LECTURE C

Non-autonomous first order ODEs

We will study the following non-autonomous first order ODEs:

Section 1: linear homogeneous ODEs: $x' = f(t)x$

Section 2: linear non-homogeneous ODEs: $x' = f(t)x + g(t)$

Sections 3 and 4: equations with separable variables: $x' = f(t)g(x)$

Sections 5,6,7,8 : certain ODEs which can be reduced to linear ODES or ODEs with separable variables by a certain SUBSTITUTION

1. Linear homogeneous equations:

$$x' = f(t)x, \quad f(t) \in C^0(\mathbb{R}). \tag{1}$$

Theorem 1. *Fix any initial condition $x(t_0) = x_0$. Equation (1) has a solution $x(t)$ satisfying this initial condition and defined for all t. This solution is unique, and it is as follows:*

$$x(t) = x_0 \cdot \exp\Big(\int_{t_0}^{t} f(s)ds \Big). \tag{2}$$

Proof. It is clear that (2) satisfies (1) and satisfies $x(t_0) = x_0$. The uniqueness holds since $f(t) \in C^0(\mathbb{R})$: due to this condition the assumptions of the existence and uniqueness theorem (Lecture Notes, Lecture A, Theorem 1) hold. □

Theorem 2. *Fix any function $F(t)$ such that $F'(t) = f(t)$. The general solution of (1) (i.e. the set of all solutions defined for all t) is*

$$C \cdot \exp\big(F(t)\big), \quad C \in \mathbb{R}. \tag{3}$$

Proof. It is clear that (3) is a solution of (1) for any $C \in \mathbb{R}$. To prove the theorem we have to show that any fixed solution $x(t)$ has form (3). Let $C_1 = x(0)$. By Theorem 1 we have $x(t) = C_1 \cdot \exp\big(\int_0^t f(s)ds \big)$. The functions $\int_0^t f(s)ds$ and $F(t)$ have the same derivative $f(t)$. Therefore $\int_0^t f(s)ds = F(t) + C_2$ for some C_2. It follows that $x(t)$ has form (3) with $C = C_1 e^{C_2}$. □

2. Linear non-homogeneous equations:

$$x' = f(t)x + g(t), \quad f(t), g(t) \in C^0(\mathbb{R}). \tag{4}$$

Theorem 3 (variation of constant). *Fix any initial condition $x(t_0) = x_0$. Equation (4) has a solution $x(t)$ satisfying this initial condition and defined for all t. This solution is unique. Fix any function $F(t)$ such that $F'(t) = f(t)$. Then $x(t)$ has the form*

$$C(t) \cdot \exp\big(F(t)\big), \tag{5}$$

where $C(t)$ is a certain function.

This theorem is called “variation of constant“ since (5) differs from the general solution (3) of the corresponding homogeneous equation (1) by replacing the constant C by a function $C(t)$.

Proof. Since $f(t), g(t) \in C^0(\mathbb{R})$ then the equation satisfies the assumptions of the existence and uniqueness theorem (Lecture Notes, Lecture A, Theorem 1). Therefore if there exists a solution defined for all t and satisfying $x(t_0) = x_0$, it is unique. This reduces the theorem to the following claim: there exists a function $C(t) \in C^1(\mathbb{R})$ such that (5) is a solution of (4) satisfying $x(t_0) = x_0$.

To prove this claim we substitute (5) to (4). We obtain:

$$C'(t) \cdot \exp\big(F(t)\big) + C(t) \cdot \Big(\exp\big(F(t)\big)\Big)' = f(t)C(t)(\exp\big(F(t)\big) + g(t).$$

