Points and curves in the Monster tower

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Abstract. Cartan introduced the method of prolongation which can be applied either to manifolds with distributions (Pfaffian systems) or integral curves to these distributions. Repeated application of prolongation to the plane endowed with its tangent bundle yields the Monster tower, a sequence of manifolds, each a circle bundle over the previous one, each endowed with a rank 2 distribution. In [**MZ**] we proved that the problem of classifying points in the Monster tower up to symmetry is the same as the problem of classifying Goursat distribution flags up to local diffeomorphism. The first level of the Monster tower is a three-dimensional contact manifold and its integral curves are Legendrian curves. The philosophy driving the current work is that all questions regarding the Monster tower (and hence regarding Goursat distribution germs) can be reduced to problems regarding Legendrian curve singularities. Here we establish a canonical correspondence between points of the Monster tower and finite jets of Legendrian curves. We show that each point of the Monster can be realized by evaluating the k-fold prolongation of an analytic Legendrian curve. Singular points arise from singular curves. The first prolongation of a point, i.e. a constant curve, is the circle fiber over that point. These curves are called vertical curves. The union of the vertical curves and their prolongations form the abnormal curves (in the sense of sub-Riemannian geometry) for the Monster distribution. Using these curves we define three types of points - regular (R), vertical (V) , and tangency (T) and from them associated singularity classes, the RVT classes. The RVT classes corresponds to singularity classes in the space of germs of Legendrian curves. All previous classification results for Goursat flags (many obtained by long calculation) now follow from this correspondence as corollaries of well-known results in the classification of Legendrian curve germs. Using the same correspondence we go beyond known results and obtain the determination and classification of all simple points of the Monster, and hence all simple Goursat germs. Finally, as spin-off to these ideas we prove that any plane curve singularity admits a resolution via a finite number of prolongations.

Contents

Preface

This paper is a natural continuation of the previous paper [**MZ**] where we studied a class of objects called Goursat distributions – certain 2-plane fields in n-space – using Cartan's method of prolongation. The class of Goursat germs have interesting singularities which get exponentially deeper and more complicated with increasing n. In that paper we constructed a sequence of circle bundles called the "Monster tower" such that any Goursat singularity in dimension n can be found in the tower at the same dimension. After writing that paper it became clear that their must be a dictionary between singularities of Legendrian curves (dimension 3), Goursat singularities, and points of the Monster tower (any dimension). The current paper develops this dictionary and uses it to prove a host of new classification results concerning Goursat singularities. Simultaneously we develop the geometry of the Monster tower and use it for resolving singularities of plane and Legendrian curves by prolonging them to the Monster.

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CHAPTER 1

Introduction

1.1. The Monster Construction

The Monster is a sequence of circle bundle projections

$$\ldots \to \mathbb{P}^{i+1}\mathbb{R}^2 \to \mathbb{P}^i\mathbb{R}^2 \to \ldots \to \mathbb{P}^2\mathbb{R}^2 \to \mathbb{P}^1\mathbb{R}^2 \to \mathbb{P}^0\mathbb{R}^2 = \mathbb{R}^2$$

between manifolds $\mathbb{P}^i\mathbb{R}^2$ of dimension $i+2$, $i = 0, 1, 2, \ldots$, each endowed with a rank 2 distribution $\Delta^i \subset T\mathbb{P}^i\mathbb{R}^2$. The construction of $(\mathbb{P}^i\mathbb{R}^2, \Delta^i)$ is inductive.

(1) $\mathbb{P}^0\mathbb{R}^2 = \mathbb{R}^2$, $\Delta^0 = T\mathbb{R}^2$.

(2) $\mathbb{P}^{i+1}\mathbb{R}^2$ is the following circle bundle over $\mathbb{P}^i\mathbb{R}^2$: a point of $\mathbb{P}^{i+1}\mathbb{R}^2$ is a pair (m, ℓ), where $m \in \mathbb{P}^i\mathbb{R}^2$ and ℓ is a 1-dimensional subspace of the plane $\Delta^i(m)$.

The rank 2-distribution Δ^{i+1} is defined in terms of smooth curves tangent to Δ^{i+1}. A smooth curve $\gamma : (a, b) \to \mathbb{P}^i\mathbb{R}^2$ is said to be *tangent* to Δ^i, or *integral*, if $\gamma'(t) \in \Delta^i\big(\gamma(t)\big)$, $t \in (a, b)$.

(3) a curve $t \to \big(m(t), \ell(t)\big)$ in $\mathbb{P}^{i+1}\mathbb{R}^2$ is tangent to Δ^{i+1} if the curve $t \to m(t)$ in $\mathbb{P}^i\mathbb{R}^2$ is tangent to Δ^i, and $m'(t) \in \ell(t)$ for all t.

The construction of $\mathbb{P}^{i+1}\mathbb{R}^2$ from $\mathbb{P}^i\mathbb{R}^2$ is an instance of a general procedure called *prolongation* due to E. Cartan [**C1, C2, C3**] and beautifully explained in section 3 of [**B**]. (Cartan invented several procedures now known as "prolongation". At a coordinate level, these prolongation procedures involve extending previously defined objects by adding derivatives.) The description we have given is repeated from our earlier work [**MZ**]. The symbol $\mathbb{P}$ is used to denote projectivization : $\mathbb{P}^{i+1}\mathbb{R}^2$ is the projectivization $\mathbb{P}\Delta^i$ of the rank 2 vector bundle Δ^i over $P^i\mathbb{R}^2$. When we refer to the Monster at level i, or the ith level of the Monster, we mean $\mathbb{P}^i\mathbb{R}^2$ endowed with the distribution Δ^i.

Convention. Throughout the paper, all objects (diffeomorphisms, curves, etc.) are assumed to be *real-analytic*.

1.2. Coordinates and the contact case

The first level of the Monster, $\mathbb{P}^1\mathbb{R}^2$, is a well-known contact 3-manifold. See for example [**A1**]. It is the space of lines in the plane, and is diffeomorphic to $\mathbb{R}^2 \times \mathbb{RP}^1$. A point $p \in \mathbb{P}^1\mathbb{R}^2$ is a pair (m, ℓ), $m \in \mathbb{R}^2$ and ℓ is a line in the tangent plane $T_m\mathbb{R}^2$. Standard local coordinates near p are (x, y, u), where (x, y) are Cartesian coordinates for $\mathbb{R}^2$ and u is an affine coordinate on the fiber $\mathbb{RP}^1$ of $\mathbb{P}^1\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{RP}^1$. To construct u, suppose that ℓ is not parallel to the y-axes: $dx|_\ell \neq 0$ and set $u = dy(v)/dx(v)$, where v is any vector spanning ℓ. In other words, u is the slope of the line ℓ and has the "hidden" meaning of dy/dx. Rearranging

$u = dy/dx$ we have $dy - udx = 0$ which defines the contact structure Δ^1 near p. Relative to these coordinates the projection $\mathbb{P}^1\mathbb{R}^2 \to \mathbb{R}^2$ is $(x, y, u) \to (x, y)$. If ℓ is parallel to the y-axis then we use instead $\tilde{u} = dx(v)/dy(v)$ and $(x, y, \tilde{u})$ are coordinates near p. A similar construction, yields coordinates on the higher Monsters $\mathbb{P}^i\mathbb{R}^2$, $i > 1$. See Chapter 7.

1.3. Symmetries. Equivalence of points of the Monster

A local symmetry of the Monster at level i is a local diffeomorphism $\Phi: U \to \widetilde{U}$, where U and $\widetilde{U}$ are open sets of $\mathbb{P}^i\mathbb{R}^2$ and $d\Phi$ brings Δ^i restricted to U to Δ^i restricted to $\widetilde{U}$.

Two points $p, \tilde{p}$ of the Monster manifold $\mathbb{P}^i\mathbb{R}^2$ are equivalent if there exists a local symmetry of $\mathbb{P}^i\mathbb{R}^2$ sending p to $\tilde{p}$.

1.4. Prolonging symmetries

The prolongation of a local symmetry Φ at level i is the local symmetry Φ^1 at level $i + 1$ defined by

$$\Phi^1(m, l) = \big(\Phi(m), d\Phi_m(\ell)\big).$$

Prolongation preserves the fibers of the fibration $\mathbb{P}^{i+1}\mathbb{R}^2 \to \mathbb{P}^i\mathbb{R}^2$. The process of prolongation can be iterated. The k-step-prolongation Φ^k of Φ is the one-step-prolongation of Φ^{k-1}.

1.5. The basic theorem

Our whole approach hinges on the basic theorem:

THEOREM 1.1. *For $i > 1$ every local symmetry at level i is the prolongation of a symmetry at level $i - 1$.*

Upon applying the theorem repeatedly, we eventually arrive at level 1, which is the contact manifold $(\mathbb{P}^1\mathbb{R}^2, \Delta^1)$. (See for example [**A1**] regarding this contact manifold. See also section 1.2). The symmetries of a contact manifold are called contact transformations, or contactomorphisms. Thus Theorem 1.1 asserts that the $(i - 1)$-fold prolongation is an isomorphism between the pseudogroup of contact transformations (level 1) and the pseudogroup of local symmetries at level i. The theorem expressly excludes the isomorphism between the $i = 0$ and $i = 1$ pseudogroups. Indeed the pseudogroup of contact transformations is strictly larger than the first prolongation of the pseudogroup of local diffeomorphisms of the plane, see [**A1**].

Theorem 1.1 can be deduced from our earlier work [**MZ**], namely from the "sandwich lemma" for Goursat distribution and the theorem (Theorem 1.2 in the next section) on realizing all Goursat distribution germs as points within the Monster tower. In order to be self-contained we present a simple proof of Theorem 1.1 in section 1.8 in purely Monster terms.

REMARK. Our Monster construction started with 1-dimensional contact elements for the plane. The construction generalized by starting instead with k-dimensional contact elements on an n-manifold. The generalization of Theorem 1.1 remains valid, and holds even for $i = 1$ when $k > 1$. This generalization is sometimes called the Backlund theorem, and is tied up with the local symmetries associated to the canonical distribution for jet spaces. See for example [**B**] and [**Y**].

1.6. The Monster and Goursat distributions

Our earlier work [**MZ**], where the Monster was introduced, was motivated by the problem of classifying Goursat distributions. Given a distribution $D \subset TM$ on a manifold M we can form its "square" $D^2 = [D, D]$, where $[\cdot, \cdot]$ denotes Lie bracket. Iterate, forming $D^{j+1} = [D^j, D^j]$. The distribution is called "Goursat" if the D^j have constant rank, and this rank increase by one at each step $\mathrm{rank}(D^{j+1}) = 1 + \mathrm{rank}(D^j)$, up until the final step j at which point $D^j = TM$.

