Lecture G. Linear ODEs of order k with constant coefficients

We will study equations of the form

$$x^{(k)} + a_{k-1}x^{(k-1)} + a_{k-2}x^{(k-2)} + \cdots + a_2x'' + a_1x' + a_0x = 0, \tag{1}$$

and of the form

$$x^{(k)} + a_{k-1}x^{(k-1)} + a_{k-2}x^{(k-2)} + \cdots + a_2x'' + a_1x' + a_0x = f(t), \tag{2}$$

where $x = x(t)$, $x^{(i)}$ denotes the ith derivative, and $a_0, a_1.,,,, a_{k-1}$ are real numbers, and in (2) $f(t)$ is a continuous function defined for all t. The initial conditions are as follows:

$$x(t_0) = c_0,\ x'(t_0) = c_1,\ \cdots,\ x^{(k-1)}(t_0) = c_{k-1}, \tag{3}$$

where $c_0, ..., c_{k-1}$ are given real numbers.

Theorem 0.1.

1. *Fix equation (1) or equation (2) and fix initial conditions (3). The equation has a solution $x(t)$ defined for all t and satisfying the initial conditions. Such solution is unique.*

2. *The set of all solutions of (2) has the form $x^*(t) + x(t)$, where $x^*(t)$ is ANY FIXED SINGLE solution of this equation and $x(t)$ is an ARBITRARY solution of the corresponding homogeneous equation (1).*

3. *The set of real-valued solutions of (1) is a k-dimensional vector space over the field $\mathbb{R}$ (it is a subspace of the space of C^∞ real-valued functions).*

4. *The set of complex-valued solutions of (1) is a k-dimensional vector space over the field $\mathbb{C}$ (it is a subspace of the space of C^∞ complex-valued functions). If $a_0, ..., a_{k-1}$ are real numbers and $x(t)$ is a complex-valued solution then the complexly-conjugate function $\bar{x}(t)$ is also a solution.*

To find a basis of the space of all solutions of (1) and to find a partial (i.e. any single) solution of (2) it is convenient to introduce the following notation.

Notation. Given a polynomial

$$P(\lambda) = \lambda^k + a_{k-1}\lambda^{k-1} + a_{k-2}\lambda^{k-2} + \cdots + a_2\lambda^2 + a_1\lambda + a_0 \tag{4}$$

we will denote by $P(\frac{d}{dt})$ the linear operator in the space of all C^∞ functions defined for all t such that

(5)

$$P\Big(\frac{d}{dt}\Big)(f(t)) = \Big(\frac{d^k}{dt^k} + a_{k-1}\frac{d^{k-1}}{dt^{k-1}} + a_{k-2}\frac{d^{k-2}}{dt^{k-2}} + \cdots + a_2\frac{d^2}{dt^2} + a_1\frac{d}{dt} + a_0I\Big)(f(t)).$$

Here $\frac{d^i}{dt^i}$ denotes the ith derivative and I denotes the identity operator.

Example 0.2. Let $P(\lambda) = \lambda^3 - 4\lambda^2 + 5\lambda - 8$. Then

$$P\Big(\frac{d}{dt}\Big)(sint) = (sint)''' - 4(sint)'' + 5(sint)' - 8sint = 4\cdot(cost - sint).$$

We can write equations (1) and (2) in the form

$$P\Big(\frac{d}{dt}\Big)(x(t)) = 0, \tag{6}$$

$$P\Big(\frac{d}{dt}\Big)(x(t)) = f(t), \tag{7}$$

where P is the polynomial (4). It is easy to prove

Lemma 0.3. *For any $\lambda_0 \in \mathbb{C}$ one has*

$$P\Big(\frac{d}{dt}\Big)(e^{\lambda_0 t}) = P(\lambda_0) \cdot e^{\lambda_0 t}.$$

An immediate corollary of this lemma is:

Lemma 0.4. *If λ_0 is a root of polynomial (4) then $e^{\lambda_0 t}$ is a solution of (6). If $a_0, ..., a_{k-1}$ are real then $e^{\bar{\lambda}_0 t}$ is also a solution of (6), as well as the real-valued functions $Re(e^{\lambda_0 t})$ and $Im(e^{\lambda_0 t})$.*

Now we need the following statement.