Since $\exp\big(F(t)\big)$ is a solution of the homogeneous equation (1), one has $\Big(\exp\big(F(t)\big)\Big)' = f(t)C(t)(\exp\big(F(t)\big)$ and this equation takes the form

$$C'(t) = \exp\big(-F(t)\big)g(t). \tag{6}$$

Fix any function $H(t)$ such that $H'(t) = \exp\big(-F(t)\big)g(t)$. Then $C(t) = H(t) + D$ is a solution of (6) for any constant D. We have proved that for any $D \in \mathbb{R}$ the function

$$(H(t) + D) \cdot \exp\big(F(t)\big), \quad D \in \mathbb{R}.$$

is a solution of equation (4). It remains to find D such that this function satisfies the initial condition $x(t_0) = x_0$. Substituting t_0 we obtain the following equation for D:

$$(H(t_0) + D) \cdot \exp\big(F(t_0)\big) = x_0. \tag{7}$$

Obviously this equation has a solution $D \in \mathbb{R}$. □

Note that the solution of (4) satisfying $x(t_0) = x_0$ can be given by an explicit formulae (involving integrals). To obtain this formula take in the proof above

$$F(t) = \int_{t_0}^{t} f(s)ds.$$

$$H(t) = \int_{t_0}^{t} \exp\big(-F(s)\big)g(s)ds.$$

Then $F(t_0) = H(t_0) = 0$ and equation (7) reduces to $D = x_0$. We obtain:

$$x(t) = \Big(x_0 + \int_{t_0}^{t} \exp\Big(-\int_{t_0}^{s} f(r)dr\Big)g(s)ds\Big) \cdot \exp\Big(\int_{t_0}^{t} f(s)ds\Big)\Big).$$

3. Equations with separable variables:

$$x' = f(t) \cdot g(x), \qquad f(t) \in C^0(I),\ f(t) \not\equiv 0,\ \ g(x) \in C^1(\tilde{I}). \tag{8}$$

Here I and $\tilde{I}$ are open intervals in the t-axes and the x-axes (including the case that I and/or $\tilde{I}$ is the whole $\mathbb{R}$).

Like for autonomous ODEs (see Lecture Notes, Lecture B) for such equations it is worth to define singular points as follows.

Definition. A singular point of (8) is a point x^* in the x-axes such that $g(x^*) = 0$.

The singular points correspond to constant solutions: if x^* is a singular point then obviously $x(t) \equiv x^*$ is a solution of (8). The converse is true unless $f(t) \equiv 0$ on the interval of definition of a solution $x(t)$: if $x(t) \equiv x^*$ is a solution defined on (a, b) and $f(t) \not\equiv 0$ on (a, b) then x^* is a singular point.

Theorem 4. *Let $x(t)$ be the solution of (8) satisfying the initial conditions $x(t_0) = x_0$, where x_0 is not a singular point, and defined for $t \in (a, b)$. Then*

1. One has $g(x(t)) \neq 0$ for any $t \in (a, b)$.

2. For any $t \in (a, b)$ and for any $s \in [x_0, x(t)]$ one has $g(s) \neq 0$.

3. One has

$$\int_{x_0}^{x(t)} \frac{ds}{g(s)} = \int_{t_0}^{t} f(s)ds, \quad t \in (a, b). \tag{9}$$

Remark. The second statement implies that the integral in the left hand side of (9) is well-defined.

Theorem 5. *Let $x(t)$ be a function defined for $t \in (a, b)$ and such that $g(x(t)) \neq 0, t \in (a, b)$. If (9) holds then $x(t)$ is a solution of (8)* satisfying $x(t_0) = x_0$.

Proof of Theorem 4. To prove the first statement assume (to get contradiction) that $g(x(t^*)) = 0$ for some $t^* \in (a, b)$. Let $x^* = x(t^*)$. Then x^* is a singular point and there is a constant solution $\tilde{x}(t) \equiv x^*$. Now we use the uniqueness theorem (it holds due to the assumptions $f(t) \in C^0(\mathbb{R}), g(x) \in C^1(\mathbb{R})$). Since $x(t^*) = \tilde{x}(t^*) = x^*$, one has $x(t) = \tilde{x}(t) = x^*$ for any $t \in (a, b)$. It follows that $x_0 = x(t_0) = x^*$ and then x_0 is a singular point. This contradicts to the assumption of the theorem.

