Lecture F

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Two body problem on a line

You will meet two body problem in few years, when your girlfriend (boyfriend) will be offered a good job in Tel Aviv (or USA) and you – in Haifa, or vise a versa. I hope you will solve this problem somehow. I also hope that you will not deal with a three body problem in this sense.

In math and physics the two body problem is as follows: one (big) body is located all the time at the center of the coordinate system [1] and attracts (or pushes away) another (small) body with a certain force whose direction belongs to the line joining the two bodies. The module of the force depends on the distance between the bodies only.

The classical two body problem is the dynamics (movement) of our Earth (or any other planet) under the gravity of Sun. Let p be the point at which the Earth is located at some time-moment t_0 and let v_0 be the vector of its velocity at the same time-moment. Denote by O the point of the location of Sun. Then it is clear physically (and can be proved mathematically) that all the time the Earth will move in the fixed plane P containing the point O and the vector v. Therefore Earth-Sun is a two body problem on a plane.

Though this problem is the most applicable (people use its solution daily) we have not developed techniques to solve it. The thing is that it requires a *system* of second order ODEs rather than one second order ODE. In fact, the dynamics of a body in a plane is described by two functions $x_1(t)$ and $x_2(t)$.

We will learn a simpler two body problem – on a line. On the other hand, we will not restrict ourselves to a concrete force, it will be given by an arbitrary function.

Namely, fix the following. One (big) body stays all the time at the point $x = 0$ of the x-axes. Another (small) body moves along the x-axes only due to two reasons:

(a) initial position $x_0 > 0$ and initial velocity v_0

(b) the big body attracts (or pushes away) the small one with a force $F = F(x)$ which depends on the coordinate x of the small body only.

The most interesting cases are:

1. Attraction with a force $F(x)$ tending to 0 as $x \to \infty$ and $v_0 > 0$.
2. Pushing away with a force $F(x)$ tending to 0 as $x \to 0$ and $v_0 < 0$.

The most important question, in each of these problems, is as follows: will the bodies meet or not? A particular case of problem 1 is a rocket launched from the surface of the Earth, vertically up, with initial velocity v_0. Then, as you know, if v_0 is sufficiently big then the rocket will not return to the Earth, will tend to ∞ (provided that we think about the Earth and the rocket only, forgetting about other bodies in the Universe). If v_0 is smaller than certain "critical" value then the rocket will return. In the case of the rocket the force $F(x)$ is proportional to $1/x^2$. What will be if $F(x)$ is another function? Will it also be a certain "critical" velocity? Or the bodies will meet however is big v_0?

[1] sometimes this assumption requires a moving coordinate system

Similarly, in problem 2: will the small body tend all the time to the big one and approach it (in finite or infinite time) or, another possibility, the small body after some time will change its velocity from negative to positive and then certainly it will go to ∞?

Of course, except these questions there are many others. For example, assume that in problem 1 the bodies will meet. What is the maximal distance between them? In which time the distance will be maximal? In problem 2: if the bodies do not meet, what is the minimal distance between them? In which time the distance will be minimal?

There are many other natural questions, and we will learn how to answer. I will concentrate on Problem 1. Problem 2 can be solved by a similar method, therefore many questions related to Problem 2 will be left for exercises and tests (of course tests will contain Problem 1 too).

Let us formulate Problem 1 in math terms. The acceleration of the small body is $x''(t)$, where $x(t)$ is its coordinate at time-moment t. The acceleration is equal to force/mass – the second Newton law. The mass m is constant. Let $f(x) = F(x)/m$, where $F(x)$ is the force. We obtain the equation

$$x'' = -f(x). \tag{0.1}$$

We need certain assumptions on the function $f(x)$. At first, we will assume that it is defined for $x > 0$ only. Physically this corresponds to the fact that the small body cannot "go through" the big body. Secondly, since we put the sign minus in the equation, we will assume that $f(x) > 0$ fort any $x > 0$ – this corresponds to the case that the big body attracts the small one. Finally, in order to pass from physical intuition to math theorems, we have to assume that $f(x)$ is not too bad, namely that it satisfies the assumptions of the existence and uniqueness theorem, see Lecture D. Therefore we assume:

$$f(x) \in C^1(0, \infty), \quad f(x) > 0 \text{ for any } x > 0. \tag{0.2}$$

We also fix the initial condition corresponding to the initial position of the small body and its initial velocity like in Problem 1 (against the direction of the force):

$$x(t_0) = x_0 > 0, \quad x'(t_0) = v_0 > 0. \tag{0.3}$$

Now we formulate Problem 1 in math terms:

to analyze solutions of equation (0.1) satisfying the initial conditions (0.3) and defined for $t \geq t_0$; the function $f(x)$ in the equation satisfies (0.2).