Proposition 0.5. *Let $\lambda_1, ..., \lambda_k$ be* **distinct** *complex numbers. Then the functions $e^{\lambda_1 t}, ..., e^{\lambda_k t}$ are linearly independent over $\mathbb{C}$.*

This proposition, Theorem 0.1 and Lemma 0.4 imply the following way of finding a basis of the space of all solutions of (6).

Theorem 0.6. *Assume that the polynomial (4) has k distinct complex roots $\lambda_1, ..., \lambda_k$. Then the tuple $e^{\lambda_1 t}, ..., e^{\lambda_k t}$ is an example of a basis of the space of all complex-valued solutions of (6). If $a_0, ..., a_{k-1}$ are real numbers then to present an example of a basis of the space of all real-valued solutions of (6) one has to separate the tuple $\lambda_1, ..., \lambda_k$ onto two groups:*

$$\{\lambda_1, ..., \lambda_k\} = \{\mu_1, ..., \mu_p, \nu_1, \bar{\nu}_1, ..., \nu_s, \bar{\nu}_s\},$$

where $\mu_1, ..., \mu_p$ are real and $\nu_1, ..., \nu_s$ are not real (note that $k = p + 2s$). Then the tuple

$$e^{\mu_1 t}, ..., e^{\mu_p t},\ Re\big(e^{\nu_1 t}\big), Im\big(e^{\nu_1 t}\big), \cdots, Re\big(e^{\nu_s t}\big), Im\big(e^{\nu_s t}\big)$$

is an example of a basis of the space of all real-valued solutions.

Example 0.7. Consider the equation

$$x'''(t) + 3x''(t) + 4x'(t) + 2x(t) = 0.$$

It can be written in the form (6) with

$$P(\lambda) = \lambda^3 + 3\lambda^2 + 4\lambda + 2.$$

The polynomial $P(\lambda)$ has 3 distinct roots $-1, -1 \pm i$. Therefore an example of a basis of the space of all real valued solutions is

$$e^{-t},\ e^{-t}cost,\ e^{-t}sint.$$

Example 0.8. Consider the equation $x^{(10)}(t) = x(t)$. It can be written in the form (6) with

$$P(\lambda) = \lambda^{10} - 1.$$

The polynomial $P(\lambda)$ has 10 distinct roots, namely

$$1,\ -1,$$
$$\mu_1 = cos(\pi/5) + isin(\pi/5), \quad \mu_2 = cos(2\pi/5) + isin(2\pi/5),$$
$$\mu_3 = cos(3\pi/5) + isin(3\pi/5), \quad \mu_4 = cos(4\pi/5) + isin(4\pi/5),$$
$$\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3, \bar{\mu}_4.$$

Therefore we obtain the following example of a basis of the space of all real-valued solutions:

$$e^t,\ e^{-t},$$
$$e^{cos\left(\frac{j\pi t}{5}\right)} \cdot cos\left(sin\left(\frac{j\pi t}{5}\right)\right), \quad e^{cos\left(\frac{j\pi t}{5}\right)} \cdot sin\left(sin\left(\frac{j\pi t}{5}\right)\right), \quad j = 1, 2, 3, 4.$$

Now we pass to the general case, when the polynomial (4) has $s \leq k$ distinct complex roots $\lambda_1, ..., \lambda_s$ of multiplicities $r_1, ..., r_s$. Then $r_1 + \cdots + r_s = k$, the degree of $P(\lambda)$ (this is the basis theorem of algebra). The case we considered above is the case $s = k$ and consequently $r_1 = \cdots = r_k = 1$.

To deal with the general case we need the following proposition.

Proposition 0.9. *Let $P(\lambda)$ and $Q(\lambda)$ be polynomials of any degrees. Then*

$$P\Big(\frac{d}{dt}\Big) \circ Q\Big(\frac{d}{dt}\Big) = (P \cdot Q)\Big(\frac{d}{dt}\Big).$$

Remark. Here $\circ$ denotes the composition of two operators. In algebraic terms of rings Proposition 0.9 means that the ring of all polynomials (with operations sum and multiplication) is isomorphic to the ring of operators of the form (5) with operations: sum and composition.