The second statement is a corollary of the first one and the continuity of $x(t)$.

Now we prove the third statement. The first two statements imply that $\frac{1}{g(s)} \in C^1[x_0, x(t)]$ for any $t \in (a, b)$. It follows that the function $A(t) = \int_{x_0}^{x(t)} \frac{ds}{g(s)}$ is differentiable for $t \in (a, b)$. The function $B(t) = \int_{t_0}^{t} f(s)ds$ is also differentiable for $t \in (a, b)$. We have to prove that $A(t) \equiv B(t)$. Note that $A(t_0) = B(t_0) = 0$. Therefore it suffices to prove that $A'(t) \equiv B'(t)$. One has

$$B'(t) = f(t), \quad A'(t) = \frac{x'(t)}{g(x(t))}. \tag{10}$$

Since $x(t)$ is a solution of (8) then $A'(t) = f(t)$ and we are done.

Proof of Theorem 5. The assumption $g(x(t)) \neq 0, t \in (a, b)$ implies that $g(s) \neq 0$ for any $s \in [x_0, x(t)], t \in (a, b)$. Therefore $\frac{1}{g(s)}$ is a differentiable function for $s \in [x_0, x(t)]$ and then the function $A(t) = \int_{x_0}^{x(t)} \frac{ds}{g(s)}$ is differentiable for $t \in (a, b)$. The function $B(t) = \int_{t_0}^{t} f(s)ds$ is also differentiable for $t \in (a, b)$. We have $A(t) \equiv B(t)$. Then $A'(t) \equiv B'(t)$. By (10) one has $\frac{x'(t)}{g(x(t))} = f(t)$ and it follows that $x(t)$ is a solution of (8).

It remains to show that $x(t_0) = x_0$. Substituting t_0 to (9) we obtain $\int_{x_0}^{x(t_0)} \frac{ds}{g(s)} = 0$. Since $1/g(s)$ is a non-vanishing function on the interval $[x_0, x(t_0)]$, it follows that $x(t_0) = x_0$.

4. Qualitative analysis of equations (8)

I mean drawing the graph of the solution $x(t)$ of (8) satisfying a fixed initial condition $x(t_0) = x_0$ and determination of the maximal possible time-interval (t^-, t^+) on which $x(t)$ is defined. In many cases this does not require taking integrals in (9). I will present several examples. In each of them we will use the uniqueness theorem and theorem on prolongation of solutions (Lecture Notes, Lecture A, Theorems 1,2). Our arguments will be similar to those in Lecture B. On certain steps of the qualitative analysis we will also use (9), but we will not take integrals in this relation.

Example 1.

$$x' = (x^2 - 1) \cdot (t^3 + t^2 - 2t), \quad x(0) = 0.$$

Let $x(t)$ be solution of this equation defined on maximal possible time-interval (t^-, t^+).

Claim 1.1. One has $x(t) \neq \pm 1$ for any $t \in (t^-, t^+)$.

PROOF. Follows from the uniqueness theorem and the fact that ± 1 are singular points (and consequently there are constant solutions $\tilde{x}(t) \equiv 1, \hat{x}(t) \equiv -1$).

Claim 1.2. One has $x(t) \in (-1, 1)$ for any $t \in (t^-, t^+)$.

PROOF. Follows from Claim 1.1 and the initial condition $x(0) = 0$.

Claim 1.3. The solution $x(t)$ is an increasing function at any point $t \in ((-\infty, -2) \cup (0, 1)) \cap (t^-, t^+)$ and a decreasing function at any point $t \in ((-2, 0) \cup (1, \infty)) \cap (t^-, t^+)$.

PROOF. Follows from Claim 1.2 (implying that $x^2(t) - 1 < 0$ for any $t \in (t^-, t^+)$ and the fact that the polynomial $t^3 + t^2 - 2t$ is positive when $t \in (-2, 0) \cup (1, \infty)$ and negative when $t \in ((-\infty, -2) \cup (0, 1)$.

