HOMEWORK 1

In all problems below, a function should be treated as a Taylor series at $0 \in \mathbb{R}^n$. You should understand the first lecture: the method of normalization by the principal part and the proof of Morse theorem using this method.

The notation h.o.t. means higher order terms and is used as follows. If $P(x)$ is a polynomial of n variables $x_1, ..., x_n$ of degree d then $P(x)+h.o.t.$ means a function of the form $P(x)$ plus terms of order $> d$, i.e. a function of the form $P(x)+\sum_{|(\alpha)|>d} c_\alpha x^\alpha$. Here $(\alpha) = (\alpha_1, ..., \alpha_n)$, $|(\alpha)| = \alpha_1 + \cdots + \alpha_n$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and c_α are any numbers.

Problem 1. Prove that any function of two variables of the form $x^3 + y^3 + h.o.t.$ can be reduced by a change of coordinates to $x^3 + y^3$.

Problem 2. Find a normal form, as simple as possible, to which any function of two variables of the form $x^m+y^m+h.o.t.$ can be reduced by a change of coordinates. Consider at first the cases $m = 3, m = 4, m = 5$, after this any integer m.

Problem 3. Find a normal form, as simple as possible, to which any function of two variables of the form (a) $x^2y+h.o.t.$ (b) $x^3+h.o.t.$ can be reduced by a change of coordinates.

Problem 4. Prove that any function of $k+s$ variables $x_1, ..., x_k, y_1, ..., y_s$ of the form $x_1^2 \pm x_2^2 \pm \cdots \pm x_k^2 + h.o.t.$ can be reduced by a change of coordinates to the form $x_1^2 \pm x_2^2 \pm \cdots \pm x_k^2 + f(y_1, ..., y_s)$, with some function f.

Problem 5. Find a normal form, as simple as possible, to which any function of three variables of the form (a) $x^3 + y^3 + z^3 + h.o.t.$ can be reduced by a change of coordinates.

Problem 6. Prove that a function of one variable of the form $x^k + h.o.t.$ can be reduced by a change of coordinates to x^k.

Problem 7. Think about a general theorem on the normalization by the principal part, i.e. about the simplest normal form to which any function of n variables $x = (x_1, ..., x_n)$ of the form $P^{(m)}(x) + h.o.t.$ can be reduced by a change of coordinates. Here $P^{(m)}(x)$ is a homogeneous degree $m \geq 2$ polynomial. The theorem should be formulated in terms of the linear operators $L_i : (H^{(i+1)})^n \to H^{(i+m)}$, where $H^{i)}$ is the space of homogeneous degree i polynomials, and

$$L_i(\phi_1, ..., \phi_n) = \sum_{j=1}^{n} \frac{\partial P^{(m)}(x)}{\partial x_j} \cdot \phi_i(x), \qquad \phi_1(x), ..., \phi_n(x) \in H^{(i+1)}.$$