Theorem 1.2 ([**MZ**]).

(1) *The distribution Δ^i on $\mathbb{P}^i\mathbb{R}^2$ is Goursat for $i \geq 1$.*

(2) *Any germ of any rank 2 Goursat distribution on a $(2+i)$-dimensional manifold appears somewhere in the Monster manifold $(\mathbb{P}^i\mathbb{R}^2, \Delta^i)$: this germ is diffeomorphic to the germ of Δ^i at some point of $\mathbb{P}^i\mathbb{R}^2$.*

This theorem asserts that the problems of classifying points of the Monster and of classifying germs of Goursat 2-distributions are the same problem.

1.6.1. Darboux, Engel, and Cartan theorems in Monster terms. A rank 2 Goursat distribution on a 3-manifold is a contact structure. A rank 2 Goursat distribution on a 4-manifold is called an Engel structure. Classical theorems of Darboux and Engel assert that all contact structures are locally diffeomorphic and that all Engel structures are locally diffeomorphic. (See, for example [**A1**], [**VG**], [**Z3**]). In Monster terms:

Theorem 1.3 (Darboux and Engel theorems in Monster terms). *All points of $\mathbb{P}^1\mathbb{R}^2$ are equivalent. All points of $\mathbb{P}^2\mathbb{R}^2$ are equivalent.*

For $i \geq 2$ not all points of $\mathbb{P}^i\mathbb{R}^2$ are equivalent, but there is a single open dense equivalence class. Cartan found the normal form for the points of this class, i.e. he wrote down the generic Goursat germ. In Monster terms:

Theorem 1.4 (Cartan theorem in Monster terms). *There is a single equivalence class of points in $\mathbb{P}^i\mathbb{R}^2$ which is open and dense. The germ of the 2-distribution Δ^i at any point of this class is diffeomorphic to the 2-distribution described in coordinates $(x, y, u_1, \ldots, u_i)$ by the vanishing of the 1-forms*

$$dy - u_1 dx, \quad du_1 - u_2 dx, \quad du_2 - u_3 dx, \quad \cdots, \quad du_{i-1} - u_i dx.$$

1.6.2. Some history. Cartan [**C1**], [**C2**], [**C3**] asserts a version of Theorem 1.2. However, Cartan defines prolongation by taking the usual derivatives of coordinates, which is to say, his prolongation is affine and does not allow tangent lines to "go vertical". When interpreted in this affine sense, Cartan's assertions are, apparently, the (false) theorem that all Goursat distributions in dimension $k + 2$ are locally diffeomorphic to the distribution of Theorem 1.4. Indeed, in his famous five-variables paper, [**C4**] Cartan seems to assert that the only Goursat germ is the open and dense one. In [**GKR**] the authors found a counterexample to exactly this assertion, and this example is the first Goursat singularity. Goursat in his book [**Go**] presented the assertion of Theorem 1.4. A number of decades later Bryant and Hsu [**B**] redefined Cartan's prolongation in projective terms, which is the prolongation we have just used in defining the Monster. For more on the history, see the introduction to [**GKR**] and section 3.1 of [**B**].

1.6.3. Normal forms for Goursat distributions. Giaro, Kumpera, and Ruiz discovered in [**GKR**] the first Goursat germ not covered by Cartan. In so doing they initiated the study of singular Goursat germs. Kumpera and Ruiz introduced special coordinates in dimension $2+i$ with associated Goursat normal forms depending on $(i-2)$ real parameters which covered all rank 2 Goursat germs. They and their followers calculated which parameters could be "killed", which could be reduced to 1 or -1 and which must be left continuous (moduli). The outcome of these computations is that the set of equivalence classes of Goursat germs on $\mathbb{R}^{2+i}$ is finite for $i \leq 7$. Consequently the set of equivalence classes of points of $\mathbb{P}^i\mathbb{R}^2$, $i \leq 7$ is finite. This number is $2, 5, 13, 34, 93$ for $i = 3, 4, 5, 6, 7$. See the works [**GKR**], [**KR**], [**Ga**], [**Mor1**], [**Mor2**] of Giaro, Kumpera, and Ruiz ($i = 3, 4$), Gaspar ($i = 5$) and Mormul ($i = 6, 7$). Mormul discovered [**Mor2**], [**Mor3**], [**Mor7**] the first moduli, which appears at $i = 8$. The length of these computations increases exponentially with i. Beyond Cartan's theorem, the only results which are valid for all dimensions are Mormul's classifications of codimension one singularities in [**Mor4**] and his classification of the simplest codimension two singularities in [**Mor5**].

1.7. Our approach

We reduce the problem of classifying points in the Monster to a well-studied classification problem: that of germs of Legendrian curves.

1.7.1. Integral curves. By a curve in $\mathbb{P}^i\mathbb{R}^2$ we mean a map $\gamma : (a, b) \to \mathbb{P}^i\mathbb{R}^2$. A curve in $\mathbb{P}^i\mathbb{R}^2$ tangent to the 2-distribution Δ^i is called *integral curve*. Integral curves in a contact 3-manifold such as $\mathbb{P}^1\mathbb{R}^2$ are called Legendrian curves.

1.7.2. Equivalent curves. Two germs of curves $\gamma : (\mathbb{R}, 0) \to (\mathbb{P}^i\mathbb{R}^2, p)$ and $\widetilde{\gamma} : (\mathbb{R}, 0) \to (\mathbb{P}^i\mathbb{R}^2, \tilde{p})$ are called equivalent if there exists a local symmetry $\Phi : (\mathbb{P}^i\mathbb{R}^2, p) \to (\mathbb{P}^i\mathbb{R}^2, \tilde{p})$ and a local diffeomorphism $\phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ (a reparameterization of a curve) such that $\widetilde{\gamma} = \Phi \circ \gamma \circ \phi$.

Remark. When $i = 1$ we will call this equivalence RL-contact equivalence ("RL" is for Right-Left, see for example [**AVG**]).

1.7.3. Classification problems. Consider the following problems.

(1) Classify germs of integral curves in the Monster tower.

(2) Classify germs of Legendrian curves.

(3) Classify points in the Monster tower.

(4) Classify finite jets of Legendrian curves.

We will show (1) and (2) are equivalent problems, and we will show that (3) and (4) are equivalent problems. The equivalence between (3) and (4) allows us to translate well-known results regarding Legendrian germs into classification results on points of the Monster which leads to a number of new classification results and gives simple unified proofs of previously disparate results. For example, Mormul's classification [**Mor4**] of the codimension one singularities of Goursat 2-distributions now becomes a corollary of the contact classification of the simplest (A-type) singularities of Legendrian germs, see section 4.4.1 and Theorem 4.16.

1.7.4. Application of reduction theorems. Our reduction theorems give explicit (easy to be programmed) algorithms reducing the problem of equivalence and classification for points of the Monster to the problems of equivalence and classification for Legendrian curve jets. These algorithms are used to obtain a number of classification results. In particular, we solve the basic classification problem:

(*) Determine and classify the simple points of the Monster tower.

A point of the Monster is called *simple* if it is contained in a neighborhood which is the union of a *finite* number of equivalence classes. The earlier classification results for levels $i = 2, \dots, 7$ mentioned above (see section 1.6.3) comprise a small part of our solution to (*).

1.8. Proof of the basic theorem

The proof of Theorem 1.1 is based on the fact that for $i \geq 1$ the fibers of the fibration $\mathbb{P}^{i+1}\mathbb{R}^2 \to \mathbb{P}^i\mathbb{R}^2$ can be intrinsically defined and so any symmetry maps fibers to fibers. This intrinsic definition is in terms of characteristic vector fields for the distribution $(\Delta^i)^2 = [\Delta^i, \Delta^i]$.

1.8.1. The distribution $(\Delta^i)^2$. The following lemma gives a simple relation between the 2-distribution Δ^{i-1} and the 3-distribution $(\Delta^i)^2$.

LEMMA 1.5. *For $i \geq 1$ the distribution $(\Delta^i)^2$ has constant rank 3. A curve in $\mathbb{P}^i\mathbb{R}^2$ is tangent to $(\Delta^i)^2$ if and only if its projection to $\mathbb{P}^{i-1}\mathbb{R}^2$ is tangent to the 2-distribution Δ^{i-1}.*

PROOF. Fix a point $(m, \ell) \in \mathbb{P}^i\mathbb{R}^2$. Choose a local frame (X_1, X_2) of Δ^{i-1} which is defined near m and is such that $X_1(m) \in \ell$. Given a point $(\widetilde{m}, \widetilde{\ell})$ close to (m, ℓ) one has $\widetilde{\ell} = \operatorname{span}\big(X_1(\widetilde{m}) + tX_2(\widetilde{m})\big)$, where $t \in \mathbb{R}$ is a small number parameterizing the line $\widetilde{\ell}$. Choose local coordinates x on $\mathbb{P}^{i-1}\mathbb{R}^2$ defined near m. Then (x, t) form a local coordinate system on $\mathbb{P}^i\mathbb{R}^2$ near (m, ℓ). In this local coordinate system

$$\Delta^i = \operatorname{span}\Big(X_1(x) + t \cdot X_2(x),\ \partial/\partial t\Big)$$

and consequently

$$(\Delta^i)^2 = \operatorname{span}\Big(X_1(x),\ X_2(x),\ \partial/\partial t\Big) \tag{1.1}$$

which proves Lemma 1.5. □

1.8.2. Characteristic vector field for the distribution $(\Delta^i)^2$.

DEFINITION 1.6. A vector field C is a characteristic vector field for a distribution D if C is tangent to D and $[C, D] \subseteq D$.

Take local coordinates (x, t) as in the proof of Lemma 1.5. Then $\partial/\partial t$ is a characteristic vector field for the 3-distribution $(\Delta^i)^2$ since $[\partial/\partial t,\ X_1(x)] = [\partial/\partial t, X_2(x)] \equiv 0$. These relations imply that any vector field of the form $f(x,t)\partial/\partial t$ is a characteristic vector field. Now, a vector field has the form $f(x,t)\partial/\partial t$ if and only if it is tangent to the fibers of the fibration $\mathbb{P}^i\mathbb{R}^2 \to \mathbb{P}^{i-1}\mathbb{R}^2$, so we have shown that any vector field tangent to the fibration is a characteristic vector field. The following lemma states that there are no other characteristic vector fields provided that $i \geq 2$.

