1. First order ODEs. Simplest examples. Existence and uniqueness theorem

Example 1.1. Consider the equation

$$x'(t) = 1 \tag{1.1}$$

This is an *equation* because there is an unknown: the function $x(t)$. This is a *differential equation of order* 1 because it involves the first derivative of the unknown function and does not involve higher order derivatives. This is an *ordinary* differential equation since the unknown function depends on one (independent) variable only – the variable t.

Of course we may denote the unknown function as we like, as well as the independent variable. The notation t corresponds to the physical interpretation of most ODE's when the independent variable is the time. For example, equation (1.1) means, in physical language, that a body moves along the straight line, the x-axes, with the constant velocity 1.

A solution of any differential equation is a function satisfying this equation and defined on an *open* interval. Any single solution is called partial solution. The set of all solutions is called general solution.

For example, equation (1.1) has partial solutions $x(t) = t,\ x(t) = t - 2,\ x(t) = t+3$. The general solution of this equation is, as it easy to prove, $x(t) = t+C,\ C \in \mathbb{R}$.

The physical interpretation of infinitely many solutions of (1.1): different solutions correspond do different initial position of the body. If we know the coordinate of the body at any fixed time-moment t_0 then we will know its position at any time t. Mathematically the initial position means the *initial condition*

$$x(t_0) = x_0, \qquad t_0, x_0 \in \mathbb{R}. \tag{1.2}$$

Equation (1.1) has *unique* solution defined for all t and satisfying (1.2): $x(t) = x_0 + t - t_0$.

Example 1.2. Consider now the equation

$$x'(t) = k \cdot t \tag{1.3}$$

which means, in physical language, that the velocity of the body is proportional to the time t. Like for (1.1) there are infinitely many solutions because the initial condition (1.1) (physically: the initial position of the body) is not fixed. The general solution is $x(t) = kt^2/2 + C,\ C \in \mathbb{R}$. There is unique solution defined for all t and satisfying (1.2): $x(t) = k(t^2/2 - t_0^2/2) + x_0$.

Example 1.3. Equation of the form

$$x'(t) = f(t) \in C^0(\mathbb{R}) \tag{1.4}$$

generalizes examples (1.1), (1.3). The notation $C^0(\mathbb{R})$ is used for the class of continuous functions defined on the whole $\mathbb{R}$. Fix any function $F(t)$ such that $F'(t) = f(t)$. Then the general solution of (1.4) can be written in the form $x(t) = F(t) + C,\ C \in \mathbb{R}$. Like in previous examples, there is unique solution defined for all t and satisfying the initial conditions (1.2). To present this solution it is convenient to take $F(t) = \int_{t_0}^t f(s)ds$, i.e. to write down the general solution in the form $x(t) = \int_{t_0}^{t} f(s)ds + C,\ C \in \mathbb{R}$. Then it is clear that the solution satisfying (1.2) has the form $x(t) = x_0 + \int_{t_0}^{t} f(s)ds$.

Consider now another class of first order ODE's

$$x'(t) = f(x(t)), \quad f(x) \in C^0(\mathbb{R}). \tag{1.5}$$

Note that (1.4) and (1.5) are principally different. Physically (1.5) means that the velocity of the body is determined by its position. It depends on time t, but via the coordinate $x(t)$ only. If the body is located at the point $x = 5$ then its velocity is $f(5)$ independently on *when* the body is at the point $x = 5$.

Example 1.4. The simplest example of an equation of the class (1.5) is

$$x'(t) = kx(t) \tag{1.6}$$

One of solutions is $x(t) \equiv 0$. Another solution is $x(t) = e^{kt}$. Note that $e^{kt} + C$ is NOT a solution unless $C = 0$. The general solution is $Ce^{kt},\ C \in \mathbb{R}$. Substituting the initial condition (1.2) we find a solution satisfying this initial condition and defined for all t: $x(t) = x_0 e^{k(t-t_0)}$. One can prove that this there are no other solutions satisfying (1.2) and defined for all t.

Example 1.5. Our next example is another equation of the class (1.5):

$$x'(t) = x^2(t) \tag{1.7}$$

along with the initial condition

$$x(0) = x_0. \tag{1.8}$$

Theorem 1.6. *If $x_0 \neq 0$ then (1.7) does not have a solutions defined for all t and satisfying (1.8).*

Note that if $x_0 = 0$ then such a solution exists: $x(t) \equiv 0$.

