Lecture E.

Equations of the form $x''(t) = G(x(t)),\ G(x) \in C^1$.
Math form of the energy preserving law.
Critical points.
Symmetries.
Periodic solutions

Equations of the form

$$x''(t) = G(x(t)), \quad G(x) \in C^1 \tag{0.1}$$

are particular but very important case of the second order ODE's. Physically such equations mean that a body moves along a straight line under the action of a force ($= mx''$, where m is the mass) which is uniquely determined by the location of the body, i.e. when the location is fixed the force is fixed, whatever is the time t and the velocity $x'(t)$. In the next lectures we will consider the two body problem and the pendulum problem (which can turn around). In the present lecture we will study the most important properties of any equation (0.1). The assumption $G(x) \in C^1$ is required to have the existence and the uniqueness Theorems, see Lecture D, Theorem 1.

In general, $G(x)$ is defined on an open interval of the x-axes, for example (A, B) with finite A, B, or $(0, \infty)$, or $(-\infty, \infty)$.

1. The energy preserving law in the math form

Definition. Let $x(t)$ be any solution of (0.1). The kinematic energy of the solution $x(t)$ is the function

$$K(t) = \frac{(x'(t))^2}{2}.$$

The potential energy of the solution $x(t)$ is the function

$$P(t) = -\int_c^{x(t)} G(s)ds,$$

where c is any number in the interval of the definition of the function $G(x)$ (the potential energy is defined up to a constant). The full energy of the solution $x(t)$ is the function

$$E(t) = K(t) + P(t).$$

Theorem 1. (Math form of the energy preserving law). *Let $x(t)$ be any solution of (0.1) and let $E(t)$ be its full energy. Then $E(t) \equiv const.$*

Proof. We have to prove that $E'(t) \equiv 0$. To prove this calculate:

$$K'(t) = x'(t) \cdot x''(t), \quad P'(t) = -G(x(t)) \cdot x'(t).$$

Consequently

$$E'(t) = x'(t) \cdot \big(x''(t) - G(x(t))\big),$$

and since $x(t)$ is a solution of (0.1) then $E'(t) \equiv 0$. □

If the initial conditions

$$x(t_0) = x_0, \quad x'(t_0) = v_0 \tag{1.1}$$

are given then Theorem 1 leads to the following **energy equation**.

Corollary 2. (the energy equation) *Let $x(t)$ a solution of (0.1) satisfying the initial conditions (1.1). Then*

$$\frac{(x'(t))^2}{2} - \int_{x_0}^{x(t)} G(s)ds = \frac{v_0^2}{2}. \tag{1.2}$$

Proof. By Theorem 1 the left hand side in (1.2) is a constant C. To find this constant substitute $t = t_0$. We obtain $C = v_0^2/2$. □

Example 1. Let $x(t)$ be solution of the equation

$$x'' = \sqrt{x} \tag{1.3}$$

satisfying the initial conditions

$$x(t_0) = 1, \quad x'(t_0) = \pm 1. \tag{1.4}$$

Then the energy equation is as follows:

$$\frac{(x'(t))^2}{2} - \frac{2}{3}(x(t))^{3/2} = -\frac{1}{6}. \tag{1.5}$$

We see that the energy preserving law reduces the second order equation (0.1) to the first order equation: equation (1.2) can be solved with respect to $x'(t)$. For example, the energy equation (1.5) implies

$$x' = \pm\sqrt{\frac{4x^{3/2} - 1}{3}} \tag{1.6}$$

It is **worth to note that this reduction is not complete: we do not know for which t one should take the sign $+$ and for which t the sign $-$**. This requires an additional analysis, see the next two lectures. To understand that this analysis is important note that the energy equation (1.5) is exactly the same for two different solutions of (1.3) – corresponding to the initial conditions (1.4) with $+$ and with $-$. It is clear that for t sufficiently close to t_0 the sign in (1.6) is the same as the sign in (1.4), but for t far from t_0 the signs might be different!