Theorem 1. (Two cases = two classes of solutions) *Consider equation (0.1) with $f(x)$ satisfying (0.2). Fix any initial condition of form (0.3). Let $x(t)$ be solution of the equation satisfying these initial conditions and defined on interval (t_0, T), where $T \leq \infty$ is maximal possible. Then $x(t)$ is a* **concave** *(Hebrew: kaura) function belonging to one of the following two classes (cases):*

Class = case A.

$T = \infty$;

$x(t)$ is a strictly increasing function for any $t > t_0$;

$x(t) \to \infty$ as $t \to \infty$.

Class = case B.

T is a finite number;

there exists $t_1 \in (t_0, T)$ such that $x(t)$ has global maximum at the point t_1;

there are no other local maxima or minima: the function $x(t)$ is increasing as $t \in (t_0, t_1)$ and decreasing as $t \in (t_1, T)$;

$x(t) \to 0$ as $t \to T$.

The physical interpretation of cases A and B is clear. The time-moment T in case B is the time-moment when the bodies meet.

Attention. Theorem 1 does not say that there exist solutions of class A and of class B. It only says: either A or B. We will see later that there are always solutions of class B and whether there are solutions of class A – depends on the function (= force) $f(x)$.

Proof of Theorem 1.

Lemma 1.1. *Any solution $x(t)$ is a concave function.*

Proof. The equation $x''(t) = -f(x(t))$ and the condition $f(x) > 0$ imply that $x''(t) < 0$ for any t of the interval of the definition of the solution. □

Lemma 1.2. *Assume that the solution $x(t)$ has a non-vanishing derivative: $x'(t) \neq 0$ for any $t \in (t_0, T)$. Then $T = \infty$.*

Proof. Assume, to get contradiction, that $T < \infty$. The assumption $x'(t) \neq 0$ implies that $x(t)$ is an increasing function. Therefore there is a **finite** limit $\lim_{t\to T} x(t) = B_1$. Since $x(t)$ is concave (Lemma 1.1) then the first order derivative $x'(t)$ is a decreasing function. Therefore there is a **finite** limit $\lim_{t\to T} x'(t) = B_2$. Now we use the theorem on prolongation of solutions (Lecture D, Theorem 2). By this theorem the solution $x(t)$ can be prolonged to $(t_0, T+\epsilon), \epsilon > 0$. Contradiction. □

Lemma 1.3. *Assume that the solution $x(t)$ is a strictly increasing solution defined for $t \in (t_0, \infty)$. Then $\lim_{t\to\infty} x(t) = \infty$.*

Proof. An increasing function tends either to a finite number or to ∞. Assume, to get contradiction, that $x(t) \to A < \infty$ as $t \to \infty$. Then $x(t) < A$ for any $t \in (t_0, \infty)$ and consequently $x(t) \in [x_0, A]$ for any $t \in (t_0, \infty)$. The function $f(x)$ is positive, therefore it takes the minimal value $\delta > 0$ on the **closed** interval $[x_0, A]$. We obtain:

$$x''(t) < -\delta, \quad t \in (t_0, \infty), \quad \delta > 0.$$

It follows, by a simple integration:

$$x'(t) = x'(t_0) + \int_{t_0}^{t} x''(s)ds < v_0 - \delta \cdot (t - t_0).$$

Consequently $x'(t) \to -\infty$ as $t \to \infty$, i.e. $x'(t) < 0$ for sufficiently big t. This contradicts to the assumption that $x(t)$ is an increasing function. □

Lemmas 1.1, 1.2, 1.3 imply:

if $x'(t) \neq 0$ for any t of the interval of the definition of the solution $x(t)$ then one has case A in Theorem 1.

