LECTURE A

First order ODEs.
Simplest examples.
Existence and uniqueness theorem.
Theorem on prolongation of solutions.

Example 1. Consider the equation

$$x'(t) = 1 \tag{1}$$

This is an *equation* because there is an unknown: the function $x(t)$. This is a *differential equation of order* 1 because it involves the first derivative of the unknown function and does not involve higher order derivatives. This is an *ordinary* differential equation since the unknown function depends on one (independent) variable only – the variable t.

Of course we may denote the unknown function as we like, as well as the independent variable. The notation t corresponds to the physical interpretation of most ODE's when the independent variable is the time. For example, equation (1) means, in physical language, that a body moves along the straight line, the x-axes, with the constant velocity 1.

A solution of any differential equation is a function satisfying this equation and defined on an *open* interval. Any single solution is called partial solution. The set of all solutions is called general solution.

For example, equation (1) has partial solutions

$$x(t) = t, \ x(t) = t - 2, \ x(t) = t + 3.$$

The general solution of this equation is, as it easy to prove,

$$x(t) = t + C, \ C \in \mathbb{R}.$$

The physical interpretation of infinitely many solutions of (1): different solutions correspond do different initial position of the body. If we know the coordinate of the body at any fixed time-moment t_0 then we will know its position at any time t. Mathematically the initial position means the *initial condition*

$$x(t_0) = x_0, \qquad t_0, x_0 \in \mathbb{R}. \tag{2}$$

Equation (1) has *unique* solution defined for all t and satisfying (2):

$$x(t) = x_0 + t - t_0.$$

Example 2. Consider now the equation

$$x'(t) = k \cdot t \tag{3}$$

which means, in physical language, that the velocity of the body is proportional to the time t. Like for (1) there are infinitely many solutions because the initial condition (1) (physically: the initial position of the body) is not fixed. The general solution is

$$x(t) = kt^2/2 + C, \ C \in \mathbb{R}.$$

There is unique solution defined for all t and satisfying (2):

$$x(t) = k(t^2/2 - t_0^2/2) + x_0.$$

Example 3. Equation of the form

$$x'(t) = f(t) \in C^0(\mathbb{R}) \tag{4}$$

generalizes equations (1), (3). The notation $C^0(\mathbb{R})$ is used for the class of continuous functions defined on the whole $\mathbb{R}$. Fix any function $F(t)$ such that $F'(t) = f(t)$. Then the general solution of (4) can be written in the form $x(t) = F(t)+C,\ C \in \mathbb{R}$. Like in previous examples, there is unique solution defined for all t and satisfying the initial conditions (2). To present this solution it is convenient to take $F(t) = \int_{t_0}^{t} f(s)ds$, i.e. to write down the general solution in the form

$$x(t) = \int_{t_0}^{t} f(s)ds + C,\ C \in \mathbb{R}.$$

Then it is clear that the solution satisfying (2) has the form

$$x(t) = x_0 + \int_{t_0}^{t} f(s)ds.$$

Consider now another class of first order ODE's

$$x'(t) = f(x(t)), \quad f(x) \in C^0(\mathbb{R}). \tag{5}$$

Note that (4) and (5) are principally different. Physically (5) means that the velocity of the body is determined by its position. It depends on time t, but via the coordinate $x(t)$ only. If the body is located at the point $x = 5$ then its velocity is $f(5)$ independently on *when* the body is at the point $x = 5$.

DEFINITION. Equations of form (5) are called autonomous.

Example 4. The simplest example of an autonomous first order ODE is

$$x'(t) = kx(t) \tag{6}$$

One of solutions is $x(t) \equiv 0$. Another solution is $x(t) = e^{kt}$. Note that $e^{kt} + C$ is NOT a solution unless $C = 0$. The general solution is $Ce^{kt},\ C \in \mathbb{R}$. Substituting the initial condition (2) we find a solution satisfying this initial condition and defined for all t: $x(t) = x_0 e^{k(t-t_0)}$. One can prove that this there are no other solutions satisfying (2) and defined for all t.