2. Critical points of solutions of (0.1).

By definition, a point t_1 of a function $x(t)$ is critical if $x'(t_1) = 0$. Recall from Lecture B that in the case of equations $x'(t) = F(x(t))$ any non-constant solutions has no critical points – it is either increasing or decreasing. For equations (0.1) this is not so – critical points are possible. Already in the simplest example $x''(t) = 1$ one has a solution $x(t) = t^2/2$ with a critical point $t_1 = 0$. Nevertheless, all critical points of solutions of equation (0.1) are very simple.

Theorem 2.1. *Let $x(t)$ be a non-constant solution of equation (0.1) and let t_1 be a critical point: $x'(t_1) = 0$. Then $x''(t_1) \neq 0$. Consequently the function $x(t)$ has either local maximum or local minimum at t_1.*

Proof. Assume, to get contradiction, that $x''(t_1) = 0$. Let $x(t_1) = A$. Then $G(A) = 0$. Therefore $\hat{x}(t) \equiv A$ is a solution of equation (0.1). It satisfies the conditions $\hat{x}(t_1) = A$, $\hat{x}'(t_1) = 0$. The solution $x(t)$ satisfies the same conditions. By the uniqueness theorem (Lecture D, Theorem 1) one gas $x(t) \equiv A$. But we assumed that $x(t) \not\equiv const$. □

3. Symmetries

The following theorem states that any solution $x(t)$ of equation (0.1) is symmetric about any critical point.

Theorem 3. (symmetries of solutions of (0.1)) *Let $x(t)$ be a solution of (0.1). Let t_1 be a critical point of $x(t)$. Then $x(t_1 + t) = x(t_1 - t)$ provided that $t_1 \pm t$ are the points in the interval of definition of $x(t)$.*

Proof. Consider the functions $\tilde{x}(t) = x(t_1 + t)$ and $\hat{x}(t) = x(t_1 - t)$. It is clear that $\tilde{x}(t)$ is also a solution of (0.1) – exactly like for the first order ODEs $x' = F(x)$ (see Lecture B, Proposition 4: shift of time). **Unlike** first order ODES, the function $\hat{x}(t)$ is also a solution of (0.1). In fact,

$$(\hat{x}(t))'' = (-1) \cdot x'(t_1 - t))' = (-1) \cdot (-1) \cdot x''(t_1 - t) = G(x(t_1 - t)) = G(\hat{x}(t)).$$

Thus we have two solutions $\tilde{x}(t)$ and $\hat{x}(t)$ of the same equation. One has $\tilde{x}(0) = \hat{x}(0) = x(t_1)$. One also has $\tilde{x}'(0) = \hat{x}'(0) = 0$ because t_1 is a critical point. By the uniqueness theorem (Lecture D, Theorem 1) one has $\tilde{x}(t) = \hat{x}(t)$. □

4. Periodic solutions

Consider now the case that a solution $x(t)$ of (0.1) has **two** critical points t_1, t_2. Then by Theorem 3

$$x(t_1 + t) = x(t_1 - t), \quad x(t_2 + s) = x(t_2 - s) \tag{4.1}$$

for any t, s such that the points $t_1 \pm t$ and $t_2 \pm s$ are in the interval of the definition of $x(t)$. Let us show that any function $x(t)$ satisfying (4.1) is **periodic** and (one of) the periods is $T = 2(t_2 - t_1)$. In fact,

$$x(t + T) = x(t + 2(t_2 - t_1)) = x(t_2 + (t + t_2 - 2t_1)).$$

Using the second relation in (4.1) we obtain

$$x(t + T) = x(t_2 - (t + t_2 - 2t_1)) = x(2t_1 - t).$$

Now we use the first relation in (4.1):

$$x(2t_1 - t) = x(t_1 + (t_1 - t)) = x(t_1 - (t_1 - t)) = x(t),$$

therefore $x(t + T) = x(t)$. We have proved

Theorem 4. *If a solution $x(t)$ of equation (0.1) has* **two** *critical points t_1, t_2 then $x(t + T) = x(t)$ with $T = 2(t_2 - t_1)$.*