Lemma 1.7. *If $i \geq 2$ then a vector field on $\mathbb{P}^i\mathbb{R}^2$ is a characteristic vector field for the distribution $(\Delta^i)^2$ if and only if this vector field is tangent to the fibers of the fibration $\mathbb{P}^i\mathbb{R}^2 \to \mathbb{P}^{i-1}\mathbb{R}^2$.*

Note that $(\Delta^1)^2 = T\mathbb{P}^1\mathbb{R}^2$ (this follows, for example, from Lemma 1.5) and consequently Lemma 1.7 does not hold for $i = 1$.

Proof. Take local coordinates (x, t) as in the proof of Lemma 1.5. Let C be a characteristic vector field for the 3-distribution (1.1). This means that C has the form

$$C = g_1(x,t)X_1(x) + g_2(x,t)X_2(x) + f(x,t)\partial/\partial t$$

and that

$$[C,\ X_1(x)],\ [C,\ X_2(x)],\ [C,\ \partial/\partial t] \in \operatorname{span}\Big(X_1(x),\ X_2(x),\ \partial/\partial t\Big). \tag{1.2}$$

Expanding out the Lie brackets, we find that (1.2) is equivalent to the inclusions

$$g_1(x,t)\cdot[X_1(x),\ X_2(x)],\quad g_2(x,t)\cdot[X_1(x),\ X_2(x)] \in \operatorname{span}\big(X_1(x), X_2(x)\big). \tag{1.3}$$

But $X_1(x), X_2(x), [X_1(x), X_2(x)]$ are linearly independent. (See Lemma 1.5 and use that $i \geq 1$.) Therefore (1.3) holds if and only if $g_1(x,t) = g_2(x,t) \equiv 0$, which is to say, if and only if C is tangent to the fibers of the fibration. □

1.8.3. From Lemma 1.7 to Theorem 1.1. Consider any symmetry Ψ of $\mathbb{P}^{i+1}\mathbb{R}^2$, $i \geq 1$. Ψ must preserve the 3-distribution $(\Delta^{i+1})^2$ and so, by Lemma 1.7, the fibration $\mathbb{P}^{i+1}\mathbb{R}^2 \to \mathbb{P}^i\mathbb{R}^2$. Therefore Ψ induces a diffeomorphism Φ of $\mathbb{P}^i\mathbb{R}^2$. Since the push-down of $(\Delta^{i+1})^2$ is Δ^i, it is clear that Φ is a symmetry of $\mathbb{P}^i\mathbb{R}^2$. To prove Theorem 1.1 it remains to show that Ψ is the prolongation of Φ.

Take any point $(m, \ell) \in \mathbb{P}^{i+1}\mathbb{R}^2$. Let I be a small interval containing 0. Take any integral curve $\Gamma : t \in I \to (m(t), \ell(t))$ in $\mathbb{P}^{i+1}\mathbb{R}^2$ passing through (m, ℓ) at $t = 0$ and such that $m'(t) \neq 0$ for $t \in I$. (Such a curve exists.) Let $\widetilde{\Gamma} = \Psi \circ \Gamma$: $t \to (\widetilde{m}(t), \widetilde{\ell}(t))$ be the image of Γ under the symmetry Ψ. The symmetry Φ brings the projected curve $t \to m(t)$ to the curve $t \to \widetilde{m}(t)$. It follows that the prolongation Φ^1 of Φ takes Γ to a curve of the form $\widehat{\Gamma} : t \to (\widetilde{m}(t), \widehat{\ell}(t))$. Since Φ^1 and Ψ are both symmetries the curves $\widetilde{\Gamma}$ and $\widehat{\Gamma}$ are both integral. Therefore $\widetilde{m}'(t) \in \widehat{\ell}(t)$ and $\widetilde{m}'(t) \in \widetilde{\ell}(t)$. Since Φ is a diffeomorphism and $m'(t) \neq 0$ we have that $\widetilde{m}'(t) \neq 0, t \in I$ and it follows that $\widehat{\ell}(t) = \widetilde{\ell}(t) = \operatorname{span}(\widetilde{m}'(t))$ for $t \in I$. We see that $\widehat{\Gamma}$ and $\widetilde{\Gamma}$ are the same curve! In particular $\widehat{\Gamma}(0) = \widetilde{\Gamma}(0)$ which means that $\Phi^1(m, \ell) = \Psi(m, \ell)$. Since (m, ℓ) was arbitrary, $\Phi^1 = \Psi$.

1.9. Plan of the Paper

Chapter 2 is devoted to prolongations of integral curves and to realization of points of the Monster by such prolongations. In this chapter we also define critical curves and distinguish two types of critical curves and several types of points in the Monster.

In sections 2.1 - 2.4 we prove the equivalence of the classification problems (1) and (2) in section 1.7.3. The proof requires that we define the prolongation of a (non-constant) singular integral curve.

In section 2.5 we prove that any point of the Monster is touched by the prolongation of some (singular) Legendrian curve and that RL-contact equivalent Legendrian curves touch equivalent points. These facts allows us to give in section 2.6 one of several equivalent definitions of a non-singular point in the Monster: a point is non-singular if it can be realized as the prolongation of *immersed* Legendrian curve, evaluated at a certain time. The Cartan Theorem 1.4 regarding the open dense set of Goursat germs becomes a direct corollary of a well-known result on the local contact equivalence of non-singular Legendrian curves.

In section 2.7 we define and study critical curves A curve is *critical* if it is an integral curve in the Monster whose projection to the first level $\mathbb{P}^1\mathbb{R}^2$ is constant. The notion of a critical curve is critical to our approach. The simplest critical curves are the vertical curves – those lying in a fiber of $\mathbb{P}^{i+1}\mathbb{R}^2 \to \mathbb{P}^i\mathbb{R}^2$. We prove that all other critical curves are prolongations of vertical curves. We also prove that a curve is critical if and only if it is a singular (or abnormal) integral curve in the sense of sub-Riemannian geometry for the distribution Δ^i.

In section 2.8 we partition directions ℓ in Δ^i into *regular* and *critical.* A direction is critical if there exists a critical curve tangent to it ; otherwise it is called regular. We then partition points of the Monster into *regular and critical.* A point $p = (m, \ell) \in \mathbb{P}^{1+k}\mathbb{R}^2$, $m \in \mathbb{P}^k\mathbb{R}^2, \ell \subset \Delta^k(m)$ is critical or regular according to whether the direction ℓ is critical or regular in Δ^k.

In section 2.9 we define regular integral curves. An integral curve germ is regular if it is immersed and has a regular direction, i.e. not tangent to a critical curve. We prove two propositions relating regular curves and regular points.

Our regularization theorem in section 2.10 states that any singularity of an analytic well-parameterized integral curve can be resolved by prolongations in the following sense: if Γ is an analytic well-parameterized integral curve germ then for sufficiently big k the k-step-prolongation of Γ is a regular integral curve. In particular any analytic well-parameterized plane curve singularity admits a resolution (or desingularization) by prolonging enough times.

In section 2.11 we give an equivalent definition of a non-singular point: a point p in the Monster is non-singular if and only if it is a regular prolongation of a point of $\mathbb{R}^2$, i.e. all circle bundle projections of p are regular points.

In section 2.12 we further decompose the critical directions and points into vertical and tangency directions and points and we study the structure of the three types of points: regular, vertical, and tangency points. We show that the fiber over a regular point contains exactly one critical point which is the vertical point, and the fiber above a critical (i.e. vertical or tangency) point contains exactly two critical points, one of them is vertical and the other is tangency.

Our results of Chapter 2 are continued in Chapter 3 which is devoted to the stratification of the Monster into RVT classes, to the RVT-codes of plane curves, and to the relation between such codes and the classical Puiseux characteristic.

The partition of points into regular (R) vertical (V) , and tangency (T) induces a stratification of the Monster into singularity classes, the RVT classes, indexed by $(i-3)$-tuples of letters R, V, T. The RVT classes are defined in section 3.1 by successively projecting a point to lower levels of the Monster and applying the R V T partition at that level. In this way we associate to each point at level i an

$i-3$-tuple of letters R, V, T. (It is an $(i-3)$-tuple because this process stops at level 2 where there are no points of type V or T invariantly defined.)

The RVT classes, expressed in Goursat terms (see Theorem 1.2), coincide with classes defined earlier by Mormul [**Mor6**] using Kumpera-Ruiz coordinates. The RVT classes in the ith level of the Monster also coincide with certain classes defined by Jean [**J**] in the kinematic model of a car pulling $(i-1)$ trailers. (This model is isomorphic to $\mathbb{P}^i\mathbb{R}^2$, see [**MZ**], Appendix D.) Despite the work of our predecessors, our construction of these RVT classes is the first coordinate-free stratification of the Monster (and consequently of Goursat distributions). The coordinate-free nature of the stratification has several advantages over coordinate definitions. It quickly yields two more equivalent definitions of what it means for a point of the Monster to be singular (section 3.2). It immediately reduces the classification of points within an arbitrary RVT class to the classification of points within a regular RVT class, i.e. an RVT class whose RVT-code ends with R or, what is the same, an RVT class whose points are all regular. See our reduction theorem in section 3.4 that we call the method of critical sections. The coordinate-free definition of the RVT classes also allows to analyze the geometric structure of these classes as subsets of the Monster (sections 3.5 and 3.6). But the strongest advantage is realized upon combining the notion of RVT classes with prolongations of non-immersed plane curves (sections 3.7 and 3.8) and with the operations of Legendrization and Monsterization (Chapter 4). This then yields a stratification of Legendrian curves which can be compared with known singularity theory results, and then carried back from the Legendrian world to the Monster world.

In section 3.7 we define the RVT-code of a plane curve germ $c : (\mathbb{R}, 0) \to \mathbb{R}^2$. It is the RVT-code of the point $c^k(0)$ where c^k is the k-step-prolongation of the curve c and k is the regularization level of c - - the minimal integer such that c^k is a regular curve. The RVT-code is well-defined for any well-parameterized plane curve germ. It is always critical, i.e. ends with V or T, not with R. It is an invariant with respect to the RL-equivalence and, moreover, with respect to a weaker "contact" equivalence of plane curves (two plane curves are contact equivalent if their one-step-prolongations are RL-contact equivalent Legendrian curves).

