ODEs - 104285. Semester: Spring. Year: 2011

HW-7. Deadline: Monday, June 6, 2 pm

1. Find

$$(a)\ e^{\begin{pmatrix}2 & 1\\ 0 & 3\end{pmatrix}}, \quad (b)\ e^{\begin{pmatrix}-1 & -1\\ 1 & -1\end{pmatrix}}, \quad (c).\ e^{\begin{pmatrix}0 & 1\\ -4 & 4\end{pmatrix}}.$$

Solving this you should use the following way of finding e^A, where A is an $n\times n$ matrix. Consider the system $\dot{x} = Ax$. Fix $i \in 1,\dots,n$. You know how to find the solution $x(t)$ of the system satisfying the initial condition $x(0) =$ vector whose i-th component is equal to 1 and all other component are equal to 0. On the other hand, you know that $x(t) = e^{At}x(0)$. It follows that the vector $x(1)$ is exactly the i-th column of the matrix e^A.

2. Find $a \in \mathbb{R}$ such that the system

$$\dot{x}_1 = 2x_1 + x_2 + ax_3,\ \dot{x}_2 = x_1 + 2x_2 + 3x_3,\ \dot{x}_3 = 2x_2 + x_3$$

has a constant non-zero solution (that is, solution of the form $x_1(t) \equiv C_1, x_2(t) \equiv C_2, x_3(t) \equiv C_3$, where C_1, C_2, C_3 are constants, $(C_1, C_2, C_3) \neq (0,0,0)$). Fir this a, find C_1, C_2, C_3.

3. Give an example of a 2×2 real matrix A such that the system $\dot{x} = Ax$ has a solution with the first coordinate $x_1(t) = e^{-3t}\left(7sin(\sqrt{2}t) + 8cos(\sqrt{2}t)\right)$. Find the second coordinate of the solution.

4. (No complex numbers in the answers). A 3×3 real matrix A has eigenvector $\begin{pmatrix}4-2i\\ 1\\ 2+i\end{pmatrix}$ corresponding to the eigenvalue $(7-5i)$ and eigenvector $\begin{pmatrix}1\\ 1\\ 2\end{pmatrix}$ corresponding to the eigenvalue (-3).

4.1. Find a basis of all real solutions of the system $\dot{x} = Ax$

4.2. Under which condition on real numbers $\alpha_1, \alpha_2, \alpha_3$ the solution of the system $\dot{x} = Ax$ tends to $0 \in \mathbb{R}^3$ as $t \to \infty$?

4.3. Find the solution of the system $\dot{x} = Ax$ satisfying the initial condition $x(0) = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$.

5. A 5×5 real matrix A has eigenvector $\begin{pmatrix}4-2i\\ 1\\ 2+i\\ 1\\ i\end{pmatrix}$ corresponding to the eigenvalue $(-2-3i)$, eigenvector $\begin{pmatrix}1\\ 2\\ 0\\ 0\\ 1\end{pmatrix}$ corresponding to the eigenvalue 2, and eigenvector

$\begin{pmatrix} 3-i \\ 2 \\ i \\ 2+i \\ 0 \end{pmatrix}$ corresponding to the eigenvalue $(8i)$. Under which condition on real numbers $\alpha_1, \alpha_2, \alpha_3$ the solution of the system $\dot{x} = Ax$ satisfying the initial conditions $x(0) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ 0 \\ 0 \end{pmatrix}$ tends to $0 \in \mathbb{R}^5$ as $t \to \infty$?

6. Find a basis of the space of all real solutions of the system $x' = Ax$, where

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

(A has the only eigenvalue 2)