Claim 1.4. If $t^+ < \infty$ then there exists the limit $\lim_{t \to t^+} x(t) \in [-1, 1]$. If $t^- > -\infty$ then there exists the limit $\lim_{t \to t^-} x(t) \in [-1, 1]$.

PROOF. Easily follows from claims 1.3 and 1.2 (exercise on hedva: prove it).

Claim 1.5. $t^+ = \infty$ and $t^- = -\infty$.

PROOF. Follows from Claim 1.4 and the theorem on prolongation of solutions.

Claim 1.5. There exist

$$\lim_{t \to \infty} x(t) = B \in (1, -1], \qquad \lim_{t \to -\infty} x(t) = A \in (1, -1].$$

PROOF. Follows from Claims 1.3 and 1.2 (we also use, of course, Claim 1.5). Exercise: explain why $A \neq 1, B \neq 1$.

Now we will find A and B. We use Theorem 4. By this theorem

$$\int_0^{x(t)} \frac{ds}{s^2 - 1} = \int_0^t (s^3 + s^2 - 2s) ds. \tag{11}$$

Take the limits in this relation as $t \to \infty$ and as $t \to -\infty$. We obtain

$$\int_0^B \frac{dx}{x^2-1} = \int_0^\infty (t^3+t^2-2t)dt;$$

$$\int_0^A \frac{dx}{x^2-1} = \int_0^{-\infty} (t^3+t^2-2t)dt.$$

Since $\int_0^\infty (t^3+t^2-2t)dt = \int_0^{-\infty}(t^3+t^2-2t)dt = \infty$ and $A, B \in (1,-1]$ (recall that $A, B \neq 1$), it follows:

Claim 1.6. $A = B = -1$.

Example 2.

$$x' = \frac{1-x^2}{t^2+t+2}, \quad x(0) = 0.$$

Let $x(t)$ be solution of this equation defined on maximal possible time-interval (t^-, t^+).

Claim 2.1.
$t^+ = \infty, t^- = -\infty$.

$x(t)$ is an increasing function (for any $t \in \mathbb{R}$).

$\lim_{t\to\infty} x(t) = B \in (-1,1], \quad \lim_{t\to-\infty} x(t) = A \in [-1,1)$
(note that $B \neq -1$ and $A \neq 1$)

Proof. Similar to claims 1.1 - 1.5 of Example 1.

To find A and B use Theorem 4:

$$\int_0^{x(t)} \frac{ds}{1-s^2} = \int_0^t \frac{ds}{s^2+s+2}.$$

Taking the limit as $t \to \pm\infty$ we obtain

$$\int_0^B \frac{dx}{1-x^2} = \int_0^\infty \frac{dt}{t^2+t+2} = r_1 \text{ (finite number)}$$

$$\int_0^A \frac{dx}{1-x^2} = \int_0^{-\infty} \frac{dt}{t^2+t+2} = r_2 \text{ (finite number)}$$

and it follows that $B < 1$ and $A > -1$.

Example 3.

$$x' = (x^2-1)(t^2-1), \quad x(0) = 2$$

Arguing as in Example 1 we obtain:

$x(t) > 1$ for any $t \in (t^-, t^+)$;

$x(t)$ is increasing for $t \in ((-\infty,-1) \cup (1,\infty)) \cap (t^-, t^+)$ and decreasing for $t \in (-1,1) \cap (t^-, t^+)$;

$t^- = -\infty$;

there exists $\lim_{t\to-\infty} x(t) = A \geq 1$.

By Theorem 4

$$\int_2^{x(t)} \frac{ds}{s^2-1} = \int_0^t (s^2-1)ds \tag{12}$$

and taking the limit as $t \to -\infty$ we obtain

$A = 1$.

We also obtain, arguing like in Example 1:

there exists $\lim_{t\to t^+} x(t) = B, \quad 1 < B \le \infty$.