Section 3.8 is devoted to the relation between the two invariants of plane curve germs: the RVT code and the classical Puiseux characteristic . Theorem **A** states that all plane curve germs with a fixed Puiseux characteristic of the form

$$\Lambda = [\lambda_0; \lambda_1, ..., \lambda_m], \quad m \geq 1, \ \lambda_1 > 2\lambda_0 \tag{1.4}$$

have the same RVT code RVT(Λ). We explain why the requirement $\lambda_1 > 2\lambda_0$ is essential. The map $\Lambda \to \mathrm{RVT}(\Lambda)$ is constructed by explicit recursion formulae, as well as the inverse map sending a critical RVT code (α) to a Puiseux characteristic $\mathrm{Pc}(\alpha)$ of the form (1.4). The maps $\Lambda \to \mathrm{RVT}(\Lambda)$ and $(\alpha) \to \mathrm{Pc}(\alpha)$ give a 1-1 correspondence between the set of all critical RVT codes and the set of all Puiseux characteristics of the form (1.4).

In Chapter 4 we define two operations : Monsterization and Legendrization, and we prove the equivalence of problems (3) and (4) in section 1.7.3.

Monsterization and Legendrization are defined in section 4.1 where we also establish the basic properties of these operations. Monsterization sends a Legendrian curve to a point of the Monster. Legendrization reverses this operation by sending a point in the Monster to a class of Legendrian curves. These two operations are basic tools for obtaining our reduction theorems. The Legendrization can be applied to a point of the Monster or to a singularity class of points in the Monster. The Legendrization of a point is the set of Legendrian curves formed by the projections to the first level of all *regular* integral curves passing through the given, i.e. curves which are not tangent to a critical curve through that point. The Legendrization of a singularity class of points of the Monster is the union of the Legendrizations of its points. Without the tangency restriction the Legendrization operation would yield singularity classes much too large (or "deep") to be useful.

In section 4.2 we use Theorem **A** and the constructed map $(\alpha) \to \mathrm{Pc}\,(\alpha)$ (section 3.8) to give an explicit formulae for the Legendrization of any RVT class. In section 4.3 we use Legendrization to relate the equivalence problem for points in the Monster with the equivalence problem for Legendrian curve germs (problems (2) and (3) of section 1.7.3). Two theorems in this section are illustrated by the simplest classification results given in section 4.4. These theorems are the starting point for establishing the equivalence of problems (3) and (4) of section 1.7.3 in the general case.

In section 4.6 we use Legendrization to give a definition of the *jet-identification number* $r = r(p)$ of a point p in the Monster. This number is the integer such that the point p can be identified with a single r-jet of a Legendrian curve in the following sense: the Legendrization of p contains, along with any Legendrian curve germ, all Legendrian curve germs with the same r-jet *and* the set of r-jets of the curves in the Legendrization of p is exhausted, up to reparameterization, by a single r-jet. Within the set of all points for which this number is defined the equivalence problem for points is shown to be equivalent, via Legendrization, to the equivalence problem for jets of Legendrian curves (Theorem 4.23). To make use of this equivalence we must answer two questions: How do we effectively characterize the points for which the jet-identification number exist? For these points, how do we calculate this number?

The answer to the first question is as follows: the jet-identification number $r(p)$ is defined if and only if the point p is regular. Furthermore, all points of any fixed regular RVT class have the same jet-identification number, so we can define the jet-identification number of a regular RVT class.

The answer to the second question, on the calculation of the jet-identification number, requires the parameterization number of a point of the Monster and of an RVT class introduced in section 4.7. The parameterization number of a point p is the order of good parameterization of a Legendrian curve γ in the set $\mathrm{Leg}(p)$, the Legendrization of p. We prove that this number does not depend on the choice of γ. Moreover, it is the same for all points of any fixed RVT class. We prove an explicit formula for calculation of the parameterization number $d(\alpha)$ in terms of the RVT-code (α): $d(\alpha) = \lambda_m - \lambda_0$, where λ_0 and λ_m are the first and the last integers in the Puiseux characteristic $\mathrm{Pc}\,(\alpha)$ defined in section 3.8.

In section 4.8 we present Theorem 4.40 which gives an explicit formula for the jet-identification number of any regular RVT class. The jet-identification number of the open Cartan class R^{1+k} (R repeated k times) is equal to k. Any other regular RVT code can be expressed in the form $(\alpha \mathrm{R}^q)$ where $q \geq 1$ and (α) is a critical RVT code. Theorem 4.40 states that the jet-identification number of the class (αR^q) is equal to $d + q - 1$, where d is the parameterization number of the class (α). Combining Theorem 4.40 with the obtained formula for the parameterization number, we obtain an explicit formula for calculation of the jet-identification number of a regular point in terms of its RVT code (Theorem 4.41).

In sections 4.9 and 4.10 we reduce Theorem 4.40 to Theorem **B**, formulated, like Theorem **A**, in terms of prolongations of plane curves.

Although the jet-identification number is undefined for critical points and for critical RVT classes, as we said above the classification of points within a critical RVT class immediately reduces to to the classification of points within a regular RVT class (section 3.4). Therefore our results completely reduce the equivalence and the classification problems for any points of the Monster and any RVT classes to the RL-contact equivalence equivalence and the RL-contact classification problems for certain finite jets of Legendrian curves and certain singularity classes of such jets. It follows (see section 1.6) that both the equivalence and classification problems for Goursat flags reduce to these same problems for finite jets of Legendrian curves. Moreover, it is clear that our reduction theorems give explicit (easy to be programmed) algorithms reducing the problems of equivalence and classification for points of the Monster to the problems of equivalence and classification for Legendrian curve jets. These algorithms are given, with all details, in Chapter 5, sections 5.1 - 5.3. They are illustrated by many examples of classification results in sections 5.4 - 5.7.

The examples of classification results in Chapter 5 are used in Chapter 6 where we determine all tower-simple and all stage-simple points in the Monster. Recall that a point is simple if it has a neighborhood covered by a finite number of equivalence classes. The notion of "neighborhood" can be either taken within the level of the Monster which contains that point, or taken in the entire Monster tower. These two choices lead to two definitions of "simple", tower-simple and stage-simple. Examples of classification results in Chapter 5 and the determination theorems in Chapter 6 contain all known results related to the classification of Goursat distributions (see section 1.6.3) and much more.

The proofs of Theorems **A** and **B** require local coordinate systems in the Monster. In [**KR**] Kumpera and Ruiz introduced special systems of coordinates designed to fit Goursat distributions. In Chapter 7 we explain their projective meaning and use them to coordinatize the Monster. We relate the KR (Kumpera-Ruiz) coordinates with critical curves, directions, and points, and with RVT classes; we also express in terms of KR coordinates the prolongations of plane curves.

In Chapter 8 we use the KR coordinates to express prolongations of plane curves in terms of iterations of a certain operator on the space of plane curve germs. We call this operator "directional blow-up". We prove certain properties of the directional blow-up which lead to the proofs of Theorems **A** and **B**.

Chapter 9 is devoted to open questions. These questions concern unfolding versus prolongation, the classical blow-up versus prolongation, possible applications of the 1-1 correspondence between RVT classes and Puiseux characteristics, the "infinite Monster", the geometry underlying the moduli occurring in the classification of points of the Monster, and the 1-1 correspondence between RVT codes and small growth vectors.

In Appendix A we illustrate the equivalence of problems (1) and (2) in section 1.7.3. We use the reduction theorem of section 2.1, which establishes the equivalence, to classify germs of immersed integral curves in an Engel 4-manifold.

In Appendix B we summarize the known results on the RL-contact classification of Legendrian curves which we use in the present work. These results include corollaries of well-known results on the local classification of germs of plane and space curves obtained in [**BG**], [**GH**], [**A2**], as well as the results of [**Z1**], [**Z2**].

Appendix C is devoted to the proof of Theorem 2.23 which asserts that an immersed curve in the Monster is critical if and only if it is singular (= abnormal) if and only if it is locally C^1-rigid.

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CHAPTER 2

Prolongations of integral curves. Regular, vertical, and critical curves and points

2.1. From Monster curves to Legendrian curves

Throughout the paper we will use the

NOTATION. By $\pi_{i+k,i} : \mathbb{P}^{i+k}\mathbb{R}^2 \to \mathbb{P}^i\mathbb{R}^2$ we denote the bundle projection.

By Theorem 1.1, the fibers of these projections are preserved by local symmetries when $k, i \geq 1$: any local symmetry of $\mathbb{P}^{i+k}\mathbb{R}^2$ extends canonically to the fibers of $\pi_{i+k,i}$ passing through any points in its domain, and maps these fibers to other fibers.

DEFINITION 2.1. The k-step-projection of a curve $\Gamma : (a, b) \to \mathbb{P}^{i+k}\mathbb{R}^2$ is the curve $\Gamma_k : (a, b) \to \mathbb{P}^i\mathbb{R}^2$ such that $\Gamma_k(t) = \pi_{i+k,k}(\Gamma(t))$.

The k-step-projection of an integral curve in $\mathbb{P}^{i+k}\mathbb{R}^2$ is an integral curve in $\mathbb{P}^i\mathbb{R}^2$. In particular, the k-step-projection of an integral curve in $\mathbb{P}^{1+k}\mathbb{R}^2$ is a Legendrian curve in $\mathbb{P}^1\mathbb{R}^2$.

THEOREM 2.2. *Let $\Gamma : (\mathbb{R}, 0) \to (\mathbb{P}^{1+k}\mathbb{R}^2, p)$ and $\widetilde{\Gamma} : (\mathbb{R}, 0) \to (\mathbb{P}^{1+k}\mathbb{R}^2, \tilde{p})$ be germs of integral curves. Consider the germs of Legendrian curves $\gamma = \Gamma_k$ and $\widetilde{\gamma} = \widetilde{\Gamma}_k$ and assume that they are not the germs of constant curves. Then Γ and $\widetilde{\Gamma}$ are equivalent if and only if γ and $\widetilde{\gamma}$ are* RL-*contact equivalent.*

This theorem reduces the classification of integral curve germs in $\mathbb{P}^i\mathbb{R}^2, i \geq 2$ to the classification of Legendrian germs in $\mathbb{P}^1\mathbb{R}^2$. Transferring know results on the classification of Legendrian curves to $\mathbb{P}^i\mathbb{R}^2$ provides a series of new results on classification of integral curves in Goursat manifolds, perhaps the most important of which is the local classification of immersed curves in an Engel 4-manifold as presented in our Appendix A.

