ODEs - 104285. Semester: Spring. Year: 2011

HW-8. You should do it by June 16-17

1. A real 3×3 matrix A has eigenvector $\begin{pmatrix} 5-2i \\ 4-3i \\ 1+i \end{pmatrix}$ corresponding the the eigenvalue $3i$ and eigenvector $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$ corresponding to eigenvalue (-2). Let $x_a(t)$ be the solution of the system $x' = Ax$ satisfying the initial condition $x(0) = \begin{pmatrix} 0 \\ a \\ 1 \end{pmatrix}$.
Find all $a \in \mathbb{R}$ such that $x_a(t)$ is a periodic function. What is its period?
Find all $a \in \mathbb{R}$ such that $x_a(t) \to 0$ as $t \to \infty$.

2. Consider the system

$$x_1' = 2ax_1 + x_2, \quad x_2' = a^2 x_1 + (a-1)x_2.$$

For which $a \in \mathbb{R}$

the equilibrium point $0 \in \mathbb{R}^2$ is asymptotically stable?

the phase portrait is a saddle? stable node? unstable node? stable focus? unstable focus? center?

For each of the phase portraits above take any example of a such that the system has this phase portrait (if a exists) and draw the phase portrait for the chosen a.

3. A function $F(x_1, x_2)$ is called the first integral of a system $x' = Ax$, where A is a 2×2 matrix, if $F \not\equiv 0$ and $F(x_1(t), x_2(t)) = const$ for any solution $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$. If the phase portrait is a center then there exists a first integral of the form $F = Ax_1^2 + Bx_1x_2 + Cx_2^2$ (such that each of the phase curves $F = const$ is an ellipse). Find a first integral for the system $x' = Ax$ where $A = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$ (the eigenvalues are $\pm 2i$). The simplest way is to solve this problem is to express the condition $\frac{d}{dt}F(x_1(t), x_2(t)) \equiv 0$ as a system of linear equations for A, B, C.

4. For which real numbers $a, b \neq 0$ the system

$$x_1' = ln(x_1 x_2), \quad x_2' = ax_1 - b$$

has an asymptotically stable equilibrium point?

5. Give an example of a function $f(x_1) \in C^\infty(\mathbb{R})$ such that the system

$$x_1' = x_2, \quad x_2' = f(x_1) + a \cdot sin(x_2)$$

has exactly 3 asymptotically stable equilibrium points and exactly 2 equilibrium points which are not asymptotically stable.