In this example for finding t^+ it is not enough to use the uniqueness theorem and the theorem on prolongation of solutions. One also should use Theorem 4. Taking the limit in (12) as $t \to t^+$ we obtain

$$\int_2^B \frac{dx}{x^2-1} = \int_0^{t^+} (s^2-1)ds.$$

The integral in the left hand side is a finite number, even if $B = \infty$. Therefore the integral in the right hand side is also a finite number. We obtain:

Claim: $t^+ < \infty$.

Now we use the theorem on prolongation of solutions. Since $t^+ < \infty$, it implies:

Claim: $B = \infty$.

Now t^+ can be found from the equation

$$\int_2^\infty \frac{dx}{x^2-1} = \int_0^{t^+} (s^2-1)ds.$$

Example 4.

$$x' = \frac{x^2+1}{1-t^2}, \quad x(0) = 0$$

Note that unlike the previous examples, this equation is defined in the domain $t \in (-1,1), x \in \mathbb{R}$ (not in the whole plane $\mathbb{R}^2$). Therefore A PRIORI $t^\pm \in [-1,1]$.

In this example there are no singular points. The equation implies that $x(t)$ increases for all $t \in (t^-, t^+)$. Therefore there

$$\lim_{t\to t^+} x(t) = B \in (0,\infty], \quad \lim_{t\to t^-} x(t) = A \in [-\infty, 0].$$

At this moment we do not know if $t^\pm = \pm 1$, and we do not know if A, B are finite or not. The only fact we know (without using Theorem 4) is as follows: if $t^+ < 1$ then $B = \infty$ and if $t^- > 1$ then $A = -\infty$. This follows from the theorem on prolongation of solutions for a domain with a boundary (see Lecture Notes, Lecture A, Th. 3).

To get more information we use Theorem 4. It gives

$$\int_0^{x(t)} \frac{ds}{s^2+1} = \int_0^t \frac{ds}{1-s^2}.$$

Taking the limits as $t \to t^+$ and as $t \to t^-$ we obtain

$$\int_0^B \frac{dx}{x^2+1} = \int_0^{t^+} \frac{dt}{1-t^2}; \quad \int_0^A \frac{dx}{x^2+1} = \int_0^{t^-} \frac{dt}{1-t^2}.$$

The integrals in the left hand sides of these relations are finite for any A, B including $A, B = \pm\infty$. It follows that the integrals in the right hand side of these relations are also finite. This is possible only if $t^+ < 1$ and $t^- > -1$.

We have proved that $t^+ < 1$ and $t^- > -1$. Now the theorem on prolongation of solutions for a domain with a boundary (see Lecture Notes, Lecture A, Theorem 3) implies:

$B = \infty$, $A = -\infty$.

Consequently $t^{\pm}$ satisfy the equations

$$\int_0^{\infty} \frac{dx}{x^2+1} = \int_0^{t^+} \frac{dt}{1-t^2}; \quad \int_0^{-\infty} \frac{dx}{x^2+1} = \int_0^{t^-} \frac{dt}{1-t^2}.$$

Remark. In the qualitative analysis like in the given examples it is worth to use the following theorem (from INFI):

Theorem 6. *Let*

$$f(x) \in C^{\infty}(\mathbb{R}), \quad f(x_0) = 0, \quad f(x) \neq 0 \text{ as } x \in [a, x_0).$$

Then $\int_a^{x_0} \frac{dx}{f(x)} = \infty$.

5. Equations of the form $x'(t) = f(ax(t) + bt + c)$

Here f is any C^1 function of one variable. The equation can be solved by the substitution

$$y(t) = ax(t) + bt + c.$$

In fact, one has

$$y' = ax' + b = af(y) + b,$$

and we get an autonomous equation (for $y(t)$).