Our next example is another equation of the class (5):

$$x'(t) = x^2(t) \tag{7}$$

along with the initial condition

$$x(0) = x_0. \tag{8}$$

Lemma 1. *If $x_0 \neq 0$ then (7) does not have a solutions defined for all t and satisfying (8).*

Note that if $x_0 = 0$ then such a solution exists: $x(t) \equiv 0$.

I will prove Lemma 1 for the case $x_0 > 0$. Assume, to get contradiction, that $x(t)$ is a solution of (7) defined for all t. The equation implies that $x'(t) \geq 0$. Therefore

$$x(t) \geq x_0 \text{ as } t \geq 0.$$

Since, as I assumed, $x_0 > 0$ then the equation implies $x'(t) \geq x_0^2$. (Note that if $x_0 < 0$ then this is not so). It follows that for $t > 0$ the function $x(t)$ is strictly increasing. It also follows that $x(t) \to \infty$ as $t \to \infty$. [1] Therefore the *inverse function* $t(x)$ is well defined on the interval (x_0, ∞) and one has

$$t(x_0) = 0, \quad t(x) \to \infty \text{ as } x \to \infty.$$

By the theorem on the derivative of the inverse function one has $t'(s) = 1/s^2$ and consequently

$$t(x) - t(x_0) = t(x) = \int_{x_0}^{x} t'(s)ds = \int_{x_0}^{x} \frac{ds}{s^2} = -1/x + 1/x_0$$

for any $x > x_0$. Taking the limit as $x \to \infty$ we get contradiction.

Exercise 1 Prove Lemma 1 for the case $x_0 < 0$.

Exercise 2 Prove that Lemma 1 holds for equation $x' = x^N$ for any integer $N \geq 2$.

Exercise 3 Prove that the equation $x'(t) = c \cdot (x^2(t) + 1),\ c > 0$ does not have any solution defined on an interval of length $> \pi/c$.

The next example shows that there are simple ODEs with two solutions satisfying the same initial condition.

Example 5. Consider the first order ODE

$$x'(t) = \sqrt{|x(t)|} \tag{9}$$

and the initial condition

$$x(0) = 0. \tag{10}$$

The function $x(t) \equiv 0$ is a solution defined for all t and satisfying (10). Such a solution is not unique. It is easy to check that

$$\tilde{x}(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ t^2/4 & \text{if } t \geq 0 \end{cases}$$

is a function of the class $C^1(\mathbb{R})$ which is also a solution of (9). This solution $\tilde{x}(t)$ also satisfies the initial condition (10).

Exercise 4 Let $0 < \alpha < 1$. Consider the equation $x'(t) = |x(t)|^\alpha$. Prove that there exist c and β such that the function

$$\tilde{x}(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ ct^\beta & \text{if } t \geq 0 \end{cases}$$

is a solution of this equation. (It follows that there are at least two solutions satisfying the initial condition $x(0) = 0$). Explain where you used that $0 < \alpha < 1$.

The general form of the first order ODE is

$$x'(t) = f(t, x(t)), \quad f : U \to \mathbb{R}, \tag{11}$$

where f is a function of two variables, t and x, defined on some *domain* $U \subset \mathbb{R} \times \mathbb{R}$ of the (t, x)-plane. [2]. In the examples given above $U = \mathbb{R} \times \mathbb{R}$; in general this is not required. For equations (4) the function f depends on t only. For equations (5) it depends on x only. The equation $x'(t) = x(t) + t$ is an example of an equation which

[1] In fact, $x(t) = \int_0^t x'(s)ds \geq \int_0^t x_0^2 ds = tx_0^2 \to \infty$ as $t \to \infty$.

[2] a domain is an **open connected** set

belongs neither to class (4) nor to (5). A solution of (11) is a function satisfying this equation and defined on an *open* interval of the t-axes. The *graph* of any solution belongs to the open set U – the domain on which the function $f(t, x)$ is defined.

Fix the initial condition

$$x(t_0) = x_0, \quad (t_0, x_0) \in U. \tag{12}$$

Note that the initial condition (12) means that the graph contains the point (t_0, x_0).

Question A (on existence of a solution) Is it true that equation (11) has a solution satisfying (12)?