Lemma 1.4. *Assume that there exists $t_1 \in (t_0, T)$ such that $x'(t_1) = 0$. Then such t_1 is unique, the function $x(t)$ has the global max at the point t_1.*

Proof. In fact, $x(t)$ is a concave function (Lemma 1.1), and it is a simple exercise to prove that the lemma holds for **any concave function**. $\square$

Lemma 1.5. *Assume that there exists $t_1 \in (t_0, T)$ such that $x'(t_1) = 0$. Then $T < \infty$.*

Proof. Since $x(t)$ has max at the point t_1 then $x'(t_1 + \epsilon) = \delta < 0$ for sufficiently small $\epsilon > 0$. Since $x''(t) < 0$ for all $t \in (t_0, T)$ then $x'(t) < \delta < 0$ for $t \in (t_1 + \epsilon, T)$. It follows

$$x(t) = x(t_1 + \epsilon) + \int_{t_1+\epsilon}^{t} x'(s)ds < x(t_1 + \epsilon) - \delta(t - t_1 - \epsilon), \quad t \in (t_1 + \epsilon, T).$$

We see that if $T = \infty$ then for sufficiently big t one has $x(t) < 0$. This is impossible since the function $f(x)$ is defined for $x > 0$ only. Therefore T is finite. $\square$

Lemma 1.6. *Assume that there exists $t_1 \in (t_0, T)$ such that $x't_1) = 0$. Then $x(t) \to 0$ as $t \to T$.*

Proof. One has $x(t) > 0$ for any solution $x(t)$ and any t in the interval of its definition. In the interval (t_1, T) the function $x(t)$ is decreasing; its derivative $x'(t)$ is also decreasing. Therefore there are **finite** limits $\lim_{t\to T} x(t) = B_1, \lim_{t\to T} x'(t) = B_2$. We know that $B_1 \geq 0$. If $B_1 > 0$ then by the theorem on prolongation of solutions (Lecture D, Theorem 2) one can prolong the solution $x(t)$ to an interval $(t_0, T + \epsilon), \epsilon > 0$. This contradicts to the definition of T. Therefore $B_1 = 0$. $\square$

Lemmas 1.4,1.5,1.6 imply:

if there exists a point $t_1 \in (t_0, T)$ such that $x'(t_1) = 0$ then one has case B in Theorem 1.

Now Theorem 1 is completely proved.

How to distinguish cases A and B? To answer we use the energy preserving law, in math terms (Lecture E, Theorem 1, Corollary 2). One has

$$\frac{(x'(t))^2}{2} - \int_{x_0}^{x(t)} -f(s)ds = \frac{v_0^2}{2},$$

or, simplifying

$$\frac{(x'(t))^2}{2} + \int_{x_0}^{x(t)} f(s)ds = \frac{v_0^2}{2}. \tag{0.4}$$

Theorem 2.

$$\text{case A} \iff \frac{v_0^2}{2} \geq \int_{x_0}^{\infty} f(s)ds \qquad \text{case B} \iff \frac{v_0^2}{2} < \int_{x_0}^{\infty} f(s)ds$$

Proof. Relation (0.4) implies

$$\frac{v_0^2}{2} \geq \int_{x_0}^{x(t)} f(s)ds.$$

This holds for any t of the interval of definition of $x(t)$. In case A $x(t)$ is defined for arbitrarily big t and tends to ∞ as $t \to \infty$. Taking the limit as $t \to \infty$ we obtain:

Claim 2.1. *In case A one has* $\frac{v_0^2}{2} \geq \int_{x_0}^{\infty} f(s)ds$.

Consider now case B. In this case there exists t_1 such that $x'(t_1) = 0$. Substituting t_1 to (0.4) we obtain

$$\frac{v_0^2}{2} = \int_{x_0}^{x_{max}} f(s)ds, \quad \text{where } x_{max} = x(t_1).$$

since $f(x) > 0$ for any $x > 0$, this relation implies:

Claim 2.2 *In case B one has* $\frac{v_0^2}{2} < \int_{x_0}^{\infty} f(s)ds$.

