1. First order ODEs. Simplest examples.
Existence and uniqueness theorem.
Theorem on prolongation of solutions.

2. Equation of the form $x'(t) = F(x(t)), \quad F(x) \in C^1(\mathbb{R})$

See Lecture Notes, Lectures 1,2.

3. Second order ODEs. Existence and uniqueness theorem.
Theorem on prolongation of solutions.

The general form of a second order ODE is

$$x'' = G(t, x, x'), \quad G : \Omega \to \mathbb{R}, \tag{3.1}$$

where $x = x(t)$ is the unknown function and G is a function of **three** variables. Like in the case of first order ODES the the function G is defined on a domain $\Omega \subset \mathbb{R}^3$; it is possible but not necessary that $\Omega = \mathbb{R}^3$.

A solution of (3.1) is any function defined on an open interval $t \in (a, b)$ and satisfying this equation for any t of the interval. The latter requires that the *extended graph* of $x(t)$ belongs to the domain Ω. The *extended graph* of a solution $x(t), t \in (a, b)$ is, by definition, the curve $\{t, x(t), x'(t)\}_{t \in (a,b)} \subset \mathbb{R}^3$.

The physical meaning of (3.1) is as follows: a body moves along a straight line (the x-axes), its acceleration (Hebrew: teutsa) at time moment t depends on the time moment, the location of the body at this time-moment, and the velocity of the body at the same time-moment.

It is clear that (3.1) has infinitely many solution; the solution depends on the initial position of the body and its initial velocity. One can expect that if the initial position and the initial velocity are fixed then the solution is unique. Mathematically, the initial position and the initial velocity means the conditions

$$x(t_0) = x_0, \quad x'(t_0) = v_0, \quad (t_0, x_0, v_0) \in \Omega. \tag{3.2}$$

These conditions are called the initial conditions.

Theorem 3.1 (Existence and uniqueness theorem)**.** *Assume that the function $G : \Omega \to \mathbb{R}$ is continuous and has continuous first order derivatives with respect to the second and the third argument (corresponding to x and x').*

1. For any point $(t_0, x_0, v_0) \in \Omega$ equation (3.1) has a solution $x(t)$ satisfying (3.2) and defined on some time-interval (a, b) containing t_0, satisfying (3.2).

2. For a point $(t_0, x_0, v_0) \in \Omega$. Let $x(t)$ and $\tilde{x}(t)$ be two solutions of the same equation (3.1) satisfying the same condition (3.2) and defined on intervals I and $\tilde{I}$. Then $x(t) = \tilde{x}(t)$ for any point $t \in I \cap \tilde{I}$.

Remarks.

1. Like in the case of first order ODEs the existence part of the theorem does not give any information on the maximal possible length of the interval (a, b).

2. In some cases the function G satisfies the assumption of Theorem 3.1 not at all points of the domain Ω but at points of some sub-domain $\Omega' \subset \Omega$. In this case Theorem 3.1 remains true if $(t_0, x_0, v_0) \in \Omega'$ and if the *extended graphs* of the

solutions $x(t)$ and $\tilde{x}(t)$ (see the definition above) in the second statement belong to Ω'.

3. Theorem 3.1 and a similar theorem for the first order ODES (see Lecture 1) are a part of a much more general existence and uniqueness theorem for *systems of ODEs of any order* which will be explained and proved in the end of this course. Discussing the general existence and uniqueness theorem (probably you already guess what is it) we will see how the assumption on the function G can be weakened.

Important. Note that the geometric interpretation of the uniqueness part of Theorem 3.1 and the theorem on the same topic for the first order ODEs are different. In the case of first order ODEs the graphs of two solutions of the same equations cannot intersect, unless they coincide or one of the graphs is a part of the other. In the case of second order ODEs the intersection is possible. What is impossible is the *tangency* of two graphs of the same equation. In fact, two solutions satisfy the same initial condition (3.2) if and only if their (usual) graphs not only intersect but also are tangent at the point (t_0, x_0).

For second order ODEs (and for kth order ODEs) there is an analogous of the theorem on prolongation of solutions (see Lecture 1). I will restrict myself to the simple part of this theorem, which is an almost immediate corollary of the existence part of Theorem 3.1.

Theorem 3.2. *Let $x(t)$ be a solution of (3.1) defined on an interval $t \in (a, b)$. Assume that there exist $\lim_{t\to b} x(t) = B_1,\ \lim_{t\to b} x'(t) = B_2$, where B_1 and B_2 are finite numbers. If $\Omega \neq \mathbb{R}^3$ we also require that the point (t_0, B_1, B_2) belongs to the domain Ω. Then the solution $x(t)$ can be prolonged to the interval $(a, b + \epsilon)$, where $\epsilon > 0$, i.e. there exists a solution defined on the interval $(a, b + \epsilon)$ which coincides with $x(t)$ at points $t \in (a, b)$.*

Similarly, if there exist $\lim_{t\to a} x(t) = A_1,\ \lim_{t\to a} x'(t) = A_2$, where A_1 and A_2 are finite numbers such that $(a, A_1, A_2) \in \Omega$ then the solution $x(t)$ can be prolonged to the interval $(a - \epsilon, b)$, where $\epsilon > 0$.

Proof. Let us prove the first statement (prolongation from (a, b) to $(a, b+\epsilon)$). The point (b, B_1, B_2) belongs to the domain Ω, therefore we can consider the initial conditions $x(b) = B_1, x'(b) = B_2$. The existence part of Theorem 3.1 states that the equation has a solution $\tilde{x}(t)$ satisfying these conditions and defined on interval $(b - \epsilon, b + \epsilon)$, $\epsilon > 0$. Now the two solutions, $x(t)$ and $\tilde{x}(t)$ can be "pasted" together as follows:

$\hat{x}(t) = x(t)$ as $t \in (a, b)$,

$\hat{x}(t) = \tilde{x}(t)$ as $t \in [b, b + \epsilon)$.

The function $\hat{x}(t)$ is defined on the interval $(a, b+\epsilon)$ and it is clear that it satisfies equation (3.1) at any point of this interval excerpt the point b. The fact that $x(t)$ and $\tilde{x}(t)$ are solutions of this equation easily implies that the function $\hat{x}(t)$ has the second derivative at the point b and satisfies equation (3.1) at this point, i.e. all is OK at the point b too. You can easily prove this yourself or repeat the proof from the implication (a) $\Longrightarrow$ (b) in Th. 1.17, Lecture 1. $\square$