I will prove below Theorem 1.6 for the case $x_0 > 0$. Moreover, I will prove that in this case there is no solution defined on the interval $(0, \infty)$.

Assume, to get contradiction, that $x_0 > 0$ and $x(t)$ is a solution of (1.7) defined for $t \in (0, \infty)$. The equation implies that $x'(t) \geq 0$. Therefore $x(t) \geq x_0$. Since $x_0 > 0$ then the equation implies $x'(t) \geq x_0^2$. (Note that if $x_0 < 0$ then this is not so). It follows that $x(t)$ is a strictly increasing function. It also follows that $x(t) \to \infty$ as $t \to \infty$. [1] Therefore the *inverse function* $t(x)$ is well defined on the interval (x_0, ∞) and one has

$$t(x_0) = 0, \quad t(x) \to \infty \text{ as } x \to \infty.$$

By the theorem on the derivative of the inverse function one has $t'(s) = 1/s^2$ and consequently

$$t(x) - t(x_0) = t(x) = \int_{x_0}^{x} t'(s)ds = \int_{x_0}^{x} \frac{ds}{s^2} = -1/x + 1/x_0$$

for any $x > x_0$. Taking the limit as $x \to \infty$ we get contradiction.

[1] In fact, $x(t) = \int_0^t x'(s)ds \geq \int_0^t x_0^2 ds = tx_0^2 \to \infty$ as $t \to \infty$.

Exercises 1.7 (part of them - for the first TIRGUL)**.** .
1. Prove that if $x_0 > 0$ then (1.7) has no solution satisfying (1.8) and defined for $t \in (0, b)$, where $b > 1/x_0$.
2. Prove Th. 1.6 for the case $x_0 < 0$ and obtain an analogous of statement 1.
3. Prove that Theorem 1.6 holds for the equation $x'(t) = x^N(t)$ with any integer $N \geq 2$.
4. Use the inverse function to find solution of the equation $x'(t) = x^N(t)$ satisfying the initial condition $x(t_0) = x_0 \neq 0$ and defined on maximally possible interval.

Exercise 1.8. Prove that the equation $x'(t) = c \cdot (x^2(t) + 1),\ c > 0$ does not have any solution defined on an interval of length $> \pi/c$.

Example 1.9. Consider the first order ODE

$$x'(t) = \sqrt{|x(t)|} \tag{1.9}$$

and the initial condition

$$x(0) = 0. \tag{1.10}$$

The function $x(t) \equiv 0$ is a solution defined for all t and satisfying (1.10). Such a solution is not unique. Let us show that the function

$$x(t) = 0 \text{ as } t \leq 0, \quad x(t) = at^s \text{ as } t \geq 0 \tag{1.11}$$

with suitable $a \neq 0$ and $s > 1$ is also a solution of (1.9) satisfying (1.10). At first note that if $s > 1$ then the function (1.11) belongs to the class $C^1(\mathbb{R})$, i.e. defined for all t, differentiable, and has continuous derivative. It is clear that this function satisfies (1.9) for any $t \leq 0$. Therefore if $s > 1$ then (1.11) is a solution of (1.9) if and only if $ast^{s-1} = \sqrt{|at^s|}$ for all $t \geq 0$. The latter holds either if $a = 0$ or if $s - 1 = s/2, a > 0$ and $as = \sqrt{a}$, i.e. $s = 2,\ a = 1/4$. We obtain that equation (1.9) has at least two solutions defined for all t and satisfying (1.10): the zero solution $x(t) \equiv 0$ and solution (1.11) with $s = 2, a = 1/4$. In fact, equation (1.9) has *infinitely many* solutions satisfying (1.10) – see Exercise 1.12.

Exercise 1.10. Prove the following

Lemma 1.11. *If $x(t)$ is a solution of an equation $x'(t) = f(x(t))$ defined for all t then for any a the function $x_a(t) = x(t+a)$ is also a solution of the same equation.*

Explain why this statement is wrong for equations of the form $x'(t) = f(t, x(t))$.

Exercise 1.12. Use Lemma 1.11 to construct infinitely many solutions of equation (1.9) satisfying (1.10).