The proof of Theorem 2.2 is given in section 2.4. It is based on Theorem 1.1 and requires the following ingredients:

the prolongation of integral curves with singularites (section 2.2);

the prolongation and projection of symmetries of the Monster (section 2.3).

2.2. Prolonging curves

Prolongation is the inverse of projection. If $\Gamma : (a, b) \to \mathbb{P}^i\mathbb{R}^2$ is an immersed integral curve then its *first prolongation* Γ^1 is the integral curve in $\mathbb{P}^{i+1}\mathbb{R}^2$ defined by differentiating Γ. Thus $\Gamma^1(t) = \big(m(t), \ell(t)\big)$, where $m = \Gamma(t)$ and ℓ is the line spanned by the non-zero vector $\Gamma'(t)$. We note that Γ^1 is the unique integral curve in $\mathbb{P}^{i+1}\mathbb{R}^2$ which projects to Γ in $\mathbb{P}^i\mathbb{R}^2$. Since Γ^1 is immersed and integral, we can prolong it, and continue prolonging to obtain integral curves Γ^k at all higher levels.

EXAMPLE 2.3. Consider a plane curve having the form $c : x(t) = t, y(t) = f(t)$. Let $u = u_1 = dy/dx$ be the standard fiber coordinate on $\mathbb{P}^1\mathbb{R}^2$ near the point $c^1(0)$ as described in section 1.2. Then $c^1(t) = (t, f(t), f'(t))$ in these coordinates, while Δ^1 is expressed as $dy - u_1 dx = 0$. The one-forms dx, du_1 provide linear coordinates on the two-planes $\Delta^1(m)$ for m in a neighborhood of $c^1(0)$. We have $dx(dc^1/dt) = 1$ so that if $p = (m, \ell) \in \mathbb{P}^2\mathbb{R}^2$ is any point sufficiently close to the point $c^2(0)$, then $dx \neq 0$ on $\ell \subset \Delta^1(m)$. It follows that the function $u_2 = du_1(v)/dx(v)$ for v spanning ℓ is well-defined near $c^2(0)$ and that x, y, u_1, u_2 coordinatize $\mathbb{P}^2\mathbb{R}^2$. In these coordinates Δ^2 is described by the vanishing of the two one-forms $dy - u_1 dx$ and $du_1 - u_2 dx$. Repeating this construction we obtain local coordinates $u_3 = du_2/dx, \cdots, u_i = du_{i-1}/dx$ in a neighborhood of the point $c^i(0) \in \mathbb{P}^i\mathbb{R}^2$ such that the distribution Δ^i is described in this neighborhood by the vanishing of the one-forms as per Theorem 1.4. In these coordinates c^i has the form

$$c^i : x(t) = t, \quad y(t) = f(t), \quad u_1(t) = f'(t), \quad \cdots, \quad u_i = f^{(i)}(t).$$

Thus the point $c^i(0)$ can be identified with the i-jet at $t = 0$ of the function $f(t)$.

It is essential to extend prolongation so as to operate on singular curves.

DEFINITION 2.4. Let $\Gamma : (a, b) \to \mathbb{P}^i\mathbb{R}^2$ be a non-constant integral analytic curve. Its prolongation Γ^1 is the non-constant analytic integral curve $(a, b) \to \mathbb{P}^{i+1}\mathbb{R}^2$ defined at $t_0 \in (a, b)$ as follows:

1. If $\Gamma'(t_0) \neq 0$ then we define $\Gamma^1(t_0)$ as we did for immersed curve.

2. If $\Gamma'(t_0) = 0$, then, since Γ is analytic and not constant we have $\Gamma'(t) \neq 0$ for $t \neq t_0$ close to t_0, and so $\Gamma^1(t)$ for these t is defined by step 1. Set

$$\Gamma^1(t_0) = \lim_{t \to t_0} \Gamma^1(t). \tag{2.1}$$

THEOREM 2.5. *The limit in (2.1) is well-defined. The corresponding curve $\Gamma^1 : (a, b) \to \mathbb{P}^{i+1}\mathbb{R}^2$ is analytic, integral, and not constant.*

Again, we can iterate prolongation so as to form $\Gamma^k = (\Gamma^{k-1})^1$, for any non-constant analytic curve germ Γ in $\mathbb{P}^{i+k}\mathbb{R}^2$.

PROOF. Let $\Gamma(t)$ be a non-constant analytic integral curve at level i which is not immersed at $t = t_0$. Fix analytic vector fields v_1, v_2 spanning the rank 2 distribution $\triangle^i$ near the point $\Gamma(t_0)$. Then for any t near t_0 one has

$$\Gamma'(t) = f_1(t) v_1\big(\Gamma(t)\big) + f_2(t) v_2\big(\Gamma(t)\big),$$

for analytic functions germs $f_1(t), f_2(t)$. Because Γ is not immersed at t_0 we have that $f_1(t_0) = f_2(t_0) = 0$. Because Γ is analytic and not constant at least one of the function germs $f_1(t), f_2(t)$ is not the zero germ. Therefore there exist a finite number r and analytic function germs $g_1(t), g_2(t)$ such that

$$f_1(t) = (t - t_0)^r g_1(t), \ f_2(t) = (t - t_0)^r g_2(t), \quad \big(g_1(t_0), g_2(t_0)\big) \neq (0, 0).$$

At least one of the germs $g_1(t)/g_2(t)$, $g_2(t)/g_1(t)$ is a well defined analytic germ. This implies the existence of the limit in the definition of $\Gamma^1(t_0)$ and the analyticity of the curve Γ^1. The integrability of Γ^1 follows from its construction. The curve Γ^1 is not constant since its projection Γ to $\mathbb{P}^i\mathbb{R}^2$ is not constant. $\square$

Proposition 2.6. *Projection and prolongation are inverses: that is, if* $\Gamma : (a,b) \to \mathbb{P}^i\mathbb{R}^2$ *is a non-constant integral curve at level* i *then*

$(\Gamma^k)_k = \Gamma$ *for* $k \geq 0$ *;* *if* Γ_k *is not constant for* $k \leq i$ *then* $(\Gamma_k)^k = \Gamma$.

Proposition 2.7. *Projection and prolongation commute with reparameterizations including singular reparameterizations: that is, if* $\Gamma : (a,b) \to \mathbb{P}^i\mathbb{R}^2$ *is a non-constant integral curve at level* i *and* $\phi : (c,d) \to (a,b)$ *is a non-constant analytic map then*

$(\Gamma \circ \phi)^k = \Gamma^k \circ \phi$*;* $(\Gamma \circ \phi)_k = \Gamma_k \circ \phi$ *for* $k \leq i$.

Proof of Propositions 2.6 and 2.7. We first do the case $k = 1$. The definition of projection and prolongation implies the validity of Proposition 2.6 for $k = 1$ at all point $t \in (a,b)$ such that $\Gamma'(t) \neq 0$ and the validity of Proposition 2.7 for $k = 1$ at all points $t \in (c,d)$ such that $\phi'(0) \neq 0$ and $\Gamma'(\phi(t)) \neq 0$. Since Γ and ϕ are analytic and not constant the points where $\Gamma'(t) = 0$ and where $\phi'(0) = 0$ are discrete. By continuity (see Theorem 2.5) Propositions 2.6 and 2.7 also hold for $k = 1$ at all these points, hence for all $t \in (a,b)$. Repeating this argument $k - 1$ times, we see that the validity of Propositions 2.6 and 2.7 for $k = 1$ implies their validity for any k under the given constraints on k.

Note that Propositions 2.6 and 2.7 also imply:

Proposition 2.8. *If* c *and* $\tilde{c}$ *are* RL*-equivalent non-constant plane curve germs then for any* $k \geq 1$ *their* k*-step-prolongations are equivalent integral curves in the Monster (in particular,* RL*-contact equivalent Legendrian curves when* $k = 1$*)*

The converse is not true even for $k = 1$, as we explained in section 1.5.

2.3. Projections and prolongations of local symmetries

In section 1.4 we defined the prolongations of a local symmetry of the Monster. The projections Φ_k of a local symmetry Φ of $\mathbb{P}^i\mathbb{R}^2$ are defined by Theorem 1.1, but they are only defined when $i - k \geq 1$. Here are the details.

Suppose $i \geq 2$. By Theorem 1.1 Φ is the one-step-prolongation of a symmetry $\widetilde{\Phi}$ at level $(i-1)$. It is clear that $\widetilde{\Phi}$ is unique. We denote $\widetilde{\Phi}$ by Φ_1 and say that Φ_1 is the one-step projection of Φ. If $i \geq 3$ then, again by Theorem 1.1 Φ_1 is the one-step-prolongation of a symmetry $\widehat{\Phi}$ at level $(i-2)$; $\widehat{\Phi}$ is unique. We denote $\widehat{\Phi}$ by Φ_2 and say that Φ_2 is the two-step projection of Φ. Iterating we define, for any $k \leq i - 1$, the projection Φ_k. It is a local symmetry at level $(i - k)$.

We do not use the projection Φ_i onto level 0. This projection does not respect symmetries, since the general contact transformation does not preserve the fibers of $\mathbb{P}^1\mathbb{R}^2 \to \mathbb{R}^2$. (See section 1.5).

Proposition 2.9. *Let* Φ *be a local symmetry at level* i.

(i) *Projection is inverse to prolongation:*

$(\Phi^k)_k = \Phi$ *for* $k \geq 0$*;*

if $k \leq i - 1$ *then* $(\Phi_k)^k = \Phi$.

(ii) *Projection and prolongation of integral curves commute with symmetries:*

$(\Phi \circ \Gamma)^k = \Phi^k \circ \Gamma^k$ *for* Γ *a non-constant integral curve at level* i*;*

if $k \leq i - 1$ *then* $(\Phi \circ \Gamma)_k = \Phi_k \circ \Gamma_k$.

The proof is almost verbatim the same as the proof of Propositions 2.6 and 2.7.

2.4. Proof of Theorem 2.2

Assume that the integral curve germs Γ and $\widetilde{\Gamma}$ are equivalent so that there exist a local symmetry $\Phi : (\mathbb{P}^{1+k}\mathbb{R}^2, p) \to (\mathbb{P}^{1+k}\mathbb{R}^2, \tilde{p})$ and a local diffeomorphism $\phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $\widetilde{\Gamma} = \Phi \circ \Gamma \circ \phi$. Let $\Psi = \Phi_k$. Then Ψ is a local contactomorphism of $\mathbb{P}^1\mathbb{R}^2$. By Proposition 2.9, (ii)

$$\widetilde{\gamma} = \widetilde{\Gamma}_k = (\Phi \circ \Gamma \circ \phi)_k = \Phi_k \circ \Gamma_k \circ \phi = \Psi \circ \gamma \circ \phi,$$

i.e. their Legendrian projections γ and $\widetilde{\gamma}$ are RL-contact equivalent.

