LECTURE B

Equations of the form $x' = V(x)$ (autonomous first order ODEs)

ATTN.: UNDERSTANDING THIS LECTURE REQUIRES DRAWING MANY GRAPHS.

In this section we use the theorems on existence, uniqueness, and prolongation of solutions (Theorems 1 and 2 from Lecture A) to present complete theory of equations of the form

$$x' = V(x), \quad V(x) \in C^1(\mathbb{R}). \tag{1}$$

Definition. *A point x^* of the x-axes is a singular point of Eq. (1) if $V(x^*) = 0$.*

The singular points correspond to solutions $x(t) \equiv const$.

Proposition 1. *A function $x(t) \equiv x^*$ is a solution of equation (1) if and only if x^* is a singular point.*

PROOF. This statement is obvious. □

Proposition 2. *Any non-constant solution $x(t)$ of (1), defined on any time-interval (α, β), is a strictly monotonic function. Moreover $x'(t) \neq 0$ for any $t \in (\alpha, \beta)$.*

PROOF. It suffices to prove that $x'(t) \neq 0$. Assume (to get contradiction) that $x'(t_0) = 0$, where $t_0 \in (\alpha, \beta)$. Then $V(x(t_0)) = 0$ which means that $x(t_0)$ is a singular point. Therefore we have a constant solution $\tilde{x}(t) \equiv x(t_0), t \in \mathbb{R}$ (see Proposition 1). The uniqueness theorem (which holds since $V(x) \in C^1$) implies $x(t) = \tilde{x}(t) = x(t_0)$ for any $t \in (\alpha, \beta)$. This contradicts to the assumption $x(t) \not\equiv const$. □

Notation. In what follows $x(t)$ is a solution of (1) satisfying the initial condition

$$x(t_0) = x_0 \tag{2}$$

and defined on maximal possible time-interval $t \in (t^-, t^+)$.

Proposition 3. *The solution $x(t)$ is an increasing function if $V(x_0) > 0$ and a decreasing function if $V(x_0) < 0$.*

PROOF. Follows from Proposition 2. □

Now we consider the following(all possible) cases.

CASE A. $x_0 \in (a, b)$, where a and b are singular points, $a < b$. There are no singular points between a and b.

Lemma A1. *In case* **A** *one has: $a < x(t) < b$ for any $t \in (t^-, t^+)$.*

PROOF. Follows from Proposition 1 and the uniqueness theorem. □

Lemma A2. *In case* **A** *there are the limits*

$$\lim_{t \to t^+} x(t) \in [a, b], \quad \lim_{t \to t^-} x(t) \in [a, b].$$

PROOF. Follows from Lemma A1 and Proposition 3. □

Lemma A3. *In case* **A** *one has $t^+ = \infty,\ t^- = -\infty$.*

Proof. Follows from Lemma A2 and the theorem on prolongation of solutions. □

Lemma A4. *In case* **A** *the limits in Lemma A2 are as follows:*

(a) if $V(x_0) > 0$ *then:* *(a1)* $\lim_{t\to\infty} x(t) = b$; *(a2)* $\lim_{t\to-\infty} x(t) = a$.

(b) if $V(x_0) < 0$ *then:* *(b1)* $\lim_{t\to\infty} x(t) = a$; *(b2)* $\lim_{t\to-\infty} x(t) = b$.

Proof. Let us prove (a1) (the proofs of (a2), (b1), (b2) are similar). We know (Proposition 3, Lemmas A2, A3) that:

(i) $x(t)$ is an increasing function;

(ii) $\lim_{t\to\infty} x(t) = B$, where $x_0 < B \leq b$.

We have to prove that $B = b$. Note that (i) and (ii) imply that there exists a sequence $t_k \to \infty$ such that $x'(t_k) \to 0$. [1] [2] Substituting these points to the equation $x' = V(x)$ we obtain $x'(t_k) = V(x(t_k))$. Taking the limit as $k \to \infty$ we obtain $0 = V(B)$. Since $x_0 < B \leq b$, it follows $B = b$. [3] □

Conclusion for the case A: One has $t^+ = \infty$, $t^- = -\infty$. If $V(x_0) > 0$ then $x(t)$ is an increasing function tending to b as $t \to \infty$ and to a if $t \to -\infty$. If $V(x_0) < 0$ then $x(t)$ is a decreasing function tending to a as $t \to \infty$ and to b if $t \to -\infty$.