Exercise 1.13. Let $0 < \alpha < 1$. Construct infinitely many solutions of the equation $x'(t) = |x(t)|^\alpha$ satisfying the initial condition $x(0) = 0$. Explain why the construction does not work if $\alpha \geq 1$.

The general form of the first order ODE is

$$x'(t) = f(t, x(t)), \quad f : U \to \mathbb{R}. \tag{1.12}$$

where f is a function of two variables, t and x, defined on some *domain* $U \subset \times\mathbb{R}$ of the (t, x)-plane. [2]. In the examples given above $U = \mathbb{R} \times \mathbb{R}$; in general this is not required. For equations (1.4) the function f depends on t only. For equations (1.5)

[2] a domain is an **open connected** set

it depends on x only. The equation $x'(t) = x(t) + t$ is an example of an equation which belongs neither to class (1.4) nor to (1.5). A solution of (1.12) is a function satisfying this equation and defined on an *open* interval of the t-axes. The *graph* of any solution belongs to the open set U – the domain on which the function $f(t, x)$ is defined.

Fix the initial condition

$$x(t_0) = x_0, \quad (t_0, x_0) \in U. \tag{1.13}$$

Note that the initial condition (1.13) means that the graph contains the point (t_0, x_0).

Question A Is it true that equation (1.12) has a solution satisfying (1.13)?

Note that if $x(t)$ is a solution defined on an interval (a, b) satisfying (1.13) then there are infinitely many solutions satisfying (1.13) – the restrictions of $x(t)$ to any sub-interval of (a, b) which contains, like (a, b), the point t_0. Therefore the right question about the uniqueness of solution of (1.12) satisfying (1.13) is as follows:

Question B Let $x(t)$ and $\tilde{x}(t)$ be solutions of the same equation (1.12), satisfying the same initial condition (1.13) and defined on the open intervals I and $\tilde{I}$ respectively. Is it true that $x(t) = \tilde{x}(t)$ for any $t \in I \cap \tilde{I}$?

The following theorem gives sufficient conditions on the function $f(t, x)$ for the positive answer to Questions A and B.

Theorem 1.14 (Existence and uniqueness theorem for first order ODEs)**.** *Assume that at any point of U the function $f(t, x)$ is*

(a) continuous with respect to (t, x)

(b) differentiable with respect to x, and the derivative $\frac{\partial f(t,x)}{\partial x}$ is continuous.

Then for any point $(t_0, x_0) \in U$ the answers to Questions A and B are positive.

This theorem is a part of a much more general existence and uniqueness theorem for *systems of ODEs of any order* which will be proved in the end of this course. Example 1.9 (and exercises after it) show that *condition (b) cannot be taken away.* But this condition can be weakened – we will see how when proving the general existence and uniqueness theorem.

Exercise 1.15. Consider the equation $x'(t) = (x(t) - 1 + t)^{1/3} \cdot g(t, x)$, where $g(t, x)$ is a function of the class C^∞ (as a function of two variables). In each of the figures 1.a – 1.d there are graphs of two solutions of this equation. Which of these pictures are impossible?

Theorem 1.14 says nothing on the interval on which the solution is defined. The maximal possible interval on which the a solution is defined might be very small even if $U = \mathbb{R} \times \mathbb{R}$, see Exercise 1.8. On the other hand, using Theorem 1.14 we can understand the only reason why a solution defined an some interval (a, b) cannot be prolonged to a bigger interval.

Definition 1.16. Let $x(t)$ and $\tilde{x}(t)$ be solutions of the same ODE such that $x(t)$ is defined on an interval I and $\tilde{x}(t)$ is defined on an interval $\tilde{I}$ such that $I \subset \tilde{I}$. We will say that the solution $\tilde{x}(t)$ is a prolongation of the solution $x(t)$ from I to $\tilde{I}$; we will also say that the solution $x(t)$ can be prolonged from I to $\tilde{I}$.

Theorem 1.17 (prolongation of solutions)**.** *Assume that the function $f(t,x)$ satisfies the assumptions of Theorem 1.14 with $U = \mathbb{R} \times \mathbb{R}$. Let $x(t)$ be a solution of equation (1.12) defined on an interval (a,b).*

1. If $b < \infty$ and the function $x(t)$ is bounded in a neighborhood of the point b [3] *then*

(a) there exists a finite limit $\lim_{t\to b} x(t) = B$

(b) the solution $x(t)$ can be prolonged from (a,b) to $(a, b+\epsilon)$ for some $\epsilon > 0$.