We also need the following simple calculation. Denote

$$L_{\lambda_0} = \lambda - \lambda_0.$$

Then

$$L_{\lambda_0}\Big(\frac{d}{dt}\Big)(e^{\lambda_0 t}) = 0.$$

Calculate also

$$L_{\lambda_0}\Big(\frac{d}{dt}\Big)(te^{\lambda_0 t}) = e^{\lambda_0}t.$$

Calculate now

$$L^2_{\lambda_0}\Big(\frac{d}{dt}\Big)(te^{\lambda_0 t}) = \text{ (by Proposition 0.9) } =$$
$$= L_{\lambda_0}\Big(\frac{d}{dt}\Big)\bigg(L_{\lambda_0}\Big(\frac{d}{dt}\Big)(te^{\lambda_0 t})\bigg) = L_{\lambda_0}\Big(\frac{d}{dt}\Big)(e^{\lambda_0 t}) = 0.$$

Continuing calculations in this way it is easy to prove (by induction) the following

Lemma 0.10. *For any $j \geq 0$ one has*

$$L^j_{\lambda_0}\Big(\frac{d}{dt}\Big)(t^j e^{\lambda_0 t}) = j! \cdot e^{\lambda_0 t} \tag{8}$$

$$L^{j+1}_{\lambda_0}\Big(\frac{d}{dt}\Big)(t^j e^{\lambda_0 t}) = 0. \tag{9}$$

Assume that λ_0 is a root of $P(\lambda)$ of multiplicity r. This means that $P(\lambda)$ can be expressed in the form

$$P(\lambda) = (\lambda - \lambda_0)^r \cdot Q(\lambda), \quad Q(\lambda_0) \neq 0,$$

where $Q(\lambda)$ is another polynomial. By Proposition 0.9 we have:

$$P\Big(\frac{d}{dt}\Big)(t^j e^{\lambda_0 t}) = Q\Big(\frac{d}{dt}\Big)\bigg(L^r_{\lambda_0}\Big(\frac{d}{dt}\Big)(t^j e^{\lambda_0 t})\bigg).$$

Now equation (9) implies that if $j \leq r-1$ then $P\Big(\frac{d}{dt}\Big)(t^j e^{\lambda_0 t}) = 0$. Therefore we obtain the following:

Proposition 0.11. *Let $\lambda_1, ..., \lambda_s$ be distinct complex roots of $P(\lambda)$ of multiplicities $r_1, ..., r_s$ respectively. Then the functions*

$$\begin{aligned} &e^{\lambda_1 t}, te^{\lambda_1 t}, ..., t^{r_1 - 1} e^{\lambda_1 t}, \\ &e^{\lambda_2 t}, te^{\lambda_2 t}, ..., t^{r_2 - 1} e^{\lambda_1 t}, \\ &\qquad \cdots \\ &e^{\lambda_s t}, te^{\lambda_s t}, ..., t^{r_s - 1} e^{\lambda_s t} \end{aligned} \tag{10}$$

are solutions if the equation (6).

The number of solutions in (10) is $r_1 + \cdots + r_s = k$ which is the dimension of the space of all solutions.

Theorem 0.12. *The functions (10) are linearly independent over $\mathbb{C}$ and consequently (10) is an example of a basis of the space of all complex-valued solutions of the equation (6).*

This basis of the space of complex-valued solutions can be transferred to a basis of the space of all real-valued solutions in the same way as above.

Example 0.13. Consider the equation

$$P\Big(\frac{d}{dt}\Big)(x(t)) = 0, \quad P(\lambda) = (\lambda^3 - 1)^4 \cdot (\lambda^2 - 1)^2.$$

This equation has order 16 (the degree of $P(\lambda)$). The polynomial $P(\lambda)$ has 4 distinct roots:

$\lambda_1 = 1$ multiplicity 6

$\lambda_2 = -1$ multiplicity 2

$\lambda_3 = (-1 + \sqrt{3}i)/2$ multiplicity 4

$\lambda_4 = \bar{\lambda}_3 = (-1 - \sqrt{3}i)/2$ multiplicity 4.