6. Equations of the form $x' = f(\frac{x}{t})$

Such equations are called homogeneous. Examples:

$x' = sin(\frac{x}{t})$

$x' = \frac{ax+bt}{cx+dt}$ (the function $\frac{ax+bt}{cx+dt}$ can be written in the form $\frac{ay+b}{cy+d}$, where $y = x/t$)

The last example can be generalized:

$$x' = \frac{\sum_{i=0}^{m} a_i x^i t^{m-i}}{\sum_{i=0}^{m} b_i x^i t^{m-i}}$$

In fact, the function in the right hand part can be written in the form

$$\frac{\sum_{i=0}^{m} a_i y^i}{\sum_{i=0}^{m} b_i y^i}, \quad y = x/t.$$

An equation of the form $x' = f(\frac{x}{t})$ can be solved by the substitution $y(t) = x(t)/t$. In fact,

$$y'(t) = \frac{x'(t) \cdot t - x(t)}{t^2} = \frac{f(y) \cdot t - x(t)}{t^2} = \frac{f(y) - y}{t}$$

and we obtain an equation with separable variables for $y(t)$.

Remark. The direct application of this method requires working in a domain in $\mathbb{R}^2(t, x)$ which does not intersect the line $t = 0$.

7. Equations of the form

$$x' = \frac{a_{11}x + a_{12}t + b_1}{a_{21}x + a_{22}t + b_2} \tag{13}$$

If $b_1 = b_2 = 0$ we have a homogeneous equation (see the previous section). Let us try to reduce b_1, b_2 to 0 by shifting the function $x(t)$ and the time t, i.e. we introduce new unknown function

$$y(t) = x(t) + \alpha$$

and new time

$$\tau = t + \beta.$$

Then

$$y'(\tau) = y'(t+\beta) = x'(t+\beta) = x'(\tau) =$$

$$= \frac{a_{11}x(\tau) + a_{12}t + b_1}{a_{21}x(\tau) + a_{22}t + b_2} = \frac{a_{11}\big(y(\tau) - \alpha\big) + a_{12}\big(\tau - \beta\big) + b_1}{a_{21}\big(y(\tau) - \alpha\big) + a_{22}\big(\tau - \beta\big) + b_2}$$

We obtain the following equation with the new (shifted) time τ and new unknown function $y(\tau)$:

$$y'(\tau) = \frac{a_{11}y + a_{12}\tau + \tilde{b}_1}{a_{21}x + a_{22}\tau + \tilde{b}_2},$$

where

$$\begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

If

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \neq 0$$

then we can solve the system of two linear equations $\tilde{b}_1 = 0, \tilde{b}_2 = 0$ (with two unknown α, β). We obtain a homogeneous equation (see the previous section).

Consider now the case

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = 0, \quad a_{11} \neq 0$$

In this case

$$(a_{21}, a_{22}) = k(a_{11}, a_{12})$$

for some $k \in \mathbb{R}$. In this case the initial equation (13) takes the form

$$x' = \frac{a_{11}x + a_{12}t + b_1}{k \cdot \big(a_{11}x + a_{12}t\big) + b_2} \tag{14}$$

This equation can be solved by the substitution

$$y(t) = a_{11}x(t) + a_{12}t.$$

In fact, we have

$$y'(t) = a_{11}x'(t) + a_{12} = a_{11} \cdot \frac{y(t) + b_1}{ky(t) + b_2} + a_{12}$$

and we obtain an autonomous equation for $y(t)$.

Consider now the case

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = 0, \quad a_{11} = 0$$

In this case either $a_{12} = 0$ or $a_{21} = 0$ (or $a_{12} = a_{21} = 0$).

If $a_{12} = 0$ then the the initial equation (13) takes the form

$$x' = \frac{b_1}{a_{21}x + a_{22}t + b_2}$$

which is an equation of the form given in section 5 (can be solved by the substitution $y(t) = a_{21}x(t) + a_{22}t + b_2$).

Finally, if $a_{21} = 0$ then the the initial equation (13) takes the form

$$x'(t) = \frac{a_{12}t + b_1}{a_{22}t + b_2}$$

which is a “baby“ equation.