2. Similarly, if $a > -\infty$ and the function $x(t)$ is bounded in a neighborhood of the point a then

(a) there exists a finite limit $\lim_{t\to a} x(t) = A$ the solution $x(t)$

(b) the solution $x(t)$ can be prolonged from (a,b) to $(a-\epsilon, b)$ for some $\epsilon > 0$.

Proof. I will prove the first statement (the second statement is similar).

At first let us show that $(a) \implies (b)$. This follows from the existence part of Theorem 1.14. By this theorem equation (1.12) has a solution $\widehat{x}(t)$ defined on an interval $(b-\epsilon, b+\epsilon), \epsilon > 0$, and satisfying the condition $\widehat{x}(b) = B$. Define

$$\tilde{x}(t) = x(t) \text{ as } t \in (a,b); \quad \tilde{x}(t) = \widehat{x}(t) \text{ as } t \in [b, b+\epsilon).$$

The function $\tilde{x}(t)$ is defined on the interval $(a, b+\epsilon)$. It is clear that it satisfies equation (1.12) for any $t \neq b$. Since $x(t)$ and $\widehat{x}(t)$ are solutions of (1.12) and the function $f(t,x)$ is continuous then $x'(t) \to f(b,B)$ as $t \to b$ (from the left). Note that $\widehat{x}'(b) = f(b,B)$. It follows that the function $\tilde{x}(t)$ is differentiable at the point b and satisfies (1.12) at the point b. In other words $\tilde{x}(t)$ is solution of (1.12) which is a prolongation of the solution $x(t)$ from (a,b) to $(a, b+\epsilon)$. We have proved that $(a) \implies (b)$.

Now let us prove (a). Assume, to get contradiction, that $x(t)$ does not have the limit as $t \to b$. Since, by the assumption of Theorem 1.17, the function $x(t)$ is bounded in a neighborhood of b then the absence of the limit as $t \to b$ implies the existence of two sequences of points $t'_k \to b, t''_k \to b$, each tends to b, such that the sequences $x(t'_k)$ and $x(t''_k)$ have *different* limits L', L'' respectively. Let $L' < L''$. Since the function $x(t)$ is continuous we obtain the following property:

(P) for any $\epsilon > 0$ and any $r \in (L', L'')$ there exists $t \in (b-\epsilon, b)$ such that $x(t) = r$.

Now we fix an arbitrary $r^* \in (L', L'')$ and use the existence part of Theorem 1.14: there exists a solution $x^*(t)$ of (1.12) defined on an interval $(b-\epsilon, b+\epsilon), \epsilon > 0$ and satisfying the condition $x^*(b) = r^*$. The property **(P)** implies that the graphs of solutions $x(t)$ and $x^*(t)$ intersect at some point. By the uniqueness part of Theorem Theorem 1.14 the solutions $x(t)$ and $x^*(t)$ coincide on the interval $(b-\epsilon, b)$. Therefore $x(t)$ has a limit as $t \to b$, it is equal to $x^*(b) = r^*$. Since r^* is an arbitrary point of the interval (L', L'') then we get contradiction.

□

Corollary 1.18. *Assume that the function $f(t,x)$ satisfies the assumptions of Theorem 1.14 with $U = \mathbb{R} \times \mathbb{R}$. Let $x(t)$ be a solution of equation (1.12) satisfying a fixed initial condition $x(t_0) = x_0$ and defined on maximally possible interval (a,b). (By Theorem 1.14 such a solution is unique). If $b < \infty$ then $x(t)$ is not bounded*

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[3] i.e. there exists M and $\delta > 0$ such that $|x(t)| < M$ for any $t \in (b-\delta, b)$

in a neighbourhood of the point b. If $a > -\infty$ then $x(t)$ is not bounded in a neighbourhood of the point a.

Exercise 1.19 (for those who understood perfectly the proof of Theorem 1.17)**.** Try to formulate and prove a theorem on prolongation of solutions for the case that $f(t,x)$ satisfies the assumptions of Theorem 1.14 in a domain $U \subset \mathbb{R} \times \mathbb{R}$.