Therefore the 16 functions

$$\begin{gathered} e^t,\ te^t,\ t^2e^t,\ t^3e^t,\ t^4e^t,\ t^5e^t \\ e^{-t},\ te^{-t}, \\ e^{(-1+\sqrt{3}i)t/2},\ te^{(-1+\sqrt{3}i)t/2},\ t^2e^{(-1+\sqrt{3}i)t/2},\ t^3e^{(-1+\sqrt{3}i)t/2} \\ e^{(-1-\sqrt{3}i)t/2},\ te^{(-1-\sqrt{3}i)t/2},\ t^2e^{(-1-\sqrt{3}i)t/2},\ t^3e^{(-1-\sqrt{3}i)t/2} \end{gathered}$$

is an example of a basis of the space of all complex-valued solutions, and the 16 functions

$$e^t,\ te^t,\ t^2e^t,\ t^3e^t,\ t^4e^t,\ t^5e^t,\ e^{-t},\ te^{-t},$$
$$e^{-t/2}cos(\sqrt{3}t/2),\ te^{-t/2}cos(\sqrt{3}t/2),\ t^2e^{-t/2}cos(\sqrt{3}t/2),\ t^3e^{-t/2}cos(\sqrt{3}t/2),$$
$$e^{-t/2}sin(\sqrt{3}t/2),\ te^{-t/2}sin(\sqrt{3}t/2),\ t^2e^{-t/2}sin(\sqrt{3}t/2),\ t^3e^{-t/2}sin(\sqrt{3}t/2)$$

is an example of a basis of the space of all real-valued solutions.

Now we have a way to solve equation (6) for any polynomial $P(\lambda)$. We also can solve equation (7) provided that we know a partial solution of this equation (i.e. any single solution). How to find a single solution of (7)? In the class I gave a way which works for any right hand side $f(t)$ (transferring to a system of first order ODEs and using the method of variations of constants). This way requires many calculations. Finding a single solution of (7) is a much simpler task if the right hand side $f(t)$ is an exponent $e^{\lambda_0 t}$ or a real or imaginary part of $e^{\lambda_0 t}, a \in \mathbb{C}$, i.e. a function of the form $e^{at}cos(bt)$ or $e^{at}sin(bt)$ with real a, b.

Theorem 0.14. *Consider the equations*

$$P\left(\frac{d}{dt}\right) = e^{\lambda_0 t}, \tag{11}$$

$$P\left(\frac{d}{dt}\right) = Re\left(e^{\lambda_0 t}\right), \tag{12}$$

$$P\left(\frac{d}{dt}\right) = Im\left(e^{\lambda_0 t}\right), \tag{13}$$

where λ_0 is a complex (in particular real) number. If λ_0 is NOT a root of the polynomial $P(\lambda)$ then the equation (11) has a solution

$$x^*(t) = \frac{e^{\lambda_0 t}}{P(\lambda_0)}.$$

If the coefficients of $P(\lambda)$ are real then equation (12) has a solution $Re(x^*(t))$, and equation (13) has a solution $Im(x^*(t))$.

Proof. The first statement is a direct corollary of Lemma 0.3. The second statement follows from the following observation: if $P(\lambda)$ has real coefficients and $P\left(\frac{d}{dt}\right)(f(t)) = g(t)$ then $P\left(\frac{d}{dt}\right)(\bar{f}(t)) = \bar{g}(t)$. $\square$

Example 0.15. Consider the equation

$$x'''(t) + 3x''(t) + 4x'(t) + 2x(t) = e^{-t} \cdot sin(2t). \tag{14}$$

It can be written in the form $P\left(\frac{d}{dt}\right)(x(t)) = Im\left(e^{(-1+2i)t}\right)$, where

$$P(\lambda) = \lambda^3 + 3\lambda^2 + 4\lambda + 2.$$

The polynomial $P(\lambda)$ has 3 distinct roots $-1, -1 \pm i$. The number $-1+2i$ is not a root of $P(\lambda)$. Therefore the equation has a solution

$$x^*(t) = Im\left(\frac{e^{(-1+2i)t}}{P(-1+2i)}\right) = e^{-t}(a^*cos(2t) + b^*sin(2t)),$$

where a^* and b^* are real numbers that can be easily calculated (I do not calculate them in this Lecture Notes). In Example 0.7 we found an example of a basis of the space of all real-valued solutions of the equation $P\Big(\frac{d}{dt}\Big)(x(t)) = 0$, it is e^{-t}, $e^{-t}cost$, $e^{-t}sint$. Therefore the general real-valued solution (the set of all solutions) of (14) is

$$e^{-t}(a^*cos(2t) + b^*sin(2t)) + C_1e^{-t} + C_2e^{-t}cost + C_3e^{-t}sint, \quad C_1, C_2, C_3 \in \mathbb{R}.$$