Theorem 2 is a **logical** corollary of Claims 2.1 and 2.2. □

What happens if the integral $\int_{x_0}^{\infty} f(x)dx$ diverges? We obtain the following corollary of Theorem 2:

Theorem 3. (direct corollary of Theorem 2).

If $\int_{x_0}^{\infty} f(x)dx = \infty$ *then in the two body problem case A is impossible, i.e. the case B holds for any* x_0, v_0. *If* $\int_{x_0}^{\infty} f(x)dx < \infty$ *then either of the cases A,B is possible. Define the "critical" velocity*

$$v_0^{\text{crit}} = \sqrt{2\int_{x_0}^{\infty} f(x)dx}.$$

The case A holds if and only if $v_0 \geq v_0^{\text{crit}}$ *and the case B holds if and only if* $v_0 < v_0^{\text{crit}}$. *(Of course,* v_0^{crit} *depends on* x_0*).*

Example 1. Let us answer the following questions for a rocket that has been launched from the surface of the Earth, vertically up, with the initial velocity v_0. Assume that the gravity to the Earth is the only acting force. Questions:

Q1. Will the rocket return to the surface of the Earth?

Q2: If yes – when?

Q3: What is the maximum height of the rocket?

Q4: In which time the rocket will reach its maximum height?

Q5: In which time the height of the rocket will be 10 km?

Solution. According to the gravity law one has the equation

$$x'' = -k/x^2,$$

where x is the distance between the rocket and the center of the Earth, and k is a certain positive coefficient. It can be easily found because we know that the acceleration near the surface of the Earth is equal to $-g$. Therefore $k/R^2 = g$, where R is the radius of the Earth. We obtain

$$x'' = -gR^2/x^2.$$

We also have the initial conditions

$$x(t_0) = R, \quad x'(t_0) = v_0.$$

Calculate

$$\int_R^\infty \frac{gR^2}{x^2} dx = gR.$$

By Theorem 3 the rocket will return if and only if

$$v_0 < \sqrt{2gR}. \tag{0.5}$$

We have answered the first question.

Now let us answer questions Q2, Q3, Q4 (assuming, of course, that the rocket will return, i.e. (0.5) holds). We use the energy preserving law (Lecture E, Theorem 1 and Corollary 2). The energy equation is as follows:

$$\frac{(x'(t))^2}{2} - \int_R^{x(t)} -\frac{gR^2}{x^2} ds = \frac{v_0^2}{2}.$$

Taking the integral we obtain

$$(x'(t))^2 - \frac{2gR^2}{x(t)} = v_0^2 - 2gR. \tag{0.6}$$

Let us find $x_{max} = x(t_1)$, where t_1 is the time-moment such that $x'(t_1) = 0$, i.e. the time-moment at which the height is maximal. To find t_1 it suffices to substitute t_1 to (0.6). We obtain

$$x_{max} = \frac{2gR^2}{2gR - v_0^2}. \tag{0.7}$$

Note that the maximum height of the rocket is $x_{max} - R$, since x is the distance to the center of the Earth.

We have answered Q3.

Question Q4: we have to find $t_1 - t_0$, where t_0 is the time-moment at which the rocket was launched. We use the inverse function $t(x)$. It is **not defined for all** t, but it is well-defined for

$$t \in (t_0, t_1), \quad x \in (R, x_{max}).$$

In this interval $x'(t) > 0$, therefore

$$\frac{dx}{dt} = +\sqrt{v_0^2 - 2gR + \frac{2gR^2}{x}},$$
$$\frac{dt}{dx} = +\frac{1}{\sqrt{v_0^2 - 2gR + \frac{2gR^2}{x}}},$$

and we obtain that the rocket will reach its maximum height in time

$$t_1 - t_0 = T = \int_R^{x_{max}} \frac{dx}{\sqrt{v_0^2 - 2gR + \frac{2gR^2}{x}}}. \tag{0.8}$$

We have answered Q4.