Case B. $x_0 > a$, where a is "the most right" singular point (i.e. there are no singular points $x^* > a$).

Lemma B1. *In case* **B** $x(t) > a$ *for any* $t \in (t^-, t^+)$, *and also the following holds:*

(a) If $V(x_0) > 0$ *then* $x(t)$ *is an increasing function,* $t^- = -\infty$, $\lim_{t\to-\infty} x(t) = a$.

(b) If $V(x_0) < 0$ *then* $x(t)$ *is a decreasing function,* $t^+ = \infty$, $\lim_{t\to\infty} x(t) = a$.

Proof. Similar to the proofs of Lemmas A1–A4. □

Lemma B2. *In case* **B** *one has the following:*

(a) If $V(x_0) > 0$ *then* $\lim_{t\to t^+} x(t) = \infty$.

(b) If $V(x_0) < 0$ *then* $\lim_{t\to t^-} x(t) = \infty$.

Remark. Statement (a) holds either for finite t^+ or for $t^+ = \infty$. Statement (b) holds either for finite t^- or for $t^- = -\infty$.

Proof. Let us proof (a) (the proof of (b) is similar). Assume (to get contradiction) that (a) does not hold. Then, since $x(t)$ is an increasing function one has $\lim_{t\to t^+} x(t) = A < \infty$. If $t^+ < \infty$ we have a contradiction with the theorem on prolongation of solutions. Therefore in the case $t^+ < \infty$ we are done. Consider now the case $t^+ = \infty$. Since in the case **B**, subcase (a) one has $V(x) > 0$ for any $x \geq x_0$, there exists $\min_{x\in[x_0,A]} V(x) = \epsilon > 0$. Since our assumptions imply $x(t) \in [x_0, A]$ for any $t \geq t_0$, one has $x'(t) = V(x(t)) \geq \epsilon > 0$ for any $t \geq t_0$. It follows $x(t) \to \infty$ as $t \to \infty$: contradiction. □

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[1] Exercise: prove this statement.

[2] One could think that (i) and (ii) imply that $x'(t) \to 0$ as $t \to \infty$. This is not so, as I explained at the lecture.

[3] This proof was suggested by one of the students (who should let me know his name!). I was going to present another proof, a bit more complicated.

In case **B** it remains to find t^+ if $V(x_0) > 0$ and t^- if $V(x_0) < 0$. This can be done using the inverse function $t(x)$ (the inverse function exists since $x(t)$ is monotonic). Since $t'(x) = \frac{1}{V(x)}$ one has

$$(3) \qquad t(x) - t(x_0) = t(x) - t_0 = \int_{x_0}^{x} \frac{ds}{V(s)} \implies t(x) = t_0 + \int_{x_0}^{x} \frac{ds}{V(s)}.$$

By Lemmas B1, B2 the inverse function $t(x)$ is defined for $x \in [a, \infty)$. Take the limit as $x \to \infty$. Since in the case $V(x_0) > 0$ one has $t^+ = t(\infty) = \lim_{x\to\infty} t(x)$ and in the case $V(x_0) < 0$ one has $t^- = t(\infty) = \lim_{x\to\infty} t(x)$, we obtain:

$$(4) \qquad V(x_0) > 0 \implies t^+ = t_0 + \int_{x_0}^{\infty} \frac{dx}{V(x)}.$$

$$(5) \qquad V(x_0) < 0 \implies t^- = t_0 + \int_{x_0}^{\infty} \frac{dx}{V(x)}.$$

Whether or not $t^+ < \infty$ in case (4)or $t^- > -\infty$ in case (5) depends on the convergence/divergence of the integrals, i.e. on the behavior of the function $V(x)$ as $x \to \infty$. Note that in case (4) the function $V(x)$ is positive for $x \in [x_0, \infty)$ and consequently $t^+ > t_0$, and in case (5) the function $V(x)$ is negative for $x \in [x_0, \infty)$ and consequently $t^- < t_0$. [4]

Conclusion for the case B: $x(t)$ is a monotonic function bounded from below by a. If $V(x_0) > 0$ then $x(t)$ is an increasing function, $t^- = -\infty$, $t^+ = (4)$, $\lim_{t\to-\infty} x(t) = a, \lim_{t\to t^+} x(t) = \infty$. If $V(x_0) < 0$ then $x(t)$ is a decreasing function, $t^+ = \infty$, $t^- = (5)$, $\lim_{t\to\infty} x(t) = a, \lim_{t\to t^-} x(t) = \infty$.