8. Equation of the form

$$x' = f(t)x^r + g(t)x \tag{15}$$

Equations of this form are called Bernoulli equations. Here r is a real number (not necessary integer!) It can be either positive or negative. If $r = 0$ then (15) is a linear (non-homogeneous) equation, see section 2. If $r = 1$ then (15) is an equation with separable variables, see section 3. Therefore in what follows we will assume

$$r \neq 0, r \neq 1.$$

Equations (15) can be solved by the substitution

$$y = x^\mu$$

with a suitable $\mu \in \mathbb{R}$. One has:

$$y' = \mu x^{\mu-1}x' = \mu x^{\mu-1} \cdot \big(f(t)x^r + g(t)x\big) =$$

$$= \mu \cdot \Big(f(t)x^{r+\mu-1} + g(t)x^\mu\Big) = \mu \cdot \Big(f(t)y^{\frac{r+\mu-1}{\mu}} + g(t)y\Big)$$

Now it is clear that one should take

$$\mu = 1 - r$$

to obtain

$$y' = (1-r) \cdot \Big(f(t) + g(t)y\Big)$$

which is a linear ODE, see section 2.

Exercises

Exercise 1. Let $x(t)$ be solution of the equation $x' = sin(t^2) \cdot x + cos(t^2)$ satisfying the initial condition $x(12) = 7$. Find $x(14)$. Integrals in the answer OK.

Exercise 2. Let $x(t)$ be solution of the equation $x' = t^3 \cdot sin(x)$ satisfying the initial condition $x(1) = 2$. Find t_1 such that $x(t_1) = 3$. Integrals in the answer OK.

Exercise 3. Let $x(t)$ be solution of the equation $x' = \sqrt{(2x+t+1)^2+1}$ satisfying the initial condition $x(0)=0$. Find t_1 such that $x(t_1)=3$. Integrals in the answer OK.

Exercise 4. Let $x(t)$ be solution of the equation

$$x' = \frac{t^3+x^3}{tx(t+x)}$$

satisfying the initial condition $x(2)=5$. Find t_1 such that $x(t_1)=3t_1$. Integrals in the answer OK.

Exercise 5. Let $x(t)$ be solution of the equation

$$x' = \frac{x+t+2}{2x-t-1}$$

satisfying the initial condition $x(0)=0$. Give a formula relating $x(t)$ and t. Integrals in this formula are OK.

Exercise 6. Let $x(t)$ be solution of the equation

$$x' = x(t^3+x)$$

satisfying the initial condition $x(2)=1$. Find $x(3)$. Integrals in the answer OK.

Exercises 7 - 10

Let $x(t)$ be solution of the equation given below, satisfying the initial condition $x(0)=x_0$ with x_0 given below, and defined on maximal possible time-interval (t^-,t^+). Find $t^\pm$. Find $B=\lim_{t\to t^+}x(t)$ and $A=\lim_{t\to t^-}x(t)$. Draw the graph of $x(t)$.

ACCEPTABLE answers (for $t^\pm, A, B$):

∞; $-\infty$; a finite number satisfying the written formula or equation (integrals in the formula or equation are OK **ONLY** if they are convergent!).

ATTENTION: all exercises and cases below are different, you should solve all of them.

Exercise 7.

$$x' = (x-4)(x-5)(x-6)(t-1)(t-2)(t-3)$$

(a) $x_0 = 7$ (b) $x_0 = 5.5$ (c) $x_0 = 4.5$ (d) $x_0 = 0$

Exercise 8.

$$x' = \frac{(x-4)(x-5)(x-6)}{(t-1)(t-2)(t-3)}, \quad x_0 = 4$$

Exercise 9.

$$x' = \frac{sin(t)}{x^8+1}, \quad x(0)=0$$

Exercise 10.

$$x' = \frac{1-x^3}{1-t^2}, \quad x(0)=0$$