Assume now that the integral curve germs γ and $\widetilde{\gamma}$ are RL-contact equivalent, i.e. there exists a local contactomorphism Ψ of $\mathbb{P}^1\mathbb{R}^2$ and a local diffeomorphism $\phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $\widetilde{\gamma} = \Psi \circ \gamma \circ \phi$. By Proposition 2.9, (i) $\Gamma = \gamma^k, \widetilde{\Gamma} = \widetilde{\gamma}^k$. Let $\Phi = \Psi^k$. Then Φ is a local symmetry at level $(1 + k)$ and by Propositions 2.6, 2.7 and 2.9

$$\widetilde{\Gamma} = \widetilde{\gamma}^k = (\Psi \circ \gamma \circ \phi)^k = \Psi^k \circ \gamma^k \circ \phi = \Phi \circ \Gamma \circ \phi,$$

i.e. the curve germs Γ and $\widetilde{\Gamma}$ are equivalent.

2.5. From curves to points

Notation 2.10. By $\mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ we mean the space of germs at $t = 0$ of non-constant analytic Legendrian curves $\gamma : (-\epsilon, \epsilon) \to \mathbb{P}^1\mathbb{R}^2$.

Prolongation k times defines a canonical map

$$\mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2) \to \mathbb{P}^{1+k}\mathbb{R}^2 : \quad \gamma \to \gamma^k(0).$$

Theorem 2.11. *Any point $p \in \mathbb{P}^{1+k}\mathbb{R}^2$ is realized by evaluating the k-step-prolongation of some germ $\gamma \in \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$, i.e. $p = \gamma^k(0)$.*

Theorem 2.12. *Let $\gamma, \widetilde{\gamma} \in \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ be equivalent germs. Then the points $p = \gamma^k(0)$, $\tilde{p} = \widetilde{\gamma}^k(0)$ are equivalent in $\mathbb{P}^{1+k}\mathbb{R}^2$.*

Theorem 2.12 is an immediate corollary of Theorem 2.2. In fact, by Theorem 2.2 the integral curve germs $\Gamma = \gamma^k$ and $\widetilde{\Gamma} = \widetilde{\gamma}^k$ are equivalent, i.e. there exists a symmetry of the Monster $\Phi : (\mathbb{P}^{1+k}\mathbb{R}^2, p) \to (\mathbb{P}^{1+k}\mathbb{R}^2, \tilde{p})$ and a reparameterization $\phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $\widetilde{\Gamma} = \Phi \circ \Gamma \circ \phi$. It follows $\widetilde{\Gamma}(0) = \Phi(\Gamma(0))$ hence the points $p = \Gamma(0)$ and $\tilde{p} = \widetilde{\Gamma}(0)$ are equivalent.

Theorem 2.11 follows from Proposition 2.6. Take an integral curve germ $\Gamma : (\mathbb{R}, 0) \to (\mathbb{P}^{1+k}\mathbb{R}^2, p)$ whose projection γ to the first level is not a constant curve. By Proposition 2.6 $\Gamma = \gamma^k$ hence $p = \Gamma(0) = \gamma^k(0)$.

This proof of Theorem 2.11 contains a gap. We must establish the existence of an integral curve Γ passing through p whose projection to the first level is not constant. Existence is less obvious than it looks and is proved in section 2.7.

Warning. The Legendrian curve germ γ in Theorem 2.11 is far from unique. If γ is immersed, then the knowledge of its first k derivatives at 0, i.e of the k-jet $j^k\gamma(0)$ determines the value of its prolongation $\gamma^k(0)$ at 0. (See Proposition 4.44 in section 4.8.) But if γ is not immersed then we may need many more of its derivatives at 0 before we can compute $\gamma^k(0)$. Related to this fact, we will see later on that when p is a "sufficiently singular" point of $\mathbb{P}^{1+k}\mathbb{R}^2$ then to realize it as $p = \gamma^k(0)$ as in Theorem 2.11 we may need to know $j^r\gamma(0)$ for $r >> k$. The determination of the minimum such r for a given p is related to the "jet-identification " and "parameterization" numbers to be introduced further on.

2.6. Non-singular points

We have several equivalent definitions of a singular point in the Monster. One of them is:

DEFINITION 2.13. A point $p \in \mathbb{P}^{1+k}\mathbb{R}^2$ is called *non-singular* (or a Cartan point) if it can be realized by evaluating the k-step-prolongation of an *immersed* Legendrian curve; thus $p = \gamma^k(0)$ for some $\gamma \in \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ with $\gamma'(0) \neq 0$. Otherwise the point p is called *singular*.

WARNING. By definition, the point p is non-singular if the class of curves $\gamma \in \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ such that $p = \gamma^k(0)$ *contains* an immersed curve. But this class of curves will also contain singular Legendrian curves. Indeed, form $\sigma = \gamma \circ \phi$ where $\phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ is a singular parameterization. Then σ is a singular Legendrian curve and $p = \sigma^k(0)$.

CLAIM 2.14. Every point in the first level $\mathbb{P}^1\mathbb{R}^2$ or the second level $\mathbb{P}^2\mathbb{R}^2$ of the Monster is non-singular.

Indeed, there is an immersed Legendrian curve passing through any fixed point p in the first level in any direction in $\Delta^1(p)$.

We have the following reformulation of Cartan's Theorem 1.4 from section 1.6.

THEOREM 2.15. *All non-singular points in* $\mathbb{P}^{1+k}\mathbb{R}^2, k \geq 0$ *are equivalent.*

PROOF. Theorem 2.15 is a direct corollary of Theorem 2.12 and the well-known theorem on local contact equivalence of all non-singular equal-dimensional integral submanifolds of a contact manifold. See for example [**AG**] for this last. □

Theorem 2.15 and Claim 2.14 imply the classical Darboux and Engel theorem formulated in Monster terms, Theorem 1.3.

2.7. Critical curves

We introduce a class of curves which play a central role in all subsequent constructions.

DEFINITION 2.16. Suppose $k \geq 2$. Then an integral curve Γ in the kth level of the Monster is called *critical* if its $(k-1)$-step-projection is a constant curve in $\mathbb{P}^1\mathbb{R}^2$. If $k = 0$ or $k = 1$, then by convention a curve is critical if and only if it is a constant curve.

To complete the proof of Theorem 2.11 we have to prove that for any point $p \in \mathbb{P}^{1+k}\mathbb{R}^2$ there exists an integral curve passing through p which is **not** critical. This follows from Proposition 2.24 below.

For $k \geq 2$ the fibers of the circle bundle projection $\mathbb{P}^k\mathbb{R}^2 \to \mathbb{P}^{k-1}\mathbb{R}^2$ are critical curves. Such a fiber can be parameterized as

$$t \to (m(t), \ell(t)) \in \mathbb{P}^k\mathbb{R}^2, \quad m(t) \equiv m \in \mathbb{P}^{k-1}\mathbb{R}^2, \quad \ell(t) \in \Delta^{k-1}(m). \tag{2.2}$$

and is immersed if $\ell'(t) \neq 0$.

Definition 2.17. Following standard bundle terminology, integral curves of the form (2.2) will be called *vertical* curves.

Remark 2.18. According to Definition 2.16, non-constant vertical curves at the first level are not to be called critical. The rationale for excluding them as critical (or "special") is that the group of contactomorphisms acts transitively on immersed Legendrian curve germs: vertical contact curves are the same as any other immersed Legendrian curve.

Vertical curves do not exhaust the class of critical curves. Prolongations of vertical curves are also critical.

Proposition 2.19. *Let V be an immersed vertical curve at level $k \geq 2$. Its prolongation $V^s, s \geq 1$ is an immersed integral critical curve at level $(k+s)$ which is neither vertical nor tangent to an immersed vertical curve in level $(k+s)$.*

Proof. That V^s is integral and critical follows immediately from the facts of prolongation and the definitions of critical and vertical curves. It is immersed since the prolongation of an immersed integral curves is immersed. It remains to prove the final statement of the proposition. Express V^s in the form $t \to (m(t), \ell(t))$ where $m(t)$ is a curve of points at level $(k+s-1)$ and $\ell(t)$ is a line in $\Delta^{k+s}(m(t))$. We must prove that $m'(t_0) \neq 0$ for any t_0 in the interval of definition of V. In other words, we must prove that the one-step projection $(V^s)_1$, being the curve $t \to m(t)$, is immersed. But $(V^s)_1 = V^{s-1}$ by Proposition 2.6 and the prolongations of an immersed curve (here V) are immersed. $\square$

Theorem 2.20. *Any immersed critical curve Γ in $\mathbb{P}^k\mathbb{R}^2, k \geq 2$ is either a vertical curve or the $(k-s)$-step prolongation of an immersed vertical curve at level $\mathbb{P}^s\mathbb{R}^2$ where $2 \leq s \leq k-1$. The integer s is uniquely determined by Γ when Γ is not vertical.*

Remark 2.21. The set of all integral lines *tangent* to a fixed point form the fiber, or vertical curve, over that point. Consequently it makes sense to say that the 1st prolongation of the constant curve m is the vertical curve projecting onto m. Theorem 2.20 thus asserts that the collection of all critical curves coincides with the prolongations (1st and higher order) of constant curves sitting at level 1 or higher.

Remark 2.22. Each critical curve has a projective parameterization, unique up to projectivity $t \mapsto at+b/(ct+d)$. Indeed, each fiber is a projective line, and symmetries preserve the fibers at level 2 or higher, and act by projective transformations on these fibers. This shows that the vertical curves have a unique projective parameterization. Use the same parameterization on the prolongations of the vertical curves to get the projective parameterization of the other critical curves.

Here is a sub-Riemannian characterization of critical curves.

Theorem 2.23. *Let Γ be an immersed curve in $\mathbb{P}^i\mathbb{R}^2, i \geq 2$. The following properties are equivalent:*

(i) Γ *is critical.*

(ii) Γ *is singular (= abnormal) in the sense of sub-Riemannian geometry.*

(iii) Γ *is locally C^1-rigid.*

This last theorem will not be used in the body of the paper. It is proved in Appendix C where the relevant definitions (singular, abnormal, and C^1-rigid) are also recalled.