Before asking the question on the uniqueness of a solution note that if $x(t)$ is a solution defined on an interval (a, b) and satisfying (12) then we can construct infinitely many solutions satisfying the same initial condition (12): we can simply restrict $x(t)$ to any sub-interval of (a, b) which contains, like (a, b), the point t_0. Therefore the right question about the uniqueness of solution of (11) satisfying (12) is as follows:

Question B (on uniqueness of solution) Let $x(t)$ and $\tilde{x}(t)$ be solutions of the same equation (11), satisfying the same initial condition (12) and defined on the open intervals I and $\tilde{I}$ respectively. Is it true that $x(t) = \tilde{x}(t)$ for any $t \in I \cap \tilde{I}$?

The following theorem gives sufficient conditions on the function $f(t, x)$ for the positive answer to Questions A and B.

Theorem 1 (Existence and uniqueness theorem for first order ODEs). *Assume that at any point of U the function $f(t, x)$ is*

(a) continuous with respect to (t, x)

(b) differentiable with respect to x, and the derivative $\frac{\partial f(t,x)}{\partial x}$ is continuous.

*Then for any point $(t_0, x_0) \in U$ the answers to Questions A and B are positive.*h

This theorem is a part of a much more general existence and uniqueness theorem for *systems of ODEs of any order* which will be proved in the end of this course. Example 5 (and Exercise 4) show that *condition (b) cannot be taken away.* But this condition can be weakened – we will see how when proving the general existence and uniqueness theorem.

Exercise 5 Consider the equation $x'(t) = (2x^2(t) + t^2 - 1)^{2/3} \cdot g(t, x)$, where $g(t, x)$ is a function of the class C^∞ (as a function of two variables) defined for all t, x. Which domains $U \subset \mathbb{R} \times \mathbb{R}$ satisfy the requirements of Theorem 1?

Theorem 1 says nothing on the interval on which the solution is defined. Even in the case $U = \mathbb{R} \times \mathbb{R}$ the maximal possible interval on which solution is defined might be not the whole t-axes (see Lemma 1); moreover it might be very small (see Exercise 3). On the other hand, using Theorem 1 we can understand the reason why a solution defined an some interval (a, b) cannot be prolonged to a bigger interval.

Definition. Let $x(t)$ and $\tilde{x}(t)$ be solutions of the same ODE such that $x(t)$ is defined on an interval I and $\tilde{x}(t)$ is defined on an interval $\tilde{I}$ such that $I \subset \tilde{I}$. We will say that the solution $\tilde{x}(t)$ is a prolongation of the solution $x(t)$ from I to $\tilde{I}$; we will also say that the solution $x(t)$ can be prolonged from I to $\tilde{I}$.

Theorem 2 (prolongation of solutions) *Assume that the function $f(t, x)$ satisfies the assumptions of Theorem 1 with $U = \mathbb{R} \times \mathbb{R}$. Let $x(t)$ be a solution of equation (11) defined on an interval (a, b).*

1. If $b < \infty$ and there exists

$$\lim_{t \to b} x(t) = B, \ \ B \neq \pm\infty$$

then the solution $x(t)$ can be prolonged from (a, b) to $(a, b + \epsilon)$ for some $\epsilon > 0$.

2. If $a > -\infty$ and there exists

$$\lim_{t \to a} x(t) = A, \ \ A \neq \pm\infty$$

then the solution $x(t)$ can be prolonged from (a, b) to $(a - \epsilon, b)$ for some $\epsilon > 0$.

Proof. I will prove the first statement (the second one is similar). The existence part of Th. 1 implies that there exists a solution $\tilde{x}(t)$ (NOT $x(t)$) satisfying the initial condition $\tilde{x}(b) = B$ and defined on the interval $(b - \epsilon, b + \epsilon)$. Now we "paste together" the solutions $x(t)$ and $\tilde{x}(t)$. Namely, we construct the function

$$x^*(t) = \begin{cases} x(t) & \text{as} \ \ t \in (a, b) \\ \tilde{x}(t) & \text{as} \ \ t \in [b, b + \epsilon) \end{cases}$$