Case C. $x_0 < b$, where b is "the most left" singular point (i.e. there are no singular points $x^* < b$).

Lemma C1. *In case* **C** $x(t) < b$ *for any* $t \in (t^-, t^+)$, *and also the following holds:*

(a) If $V(x_0) > 0$ *then* $x(t)$ *is an increasing function,* $t^+ = \infty$, $\lim_{t\to\infty} x(t) = b$.

(b) If $V(x_0) < 0$ *then* $x(t)$ *is a decreasing function,* $t^- = -\infty$, $\lim_{t\to-\infty} x(t) = b$.

PROOF. Similar to the proofs of Lemmas A1–A4. □

Lemma C2. *In case* **C** *one has the following:*

(a) If $V(x_0) > 0$ *then* $\lim_{t\to t^-} x(t) = -\infty$.

(b) If $V(x_0) < 0$ *then* $\lim_{t\to t^+} x(t) = -\infty$.

PROOF. Similar to the proof of Lemma B2. □

To find t^- in the case $V(x_0) > 0$ and t^+ in the case $V(x_0) < 0$ we use the same way as in case **B** (the inverse function $t(x)$). The difference with respect to case **B** is that now the inverse function $t(x)$ is defined for $x \in (-\infty, x_0]$ and $t^- = t^-(-\infty)$ in the case $V(x_0) > 0$ and $t^+ = t^+(-\infty)$ in the case $V(x_0) < 0$ (in the case **B** we

[4] Of course we know a priori that $t^+ > t_0$ and $t^- < t_0$; this is one of the ways to check that there is no mistake/misprint in the answer.

had $+\infty$ instead of $-\infty$). Therefore one has to take the limit in (3) as $x \to -\infty$ (not as $x \to \infty$ as in the case **B**) and we obtain:

$$\text{(6)} \qquad V(x_0) > 0 \implies t^- = t_0 + \int_{x_0}^{-\infty} \frac{dx}{V(x)} = t_0 - \int_{-\infty}^{x_0} \frac{dx}{V(x)}$$

$$\text{(7)} \qquad V(x_0) < 0 \implies t^+ = t_0 + \int_{x_0}^{-\infty} \frac{dx}{V(x)} = t_0 - \int_{-\infty}^{x_0} \frac{dx}{V(x)}.$$

Whether or not $t^- > -\infty$ in case (6) or $t^+ < \infty$ in case (7) depends on the convergence/divergence of the integrals, i.e. on the behavior of the function $V(x)$ as $x \to -\infty$. Note that in case (6) the function $V(x)$ is positive for $x \in (-\infty, x_0]$ and consequently $t^- < t_0$, and in case (7) the function $V(x)$ is negative for $x \in (-\infty, x_0]$ and consequently $t^+ > t_0$ (see the footnote on the previous page).

Conclusion for the case C: $x(t)$ is a monotonic function bounded from above by b. If $V(x_0) > 0$ then $x(t)$ is an increasing function, $t^+ = \infty$, $t^- =$ (6), $\lim_{t\to\infty} x(t) = b, \lim_{t\to t^-} x(t) = -\infty$. If $V(x_0) < 0$ then $x(t)$ is a decreasing function, $t^- = -\infty$, $t^+ =$ (7), $\lim_{t\to-\infty} x(t) = b, \lim_{t\to t^+} x(t) = -\infty$.

Case D: there are no singular points.

