\ \beta \in \mathbb{R}$

If $\beta = 1$ it is a linear first order ODE. If $\beta \neq 1$ it can be simplified to a linear ODE by introducing

$$y(t) = x^r(t) \text{ with a suitable } r.$$

We have

$$y'(t) = rx^{r-1}x'(t) = rx^{r-1}(t)\left(a(t)x(t) + b(t)x^\beta(t)\right) = r\left(a(t)y(t) + b(t)x^{\beta+r-1}(t)\right).$$

We see that taking

$$r = 1 - \beta$$

we obtain a linear equation

$$y'(t) = (1-\beta)\left(a(t)y(t) + b(t)\right).$$

We know how to solve this equation (section 8.4). We can obtain a formula for $y(t)$. We know that $x(t) = (y(t))^{\frac{1}{1-\beta}}$, therefore we can obtain a formula for $x(t)$.

Example 10.5.1. Let us find the solution of the equation

$$x' = -tx + x^{1/3}$$

satisfying the initial condition $x(0) = 2$.

Introduce

$$y(t) = x^{2/3}(t).$$

Then

$$y' = \frac{2}{3}x^{-1/3}x' = \frac{2}{3}x^{-1/3}(-tx + x^{1/3}) = -\frac{2}{3}tx^{2/3} + 2/3 = -\frac{2}{3}ty + \frac{2}{3}.$$

Thus we obtain

$$y' = -\frac{2}{3}ty + \frac{2}{3},\quad y(0) = (x(0))^{2/3} = 2^{2/3}. \tag{10.5.1}$$

The set of all solution of the homogeneous linear equation $y' = -\frac{2}{3}ty$ is

$$y' = -\frac{2}{3}ty \quad \Longrightarrow \quad y = ce^{-t^2/3}, \quad c \in \mathbb{R}.$$

A particular solution $y^*(t)$ of (10.5.1) can be found by the method of variation of constants. It has the form $y^*(t) = C(t)e^{-t^2/3}$ where the function $C(t)$ satisfies the equation

$$C'(t) = \frac{2}{3}e^{t^2/3}.$$

We can take

$$C(t) = \frac{2}{3}\int_0^t e^{s^2/3}ds.$$

We obtain the set of all solutions of equation (10.5.1):

$$y(t) = \left(c + \frac{2}{3}\int_0^t e^{s^2/3}ds\right)e^{-t^2/3}.$$

The initial condition $y(0) = 2^{2/3}$ implies $c = 2^{2/3}$. Therefore

$$y(t) = \left(2^{2/3} + \frac{2}{3}\int_0^t e^{s^2/3}ds\right)e^{-t^2/3}$$

and

$$x(t) = y^{3/2}(t) = \left(\left(2^{2/3} + \frac{2}{3}\int_0^t e^{s^2/3}ds\right)e^{-t^2/3}\right)^{3/2} =$$

$$= \left(2^{2/3} + \frac{2}{3}\int_0^t e^{s^2/3}ds\right)^{3/2} \cdot e^{-t^2/2}.$$

10.6. Exercises

1. Find a formula, without integrals, for the solution $x(t)$ of the equation $x' = x(1-x)t$ satisfying the initial condition $x(1) = 1/2$.

2. Draw the graph of the solution $x(t)$ of the given first order ODE satisfying the given initial condition and defined on maximal possible interval (t^-, t^+).
Determine in t^- and t^+ are finite or $\pm\infty$.
If t^- and/or t^+ is finite give a formula for it (integrals are OK).
Find $lim_{t\to t^+}x(t)$ and $lim_{t\to t^-}x(t)$ (integrals are OK if the limit is not $\pm\infty$)

2.1. $x' = (x^4 - 1)(t^3 - t), \quad x(0) = 0$

2.2. $x' = sin\, x \cdot cos\, t, \quad x(\pi) = 10$

2.3 $x' = \sqrt{x^8 + 1} \cdot \frac{t}{t^2+1}, \quad x(0) = 0$

2.4 $x' = \sqrt{x^8 + 1} \cdot \frac{1}{t^2+1}, \quad x(0) = 0$

2.5 $x' = (t-1) \cdot \frac{(x-1)^2}{x^2+1}, \quad x(1) = 2$

2.6 $x' = (t-1)^2 \cdot \frac{(x-1)^2}{x^2+1}. \quad x(1) = 2$

2.7. $x' = (1 - x^2)(t + 3), \quad x(0) = a > 1$

2.8. $x' = (1 - x^2)(t + 3), \quad x(0) = a < -1$

3. Find the solution of the equation $x' = \frac{x+t}{t}$ satisfying the initial condition $x(2) = 1$ and defined for all $t > 0$. No integrals in the final answer.

4. Let $x(t)$ be a solution of the equation

$$x' = \frac{tx(x^2 + t^2)}{x^4 + 2t^4}$$

satisfying the initial condition $x(2) = 1$ and defined on some interval $t \in (\alpha, \beta)$ which does not contain the point $t = 0$. Give a formula relating $x(t)$ and t. Integrals are OK.

5. Let $x(t)$ be a solution of the equation

$$x' = sin\,(2x + 3t - 1) + 2$$

satisfying the initial condition $x(0) = 0$ and defined on some interval $t \in (\alpha, \beta)$. Give a formula relating $x(t)$ and t. Integrals are OK.

6. Let $x(t)$ be a solution of the equation

$$x' = \sqrt{x} + \frac{t}{x}$$

(Bernoulli equation) satisfying the initial condition $x(1) = 1$ and defined on some interval $t \in (\alpha, \beta)$. Give an explicit formula for $x(t) : x(t) = ...$ Integrals are OK.