It is clear that this function satisfies the equation at any point of the interval $(a, b + \epsilon)$ except (logically possible) point $t = b$. Let us check that the function $x^*(t)$ is differentiable at point $t = b$ and satisfies the equation at this point. If $t \in (a, b)$ then $(x^*)'(t) = x'(t) = f(t, x(t))$ and we see that

$$\lim_{t \to b^-} (x^*)'(t) = \lim_{t \to b^-} f(t, x(t)) = f(b, B).$$

(Here $\lim_{t \to b^-}$ is the limit from the left). If $t \in (b, b + \epsilon)$ then $(x^*)'(t) = \tilde{x}'(t) = f(t, \tilde{x}(t))$ and we see that

$$\lim_{t \to b^+} (x^*)'(t) = \lim_{t \to b^+} f(t, \tilde{x}(t)) = f(b, B).$$

(Here $\lim_{t \to b^+}$ is the limit from the right). It follows that the function $x^*(t)$ is differentiable at the point $t = b$ and its derivative at this point is equal to $(x^*)'(b) = f(b, B)$. Since $B = x^*(b)$, the function x^* satisfies the equation at the point $t = b$ and consequently at any point $t \in (a, b + \epsilon)$. □

Theorem 3 (prolongation of solutions: the case that the domain U has a boundary) *Assume that the function $f(t, x)$ satisfies the assumptions of Theorem 1 with a domain $U \subset \mathbb{R} \times \mathbb{R}$ having a boundary. Let $x(t)$ be a solution of equation (11) defined on an interval (a, b).*

1. Assume that:

$b < \infty$;

there exists $\lim_{t \to b} x(t) = B$;

the point (b, B) does not belong to the boundary of the domain U.

Then the solution $x(t)$ can be prolonged from (a, b) to $(a, b + \epsilon)$ for some $\epsilon > 0$.

2. Assume that:

$a > \ -\infty$;

there exists $\lim_{t \to a} x(t) = A$;

the point (a, A) *does not belong to the boundary of the domain* U.

Then the solution $x(t)$ *can be prolonged from* (a, b) *to* $(a - \epsilon, b)$ *for some* $\epsilon > 0$.

The proof of this theorem is almost the same as the proof of Theorem 2.

Example 5. Let us illustrate Theorems 1 and 2 returning to the example $x' = x^2$. Let $x(t)$ be solution of this equation satisfying the initial condition $x(0) = 1$ and defined on maximal possible interval (t^-, t^+). We proved (Lemma 1) that $t^+ < \infty$. We also proved that $x(t)$ is an increasing function as $t \geq 0$. Therefore there exists $\lim_{t \to t^+} x(t) \leq \infty$. By Theorem 2: $\lim_{t \to t^+} x(t) = \infty$. This allows to draw the graph of $x(t)$ as $t \geq 0$.

Now let us draw the graph of $x(t)$ for $t \leq 0$. At first note that since $\tilde{x}(t) \equiv 0$ is a solution of the equation $x' = x^2$ then by the uniqueness part of Theorem 1 one has $x(t) \neq 0$ for any $t \in (t^-, t^+)$. Since $x(0) = 1 > 0$ and since $x(t)$ is a continuous function (moreover, differentiable) then $x(t) \neq 0$ for any $t \in (t^-, t^+)$. Now our equation $x' = x^2$ implies that $x(t)$ is a strictly increasing function for any $t \in (t^-, t^+)$. It follows that there exists $\lim_{t \to t^-} x(t) = A \in [0, 1)$. Now Theorem 2 allows to make conclusion that $t^- = -\infty$.

To draw the graph of $x(t)$ it remains to find A. One can "expect" that $A = 0$. This is so, but this has to be proved. We will prove it later (see Lecture B). Also, in Lecture B we will extend all arguments above to a general theory of autonomous equations $x' = V(x)$.

Exercise 6. Find t^+ in Example 5 (using the inverse function $t(x)$ as in the proof of Lemma 1).

Exercise 7. Let $\tilde{x}(t)$ be the solution of the same equation $x' = x^2$ satisfying the initial condition $x(0) = -1$ and defined on maximal possible interval $(\tilde{t}^-, \tilde{t}^+)$. Using similar arguments, find $\tilde{t}^+, \tilde{t}^-$ and draw the graph of $\tilde{x}(t)$.