ODEs - 104285. Semester: Spring. Year: 2011

HW-1. Deadline: Monday, March 14, 6 pm

1. Use the uniqueness theorem to prove that any solution of the equation $x'(t) = x(t)$ defined for all $t \in \mathbb{R}$ has the form $x(t) = Ce^t$.

2. In the lecture we checked that the equation $x'(t) = \sqrt{|x(t)|}$ has two solutions satisfying the initial condition $x(0) = 0$: the solution $x(t) \equiv 0$ and the solution $\widetilde{x}(t) = \frac{t^2}{4}$ as $t \geq 0$, $\widetilde{x}(t) = 0$ as $t \leq 0$.
How many there are other solutions of the same equation satisfying the same initial condition? To answer, consider the functions
$\widehat{x}(t) = \frac{t^2}{4}$ as $t \geq 0$, $\widehat{x}(t) = -\frac{t^2}{4}$ as $t \leq 0$;
$\widetilde{x}_r(t) = \frac{(t-r)^2}{4}$ as $t \geq r$, $\widetilde{x}_r(t) = 0$ as $t \leq r$,
where $r \in \mathbb{R}$ is a parameter. Which of the functions $\widehat{x}(t), \widetilde{x}_r(t)$ satisfy the same equation and the same initial condition? Draw the graphs of these functions and explain why there is no contradiction to the uniqueness theorem.

3.(*) (Related to problem 2). Let $y(t)$ be one of solutions of the equation $y'(t) = \sqrt{|y(t)|}$ satisfying the initial condition $y(1) = 1/4$ and defined for all $t \in \mathbb{R}$. Is it true that $y(t) = t^2/4$ for all $t \geq 0$? If yes - prove it, if no - give a counterexample.

4. Let $f(t, x)$ be a function of two variables satisfying the requirements in the existence and uniqueness theorem in the open set $U = \{(t, x) : t < 1.01\}$. Let $x(t)$ and $\widetilde{x}(t)$ be solutions of the same equation $x'(t) = f(t, x(t))$ defined on the interval $t \in (-\infty, 1)$. Assume that $\lim_{t\to 1} x(t) = \lim_{t\to 1} \widetilde{x}(t) = B$, where B is a finite number (nor $\pm\infty$). Prove that $x(t)$ and $\widetilde{x}(t)$ is the same function.

5. Given a real $r > 0$ find an autonomous ordinary differential equation which has solutions $x(t) = \frac{1}{C-rt}$ with any $C \in \mathbb{R}$.

6. (Related to problem 5). Assume that the population of cats near your house changes in time according to the equation $x'(t) = 0.01x^2(t)$ where $x(t)$ is the number of cats at time t measured in years. Assume that today there are 10 cats near your house. How many cats there will be near your house in 9 years? In 9 years and 10 months? In 9 years and 11 months? In 10 years?