The answer to Q2 (when the rocket will return) follows from the theorem on symmetries of solutions of equations $x'' = G(x)$ about a critical point (Lecture E, Theorem 3). This theorem implies that the rocket will return to the surface of the Earth in time

$$2T = 2 \cdot \int_R^{\frac{2gR^2}{2gR - v_0^2}} \frac{dx}{\sqrt{v_0^2 - 2gR + \frac{2gR^2}{x}}}.$$

We have answered Q1 - Q4.

Now we deal with Q5. We have to give the answer in each of the cases: the rocket does not return (case A); the rocket returnes (case B).

Denote by t_2 the time-moment such that $x(t_2) = 10 + R$ (i.e. the height of the rocket is 10 km, provided that R is measured in km). We have to find $t_2 - t_0$.

If $v_0 \geq \sqrt{2gR}$ then we have case A, $x(t)$ is an increasing function, and therefore there will be unique time-moment t_2 such that $x(t_2) = 10 + R$ (i.e. the height of the rocket is 10 km, provided that R is measured in km). The time-moment t_2 can be found in the same way as above:

$$t_2 = t_0 + \int_R^{10+R} \frac{dx}{\sqrt{v_0^2 - 2gR + \frac{2gR^2}{x}}}.$$

If we wish to get the answer in DAYS then R must be measured in km and g must be measured in km/day^2.

If $v_0 < \sqrt{2gR}$ then we have case B. In this case one has to calculate x_{max}, see (0.7), and dependently on v_0:

if $x_{max} < 10 + R$ then there are no solutions – the rocket will never reach the height $10km$.

if $x_{max} = 10 + R$ then $t_2 = t_1$ and $t_2 - t_0 = T$, see (0.8).

if $x_{max} > 10 + R$ then there will be **two** time-moments $t_2' < t_2''$ such that

$$x(t_2') = x(t_2'') = 10 + R.$$

To find t_2' we solve equation (0.6) on the interval $t \in (t_0, t_2')$, where $x'(t) > 0$, and exactly like above (using the inverse function) we obtain

$$t_2' = t_0 + \int_R^{10+R} \frac{dx}{\sqrt{v_0^2 - 2gR + \frac{2gR^2}{x}}}. \tag{0.9}$$

To find t_2'' there are two ways. In any case we need the time t_1 to the maximal height, which we already know, see (0.8).

First way to find t_2''. We solve equation (0.6) on the interval $t \in (t_1, t_2'')$, where $x'(t) < 0$. We obtain

$$\frac{dx}{dt} = -\sqrt{v_0^2 - 2gR + \frac{2gR^2}{x}},$$
$$\frac{dt}{dx} = -\frac{1}{\sqrt{v_0^2 - 2gR + \frac{2gR^2}{x}}},$$

$$t_2'' = t_1 + \int_{x_{max}}^{10+R} -\frac{dx}{\sqrt{v_0^2 - 2gR + \frac{2gR^2}{x}}} = t_1 + \int_{10+R}^{x_{max}} \frac{dx}{\sqrt{v_0^2 - 2gR + \frac{2gR^2}{x}}}. \tag{0.10}$$

Substituting $t_1 = (0.8)$ we obtain

$$t_2'' = t_0 + \int_R^{x_{max}} \frac{dx}{\sqrt{v_0^2 - 2gR + \frac{2gR^2}{x}}} + \int_{10+R}^{x_{max}} \frac{dx}{\sqrt{v_0^2 - 2gR + \frac{2gR^2}{x}}}. \tag{0.11}$$

Second way to find t_2''. We use the theorem on symmetries of solutions of equations $x'' = G(x)$ about a critical point (Lecture E, Theorem 3). By this theorem

$$t_2'' - t_1 = t_1 - t_2'$$

and consequently

$$t_2'' = 2t_1 - t_2'. \tag{0.12}$$

Substituting $t_1 = (0.8)$ and $t_2' = (0.9)$ we obtain

$$t_2'' = t_0 + 2\int_R^{x_{max}} \frac{dx}{\sqrt{v_0^2 - 2gR + \frac{2gR^2}{x}}} - \int_R^{10+R} \frac{dx}{\sqrt{v_0^2 - 2gR + \frac{2gR^2}{x}}}. \tag{0.13}$$

Of course (0.13) and (0.11) is the same answer (check it).