How to solve equation (11) (and consequently equations (12), (13)) if λ_0 is a root of $P(\lambda)$? In this case

$$(15) \qquad P(\lambda) = L^r_{\lambda_0} \cdot Q(\lambda), \quad L_{\lambda_0} = \lambda - \lambda_0, \quad Q(\lambda_0) \neq 0, \quad r: \text{ multiplicity of } \lambda_0.$$

Calculate

$$P\Big(\frac{d}{dt}\Big)(t^r e^{\lambda_0 t}) = \text{ (by Proposition 0.9) } =$$

$$= Q\Big(\frac{d}{dt}\Big)\bigg(L^r_{\lambda_0}\Big(\frac{d}{dt}\Big)((t^r e^{\lambda_0 t})\bigg) = \text{ (by equation (8) in Lemma 0.10) } =$$

$$= Q\Big(\frac{d}{dt}\Big)(r!e^{\lambda_0 t}) = \text{ (by Lemma 0.3) } = r!Q(\lambda_0)e^{\lambda_0 t}.$$

This calculation implies the following statement.

Theorem 0.16. *Let λ_0 be a root of $P(\lambda)$ of multiplicity r. Express $P(\lambda)$ in the form (15). The equation (11) has a solution*

$$(16) \qquad x^*(t) = \frac{t^r e^{\lambda_0 t}}{r! \cdot Q(\lambda_0)}.$$

If $P(\lambda)$ has real coefficients then $Re(x^*(t))$ and $Im(x^*(t))$ are solutions of equations (12) and (13)) respectively.

Note now that

$$r!Q(\lambda_0) = P^{(r)}(\lambda_0),$$

therefore the same partial solution can be expressed in the form

$$(17) \qquad x^*(t) = \frac{t^r e^{\lambda_0 t}}{P^{(r)}(\lambda_0)}.$$

What is better to use: (16) or (17)? To use (16) we need to divide $P(\lambda)$ over $(\lambda - \lambda_0)^r$. To use (17) we have to calculate the rth derivative of $P(\lambda)$. What is simpler depends on a problem we are solving.

Example 0.17. Let us find a partial solution of the equation

$$P\Big(\frac{d}{dt}\Big)(x(t)) = e^t, \quad P(\lambda) = (\lambda^3 - 1)^4 \cdot (\lambda^2 - 1)^2.$$

The number 1 is a root of $P(\lambda)$ of multiplicity 6. Here it is more convenient tot use (16). For that, write $P(\lambda)$ in the form

$$P(\lambda) = (\lambda - 1)^6 \cdot Q(\lambda), \quad Q(\lambda) = (\lambda^2 + \lambda + 1)^4(\lambda + 1)^2.$$

We obtain a partial solution

$$x^*(t) = \frac{t^6 e^t}{6!Q(1)} = \frac{t^6 e^t}{6! \cdot 3^4 \cdot 4}.$$

Example 0.18. Let us find a partial solution of the equation

$$x^{(6)}(t) + x^{(5)}(t) + 3x^{(4)}(t) - x'(t) - 2x(t) = sint.$$

This equation can be written in the form

$$P\Big(\frac{d}{dt}\Big)(x(t)) = Im(e^{it}), \quad P(\lambda) = \lambda^6 + \lambda^5 + 3\lambda^4 - \lambda - 2.$$

Note that $P(i) = 0$ and $P'(i) \neq 0$, i.e. i is a root of $P(\lambda)$ of multiplicity 1. In this example it is easier to use (17). We obtain a partial solution

$$x^*(t) = Im\bigg(\frac{te^{it}}{P'(i)}\bigg) = t \cdot Im\bigg(\frac{cost + isint}{6i^5 + 5i^4 + 12i^3 - 1}\bigg) =$$

$$= t \cdot Im\bigg(\frac{cost + isint}{4 - 6i}\bigg) = \frac{t}{52} \cdot Im\Big((cost + isint) \cdot (4 + 6i)\Big) = \frac{t}{52} \cdot (6cost + 4sint).$$