Proof of Theorem 2.20. The projection $\pi_{k,1} \circ \Gamma$ is a constant curve. Let s be the maximal number such that the projection $\pi_{k,s-1} \circ \Gamma$ is a constant curve. If Γ is not vertical then $2 \leq s < k-1$. Consider the projection $V = \pi_{k,s} \circ \Gamma$. It is a non-constant vertical curve in level s. By Proposition 2.6 Γ is the $(k-s)$-step-prolongation of V: $\Gamma = V^{k-s}$.

Let us prove that the constructed vertical curve V is immersed. Assume, by way of contradiction, that it is not immersed at some point t_0 of the interval of its definition. Take any immersed vertical curve germ $\widehat{V}$ at t_0 such that $\widehat{V}(t_0) = V(t_0)$. Then near t_0 one has $V(t) = \widehat{V}(\phi(t))$ for some analytic map germ $\phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $\phi'(0) = 0$. Since Γ is not constant and analytic, $\phi(t) \not\equiv 0$ and by Proposition 2.7 $\Gamma(t) = (\widehat{V}(\phi(t))^{k-s} = \widehat{V}^{k-s}(\phi(t))$. Differentiating this relation at $t = t_0$ we obtain $\Gamma'(t_0) = (\widehat{V}^{k-s})'(t_0) \cdot \phi'(t_0) = 0$. This contradicts the assumption that Γ is immersed.

We have proved, for Γ not vertical, the existence of the integer s of the theorem. We prove s is unique. We must prove that if V is an immersed vertical curve at level s and if $\widetilde{V}$ is another immersed vertical curve at level $\tilde{s}$, with $s, \tilde{s} < k$, and if $V^{k-s} = \widetilde{V}^{k-\tilde{s}}$ then $s = \tilde{s}$. Assume, by way of contradiction, that $s > \tilde{s}$. By Proposition 2.6 we have

$$V^{k-s} = \widetilde{V}^{k-\tilde{s}} \implies (V^{k-s})_{k-s} = (\widetilde{V}^{k-\tilde{s}})_{k-s} \implies V = \widetilde{V}^{s-\tilde{s}}.$$

Since $s > \tilde{s}$, by Proposition 2.19 the curve $\widetilde{V}^{s-\tilde{s}}$ is not vertical, and we get a contradiction. The proof of theorem 2.20 is completed.

If $\Gamma : (\mathbb{R}, 0) \to (\mathbb{P}^k\mathbb{R}^2, p)$ is an integral curve germ and Γ is the $(k-s)$-step-prolongation of an integral curve germs V in $\mathbb{P}^s\mathbb{R}^2$ then $V(0) = \pi_{k,s}(p)$. (See Proposition 2.6.) Note also that all immersed vertical curve germs through the same point of the Monster can be obtained from each other by reparameterization. These observations, the fact that prolongations respect reparameterizations (Proposition 2.7), Theorem 2.20 and the last statement of Proposition 2.19 imply the following characterization of the immersed critical integral curves.

Proposition 2.24. *Let $p \in \mathbb{P}^k\mathbb{R}^2, k \geq 2$. One of the following holds:*

(a) *There is, up to reparameterization, a unique immersed critical integral curve germ passing through p. It is the germ of the vertical curve.*

(b) *There are, up to reparameterizations, exactly two immersed critical integral curve germs through p. One of them is the germ of the vertical curve through p. The other is the prolongation of some vertical curve from a lower level $s = s(p)$, $2 \leq s \leq k-1$. These two critical curve germs have different directions at p.*

Remark. In section 2.12 we will prove that (a) holds for regular points and (b) - for critical points (these two types of points are defined in the next section).

Completion of the proof of theorem 2.11. Proposition 2.24 implies the existence of non-critical integral curve through any point p of the Monster: take any integral curve besides the one or two curves described in the proposition. This curve is the prolongation of some non-constant Legendrian curve. The proof of Theorem 2.11 is complete.

2.8. Critical and regular directions and points

In this section we decompose integral directions and then points of the Monster into two types: regular and critical. Recall (Definition 2.16) that an integral curve is critical if its projection to the first level is a constant curve in $\mathbb{P}^1\mathbb{R}^2$.

DEFINITION 2.25. Let $m \in \mathbb{P}^k\mathbb{R}^2$. A line ℓ in $\Delta^k(m)$ is called critical if it is tangent to an immersed critical integral curve, i.e. if there exists a critical integral curve germ $\Gamma : (\mathbb{R}, 0) \to (\mathbb{P}^k\mathbb{R}^2, m)$ such that ℓ is spanned by $\Gamma'(0)$. Otherwise the line ℓ is called regular.

PROPOSITION 2.26. *All lines in $\Delta^0 = T\mathbb{R}^2$ and Δ^1 are regular. If $p \in \mathbb{P}^k\mathbb{R}^2$ and $k \geq 2$ then the 2-plane $\Delta^k(p)$ contains at least one and at most two critical lines.*

PROOF. The first statement follows from Definition 2.16 since by this definition there are no immersed critical integral curves in $\mathbb{R}^2$ or $\mathbb{P}^1\mathbb{R}^2$. The second statement is a direct corollary of Proposition 2.24. □

DEFINITION 2.27. Let $m \in \mathbb{P}^k\mathbb{R}^2$ and ℓ be a line in $\Delta^k(m)$. The point $p = (m, \ell) \in \mathbb{P}^{1+k}\mathbb{R}^2$ is called regular or critical depending on whether or not the line $\ell \subset \Delta^k(m)$ is regular or critical.

DEFINITION 2.28. Let $m \in \mathbb{P}^k\mathbb{R}^2$. The fiber above m is the embedded projective line $\pi^{-1}_{k+1,k}(m) \subset \mathbb{P}^{i+1}\mathbb{R}^2$, i.e. the set $\{(m, \ell) \in \mathbb{P}^{1+k}\mathbb{R}^2\}$, where m is fixed and ℓ varies over $\Delta^k(m)$.

The following statement is a direct corollary of Proposition 2.26.

PROPOSITION 2.29. *All points of the first and the second level of the Monster, $\mathbb{P}^1\mathbb{R}^2$ and $\mathbb{P}^2\mathbb{R}^2$, are regular. Any higher level contains regular and critical points. Furthemore, if $k \geq 2$ and $m \in \mathbb{P}^k\mathbb{R}^2$ then the fiber above m contains at least one and at most two critical points.*

The first statement of Proposition 2.29 corresponds to the classical Darboux and Engel theorems expressed in Monster terms (Theorem 1.3.)

The reader should not confuse the partitioning of the Monster into critical or regular points with its partitioning into singular or non-singular points. The Definition 2.13 of a non-singular point (= Cartan point) is quite different from the definition of a regular point. Theorem 2.39 in section 2.11 provides an alternate equivalent definition of non-singular point, asserting that

Every non-singular point is regular. At level $i \geq 3$ regular points occur which are singular in addition to the regular non-singular points.

2.9. Regular integral curves

We have defined critical curves and used them to define regular and critical directions and points. But we have not yet defined regular curves.

DEFINITION 2.30. An integral curve germ $\Gamma : (\mathbb{R}, 0) \to (\mathbb{P}^k\mathbb{R}^2, p)$ is *regular* if it is immersed and the line spanned by its tangent $\Gamma'(0)$ is a regular line.

It is worth noting that Definition 2.30 and Proposition 2.26 imply:

A Legendrian curve in the first level $\mathbb{P}^1\mathbb{R}^2$ is regular if and only if it is immersed. At every higher level of the Monster there are immersed critical integral curves.

We have partitioned directions and points into two disjoint sets "critical" and "regular". Every point and every direction is either critical or it is regular. For integral curves this strict partitioning is not valid. At any level of the Monster there are integral curves which are neither critical nor regular. Indeed, according to Definition 2.30 any non-constant non-immersed integral curve in the first level is neither regular nor critical. According to Proposition 2.24 at any higher level there exist immersed integral curves which are neither regular nor critical, for example, non-critical curves tangent to critical directions.

The following two propositions relate regular points with regular curves.

PROPOSITION 2.31. *Let $\Gamma : (\mathbb{R}, 0) \to \mathbb{P}^k\mathbb{R}^2$ be the germ of a regular integral curve.*

(i) *Its prolongations $\Gamma^i, i \geq 0$ are regular curves.*

(ii) *The points $\Gamma^i(0), i \geq 1$ are regular.*

PROOF. Statement (ii) is an immediate corollary of (i) (recall that any regular integral curve is immersed). To prove (i) assume that Γ^i is not regular, i.e. tangent to a critical curve. Then $\Gamma = (\Gamma^i)_i$ is tangent to the i-step projection of this critical curve which is itself a critical curve. This contradicts the regularity of Γ. □

To formulate a converse to Proposition 2.31 we need the following definition.

DEFINITION 2.32. Let p be a regular point in the kth level of the Monster and let $q = \pi_{k,k-r}(p)$. If the points $\pi_{k,k-1}(p), \pi_{k,k-2}(p), \ldots, \pi_{k,k-r+1}(p)$ are all regular then we will say that p is a r-step-regular-prolongation of q. The point q might be either regular or critical.

PROPOSITION 2.33. *Let $q \in \mathbb{P}^s\mathbb{R}^2$, $s \geq 1$. If a point $p \in \mathbb{P}^k\mathbb{R}^2$ is a regular prolongation of q and $\Gamma : (\mathbb{R}, 0) \to (\mathbb{P}^k\mathbb{R}^2, p)$ is a regular integral germ curve then its projection to level s is also a regular integral curve germ $(\mathbb{R}^2, 0) \to (\mathbb{P}^s\mathbb{R}^2, q)$.*

To prove this statement we need the following lemma.