In this case either $V(x) > 0$ for all x or $V(x) < 0$ for all x. Arguing in the same way as in cases **A - C** we obtain:

If $V(x) > 0$ then $x(t)$ is an increasing function which tends to ∞ as $t \to \infty$ and which tends to $-\infty$ as $t \to -\infty$. In this case

$$t^+ = t_0 + \int_{x_0}^{\infty} \frac{dx}{V(x)}, \qquad t^- = t_0 + \int_{x_0}^{-\infty} \frac{dx}{V(x)} = t_0 - \int_{-\infty}^{x_0} \frac{dx}{V(x)}.$$

If $V(x) < 0$ then $x(t)$ is a decreasing function which tends to $-\infty$ as $t \to \infty$ and which tends to ∞ as $t \to -\infty$. In this case

$$t^+ = t_0 + \int_{x_0}^{-\infty} \frac{dx}{V(x)} = t_0 - \int_{-\infty}^{x_0} \frac{dx}{V(x)}, \qquad t^- = t_0 + \int_{x_0}^{\infty} \frac{dx}{V(x)}.$$

Note that in case **D** we might have any of the following cases:

$t^+ < \infty,\ t^- > -\infty$ (example: $V(x) = x^2 + 1$)

$t^+ = \infty,\ t^- = -\infty$ (example: $V(x) = \sqrt{x^2 + 1}$)

$t^+ < \infty,\ t^- = -\infty$ (example: $V(x) = e^x$)

$t^+ = \infty,\ t^- > -\infty$ (example: $V(x) = e^{-x}$).

I conclude this lecture with the following simple, but important theorem.

Proposition 4. (shift of time). *Let $x(t)$ be a solution of an autonomous ODE $x' = V(x)$ defined on the interval (a, b). Fix $s \in \mathbb{R}$. Then the function*

$$\tilde{x}(t) = x(t + s), \quad t \in (a - s, b - s)$$

is also a solution of the same equation.

Proof. $\tilde{x}'(t) = \frac{d}{dt} x(t + s) = x'(t + s) = V(x(t + s)) = V(\tilde{x}(t))$. □

Note that this theorem and the uniqueness theorem (Lecture A, Theorem 1) imply the following corollary: if $V(x) \in C^1(\mathbb{R})$ and $x(t), \tilde{x}(t)$ are solutions of the same equation $x' = V(x)$ defined on intervals $(a, b), (\tilde{a}, \tilde{b})$ then any horizontal shift of the graph of $\tilde{x}(t)$ (to the left or to the right) restricted to $t \in (a, b)$ either does not intersect the graph of $x(t)$ or coincides with this graph.

Exercises

Exercise 1,2,3, 4. Let $x(t)$ be the solution of the equation $x'(t) = V(x(t))$, where the function $V(x)$ is given below, satisfying the initial condition $x(t_0) = x_0$, where t_0 and x_0 are given below, and defined on maximal possible interval (t^-, t^+). Find t^-, t^+ and draw the graph of $x(t)$. Integrals are allowed **only** if they converge.

1.1. $V(x) = \sin(e^x),\ t_0 = 0,\ x_0 = -1$ 1.2. $V(x) = \sin(e^x),\ t_0 = 1,\ x_0 = -1$
1.3. $V(x) = \sin(e^x),\ t_0 = 1,\ x_0 = 1$ 1.4. $V(x) = \sin(e^x),\ t_0 = 0,\ x_0 = 2$
1.5. $V(x) = \sin(e^x),\ t_0 = -1,\ x_0 = 3$ 1.6. $V(x) = \sin(e^x),\ t_0 = -1,\ x_0 = 4$

2.1. $V(x) = (x-1)^2(x-2)^3(x+1)^4(x+2)^5,\ t_0 = 1,\ x_0 = -2$
2.2. $V(x) = (x-1)^2(x-2)^3(x+1)^4(x+2)^5,\ t_0 = 1,\ x_0 = -1.5$
2.3. $V(x) = (x-1)^2(x-2)^3(x+1)^4(x+2)^5,\ t_0 = -1,\ x_0 = -0.5$
2.4. $V(x) = (x-1)^2(x-2)^3(x+1)^4(x+2)^5,\ t_0 = -1,\ x_0 = 1$

