ODEs 104285. Bohan-2. June 6, 2012

2 problems (50 pts each). 1 hour. Final answers and short explanations are required. Attach your work to *this* page. Do not forget to write down your t/z number here:

t/z:

You can use any notes, books, calculators. If you wish to compare your and my answers *today* copy your answers and the code of your version in a separate page and take it home. Good luck!

1. Let $x = x(t)$ be solution of the equation

$$x''(t) = -\frac{1}{x^3(t)}$$

satisfying the initial conditions

$$x(0) = 2, \quad x'(0) = v_0 > 0$$

(think on the two body problem). Find a necessary and sufficient condition on $v_0 > 0$ under which there exists $t_1 > 0$ such that

$$x'(t_1) = \frac{1}{2}v_0.$$

No integrals in the answer.

Solution. Certainly t_1 exists if v_0 is smaller than the "critical velocity" $v_{0,crit}$. If $v_0 \leq v_{0,crit}$ then $x(t) \to \infty$ as $t \to \infty$ and $x'(t)$ decreases and tends to a certain number which might be either smaller or bigger than $\frac{v_0}{2}$. Therefore in the case $v_0 \geq v_{0,crit}$ the required t_1 exists only if v_0 is not too big.

The energy equation and the initial conditions give

$$(x'(t))^2 - \frac{1}{x^2(t)} = const = v_0^2 - \frac{1}{4}$$

and the required t_1 exists if and only if the equation

$$\left(\frac{1}{2}v_0\right)^2 - \frac{1}{x^2} = v_0^2 - \frac{1}{4}$$

has a solution $x > 0$. It means that $v_0 < \frac{1}{\sqrt{3}}$ which is the final answer. Note that the critical velocity is $v_{0,crit} = \frac{1}{2} < \frac{1}{\sqrt{3}}$.

See next page for Problem 2

2. A pendulum is described by the equation

$$\theta'' = -\frac{g}{l} sin\theta$$

where l is the pendulum length, the angle θ is measured in radians, the positive direction is anticlockwise, and $\theta = 0 \ mod \ 2k\pi$ corresponds to the position of stable equilibrium. At the initial time $t = 0$ the pendulum is horizontal ($\theta(0) = \theta_0 = \pi/2$) and has initial velocity $\theta'(0) = v_0 = \sqrt{g}$ rad/sec directed anticlockwise. In which time t_1 the pendulum will pass (for the first time) the position of stable equilibrium? This means that

$$t_1 = min\{t \geq 0 : \quad \theta(t_1) = 0 \quad mod \quad 2\pi k\}$$

The answer depends on the length l and you should solve the problem for

a) the length l is 1 meter b) the length l is 3 meters.

Integrals in the answers are OK.

Solution. The critical velocity for $\theta_0 = \pi/2$ is $v_{0,crit} = \sqrt{\frac{2g}{l}}$.

In case a), when $l = 1$ one has $v_0 < v_{0,crit}$, therefore the pendulum will *not* make a full rotation. In this case the energy equation has the form

$$(\theta'(t))^2 - 2gcos\theta = g$$

and at first one should find $\theta_{\max}$ substituting $t = t^*$ such that $\theta'(t^*) = 0$. We obtain $cos\theta_{max} = -\frac{1}{2}$ and consequently $\theta_{max} = \frac{2\pi}{3}$. For $\pi/2 \leq \theta \leq \theta_{max}$ one has, from the energy equation

$$\theta' = +\sqrt{g(1 + 2cos\theta)}$$

from where

$$t^* = \int_{\pi/2}^{2\pi/3} \frac{d\theta}{\sqrt{g(1 + 2cos\theta)}}.$$

For $t^* \leq t \leq t_1$ one has

$$\theta' = -\sqrt{g(1 + 2cos\theta)}$$

from where

$$t_1 - t^* = -\int_{2\pi/3}^{0} \frac{d\theta}{\sqrt{g(1 + 2cos\theta)}} = \int_{0}^{2\pi/3} \frac{d\theta}{\sqrt{g(1 + 2cos\theta)}}.$$

Therefore

$$t_1 = \int_{\pi/2}^{2\pi/3} \frac{d\theta}{\sqrt{g(1 + 2cos\theta)}} + \int_{0}^{2\pi/3} \frac{d\theta}{\sqrt{g(1 + 2cos\theta)}}.$$

In case b), when $l = 3$ one has $v_0 > v_{0,crit}$, therefore the pendulum make a full rotation. In this case the energy equation takes the form

$$(\theta'(t))^2 - \frac{2}{3}gcos\theta = g.$$

One has $\theta'(t) > 0$ for all t, therefore $\theta' = +\sqrt{g(1 + \frac{2}{3}cos\theta)}$ and we obtain

$$t_1 = \int_{\pi/2}^{2\pi} \frac{d\theta}{\sqrt{g(1 + \frac{2}{3}cos\theta)}}.$$