3.1. $V(x) = \frac{x}{x^2+1},\ t_0 = 0,\ x_0 = -1$ 3.2. $V(x) = \frac{x}{x^2+1},\ t_0 = 1,\ x_0 = 1$
3.3. $V(x) = \frac{x^2}{x^2+1},\ t_0 = 0,\ x_0 = -1$ 3.4. $V(x) = \frac{x^2}{x^2+1},\ t_0 = 1,\ x_0 = 1$
3.5. $V(x) = \frac{x^3}{x^2+1},\ t_0 = 0,\ x_0 = -1$ 3.6. $V(x) = \frac{x^3}{x^2+1},\ t_0 = 1,\ x_0 = 1$
3.7. $V(x) = \frac{x^4}{x^2+1},\ t_0 = 0,\ x_0 = -1$ 3.8. $V(x) = \frac{x^4}{x^2+1},\ t_0 = 1,\ x_0 = 1$

4.1. $V(x) = 15 + 14\sin^5 x,\ t_0 = 0,\ x_0 = 1$
4.2. $V(x) = (x-1)(x-2)(x-3),\ t_0 = 0,\ x_0 = 0$
4.3. $V(x) = (x-1)(x-2)(x-3)(x-4),\ t_0 = 0,\ x_0 = 0$
4.4. $V(x) = (1-x)(2-x)(3-x),\ t_0 = 0,\ x_0 = 4$
4.5. $V(x) = x \cdot ln(x^2+1),\ t_0 = -1,\ x_0 = -1$
4.6. $V(x) = (x+5) \cdot ln(x^2+1),\ t_0 = 1,\ x_0 = 1$
4.7. $V(x) = (1-x)\sqrt{x^2+1},\ t_0 = 0,\ x_0 = -1$
4.8. $V(x) = (2x-7) \cdot ln(x^2+1),\ t_0 = -6,\ x_0 = 3$

Exercises 5,6,7. Let $x_a(t)$ be the solution of the equation $x'(t) = V(x(t))$ satisfying the initial condition $x(0) = a$, $a \in \mathbb{R}$ and defined on maximal possible interval $(t^-(a), t^+(a))$. Give an example of $V(x) \in C^1(\mathbb{R})$ such that **each** of the requirements given below holds.

Exercise 5.

(a) $t^+(a) = \infty, t^-(a) = -\infty$ for any $a \in \mathbb{R}$
(b) $\lim_{t\to\infty} x_a(t) = \infty$ if and only if $a > 1$
(c) $\lim_{t\to\infty} x_a(t) = -\infty$ if and only if $a < -1$
(d) $\lim_{t\to\infty} x_a(t) = 0$ if and only if $a = 0$

Exercise 6.

(a) $t^+(a) = \infty, t^-(a) = -\infty$ for any $a \in \mathbb{R}$

(b) $\lim_{t\to\infty} x_a(t) \neq +\infty$ for any $a \in \mathbb{R}$

(c) $\lim_{t\to-\infty} x_a(t) \neq +\infty$ for any $a \in \mathbb{R}$

(d) $\lim_{t\to\infty} x_a(t) = -\infty$ if and only if $a < 0$

Exercise 7.

(a) $t^+(a) < \infty$ if and only if $a > 1$

(b) $t^-(a) > -\infty$ if and only if $a < 1$

Exercise 8. Find all points of inflection (NIKUDOT PITUL) of the solution $x(t)$ of the equation $x' = 2 + sin^2 x$ satisfying the initial condition $x(-1) = 3$ and defined for all $t \in \mathbb{R}$. Integrals in the answer are OK.

Exercise 9. Let $x(t) = t^3 + t^2 + t + 1$ be a solution of an autonomous ODE $x' = V(x) \in C^1(\mathbb{R})$ defined on the time-interval $t \in (-1, 1)$. Let $\tilde{x}(t)$ be another solution of the same equation given below. In some of the cases below there is a contradiction. In which?

(a) $\tilde{x}(t) = t^3 + t^2 + t + 5, \quad t \in (-1, 1)$

(b) $\tilde{x}(t) = t^3 + t, \quad t \in (1.2, 2)$

(c) $\tilde{x}(t) = t^3 + t, \quad t \in (1.5, 2)$

(d) $\tilde{x}(t) = t^3 - t, \quad t \in (-0.5, 0.5)$

(e) $\tilde{x}(t) = t^3 - t - 10, \quad t \in (-1, 1)$