LEMMA 2.34. *The one-step prolongation of a non-immersed integral curve germ Γ in the kth level of the Monster, $k \geq 1$ is not a regular curve.*

PROOF. Let $\Gamma : t \to m(t)$, $m(t) \in \mathbb{P}^k\mathbb{R}^2$, $m'(0) = 0$. The curve Γ^1 has the form $(m(t), \ell(t))$ with $\ell(t)$ a line in $\Delta^k(m(t))$. If Γ^1 is not immersed then it is not regular, according to Definition 2.30. Assume that Γ^1 is immersed. Choose a local trivialization $\Psi : U \times \mathbb{RP}^1 \to \mathbb{P}^{1+k}\mathbb{R}^2$ of the fibration $\pi_{k+1,k}$, so that $U \subset \mathbb{P}^k\mathbb{R}^2$ is an open set containing $\Gamma(0)$. Then $\phi^{-1}(\Gamma^1(t)) = (m(t), s(t))$ with $s(t)$ a curve in $\mathbb{RP}^1$. Consider the vertical curve $\widehat{\Gamma} : t \to \phi^{-1}(m(0), s(t))$. Since $m'(0) = 0$, we must have that $s'(0) \neq 0$ so that this curve is immersed. Being vertical, $\widehat{\Gamma}$ is a critical integral curve, and it is tangent to Γ^1. Therefore Γ^1 is not regular. □

PROOF OF PROPOSITION 2.33. By Lemma 2.34 the one-step-projection Γ_1 is an immersed curve. Since the point p is regular and $p = (\Gamma_1)^1(0)$, the curve Γ_1 has a regular direction at $t = 0$, i.e. Γ_1 is a regular curve. Now Lemma 2.34 implies that the two-step-projection Γ_2 is an immersed curve. Let p_1 be the one-step projection of p. Since the point p_1 is regular and since $p_1 = (\Gamma_2)^1(0)$, the curve Γ_2 is regular. Continuing in this manner we find that the projections of Γ to levels $k-1, k-2, ..., s$ are regular curves.

2.10. Regularization theorem

One might expect that by prolonging a singular integral curve Γ we make its singularity "simpler". If so, then its higher prolongations Γ^i should be even simpler. Does the singularity eventually disappear, or "resolve" as i increases? We must be careful here about what we mean by "resolve". To say that that Γ^i is immersed is not enough. Critical curves are immersed but must be considered as degenerate or "singular", coming as they do from prolongations of constant curves. We will say that the singularity Γ has been resolved after k prolongations if Γ^k is regular. Recall (Definition 2.30) that a regular curve is one that is immersed *and* not tangent to any critical curve.

Definition 2.35. Recall that an analytic curve germ $\Gamma : (\mathbb{R}, 0) \to \mathbb{M}^n$ in an n-manifold M^n is called "badly parameterized" if it can be factored as $c(t) = \mu\big(\phi(t)\big)$ with $\mu : (\mathbb{R}, 0) \to M^n$ and $\phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ analytic germs such that $d\phi/dt(0) = 0$. If Γ is not badly parameterized it is called well-parameterized.

For example, the plane curve germ $t \to (t^6, t^9 + t^{15})$ is badly parameterized and the plane curve germ $t \to (t^6, t^9 + t^{14})$ is well-parameterized.

Theorem 2.36. *With the exception of the critical curves, every analytic well-parameterized integral curve germ Γ in the Monster admits a regular prolongation, i.e. there exists k such that the prolongation Γ^k is a regular curve. In particular any analytic well-parameterized plane curve germ admits a regular prolongation. On the other hand, if Γ is badly parameterized then it does not admit a regular prolongation.*

In the final statement of the theorem the claim that Γ does not admit a regular prolongation can be replaced by a stronger claim that all prolongations of Γ are badly parameterized curves. This is a direct corollary of the following proposition.

Proposition 2.37. *If $\Gamma : (\mathbb{R}, 0) \to (\mathbb{P}^i\mathbb{R}^2)$, $i \geq 0$ is a badly parameterized curve germ in any level of the Monster then any its prolongation Γ^k is also badly parameterized.*

Proof. Follows from Proposition 2.7. Indeed, if Γ is badly parameterized then $\Gamma(t) = \widetilde{\Gamma}(\phi(t))$ where $d\phi(0) = 0$ and by Proposition 2.7 one has $\Gamma^k(t) = \widetilde{\Gamma}^k(\phi(t))$ which means that the prolongation Γ^k is also badly parameterized. $\square$

Now let us show that Theorem 2.36 for integral curves reduces to the case of plane curves. The reduction is based on Propositions 2.6, 2.31, 2.37, and the following simple lemma.

Lemma 2.38. *Let $\gamma : (\mathbb{R}, 0) \to \mathbb{R}^3$ be a space curve, let $S \subset \mathbb{R}^3$ be any non-singular surface, and let $\pi : \mathbb{R}^3 \to S$ be any projection. If $\pi \circ \gamma$ is a constant curve then the curve γ is either immersed or badly parameterized.*

Proof. We can take local coordinates x, y, z centered at the point $\gamma(0)$ such that $S = \{z = 0\}$ and $\pi : (x, y, z) \to (x, y)$. In these coordinates $\gamma(t) = (0, 0, z(t))$. If $z'(0) \neq 0$ then γ is immersed, and if $z'(0) = 0$ then γ is badly parameterized. $\square$

Reduction of the regularization Theorem 2.36 to plane curves. Assume that Theorem 2.36 is proved for plane curve germs. Then the proof for any well-parameterized non-critical integral curve germ Γ in $\mathbb{P}^i\mathbb{R}^2$ is as follows. We will assume that Γ is not a regular curve, otherwise there is nothing to prove.

Consider the plane curve germ $c = \Gamma_i$, and the Legendrian curve germ $\gamma = \Gamma_{i-1}$, the projections of Γ to the zeroth and the first level. They are related by $\gamma = c^1$. Since Γ is not a critical curve, γ is not a constant curve. By Proposition 2.6 $\Gamma = \gamma^{i-1}$.Since the prolongations of regular curves are regular (Proposition 2.31) and since Γ is not regular, we see that γ is not regular. A Legendrian curve in the first level is regular if and only if it is immersed. On the other hand, since Γ is well-parameterized so is γ (Proposition 2.37). Thus γ is a well-parameterized non-immersed curve germ in $\mathbb{P}^1\mathbb{R}^2$. Lemma 2.38 now implies that its one-step projection $\gamma_1 = c$ is a well-parameterized non-constant curve.

We have proved that $c = \Gamma_i$ is a non-constant curve. Now we can use Proposition 2.6 relating c and Γ as follows: $\Gamma = c^i$. By Proposition 2.37 the curve c is well-parameterized. Then, assuming that Theorem 2.36 holds for plane curve germs, there exists r such that the r-step prolongation c^r is a regular curve. Since $c^r = \Gamma^{r-i}$ provided that $r > i$, in the case $r > i$ the curve Γ admits a regular prolongation: the $(r - i)$-step prolongation of Γ is a regular curve. The case $r \leq i$ contradicts to our assumption that Γ is not regular. In fact, if $r \leq i$ then $\Gamma = (c^r)^{i-r}$ and since the curve c^r is regular, by Proposition 2.31 so is Γ. The reduction to plane curve germs is completed. $\square$

We have just reduced the proof of the regularization Theorem 2.36 to the case of plane curves. We postpone the proof in this case because it is not simple and requires results and constructions developed in further chapters. We will give two independent proofs of this plane curve case of the theorem. The first proof, given in section 3.8.1, is based on the Puiseux characteristic of a plane curve singularity, and results of Chapter 3, section 3.8. The second proof, given in Chapter 8, requires the introduction of special "KR" coordinates on the Monster (Chapter 7), as well as an operator on the space of plane curve germs which we call directional blow-up which we introduce at the beginning of Chapter 8.

2.11. An equivalent definition of a non-singular point

Theorem 2.39. *A point in the Monster is non-singular (*Defn. 2.13) *if and only if it is the regular prolongation* (Defn. 2.32) *of some point in the plane* $\mathbb{R}^2$.

Proof. Assume that the point p at level k is non-singular. Then there exists an immersed Legendrian curve germ $\gamma : (\mathbb{R}, 0) \to \mathbb{P}^1\mathbb{R}^2$ such that $\gamma^k(0) = p$. Any immersed integral curve in the first level is regular, so γ is a regular curve. By Proposition 2.31 the points $\gamma^i(0), i \geq 1$ are regular. By Proposition 2.6 for $i \leq k$ one has $\pi_{k,i}(p) = \pi_{k,i}(\gamma^k(0)) = \gamma^i(0)$. Hence $\pi_{k,i}(p)$ are regular points and p is the $(k-1)$-step-regular-prolongation of the point $\gamma(0) \in \mathbb{P}^1\mathbb{R}^2$. Consequently p is the k-step-regular-prolongation of the projection of this point to the plane, the point $\pi_{1,0}(\gamma(0) \in \mathbb{R}^2$.

Assume now that the point p at the kth level is the regular prolongation of a point in the plane $\mathbb{R}^2$. We must prove that p is a non-singular point as per Definition 2.13. Take a regular integral curve germ $\Gamma : (\mathbb{R}, 0) \to (\mathbb{P}^k\mathbb{R}^2, p)$. (The existence of such a Γ follows from Proposition 2.26). By definition of regular prolongation, the points $\pi_{k,i}(p), i \leq k$ are all regular points. Then by Proposition 2.33 the $(k-1)$-step-projection $\gamma = \Gamma_{k-1}$ is a regular curve, and consequently an immersed Legendrian curve in $\mathbb{P}^1\mathbb{R}^2$. By Proposition 2.6 $p = \gamma^{k-1}(0)$. Therefore, according to the definition, p is a non-singular point. $\square$

2.12. Vertical and tangency directions and points

We decompose critical directions and points into vertical and tangency directions and points. The terminology "tangency" will not be explained until section 3.6 in Theorem 3.15.

Definition 2.40. Let $m \in \mathbb{P}^k\mathbb{R}^2, k \geq 2$, and let ℓ be a critical line in $\Delta^k(m)$. The line ℓ is called vertical if it is tangent to the immersed vertical curve through m, that is, if there exists an immersed vertical curve germ $\Gamma : (\mathbb{R}, 0) \to (\mathbb{P}^k\mathbb{R}^2, m)$ such that $\ell = \text{span}\,(\Gamma'(0))$. Otherwise the line ℓ is called tangency line.

Remark. An equivalent definition of the vertical line $\ell \subset \Delta^k(m), k \geq 2$ is $\ell = ker(d\pi_{k,k-1}(m))$.

Note that for $p \in \mathbb{P}^k\mathbb{R}^2$, $k \geq 2$ the 2-plane $\Delta^k(p)$ contains a unique vertical line.

Proposition 2.41. (see Figure 1). *Let $p \in \mathbb{P}^k\mathbb{R}^2$ and $k \geq 2$. If p is a regular point then $\Delta^k(p)$ contains no tangency lines, i.e. it contains only one critical line which is the vertical line. If p is a critical point then $\Delta^k(p)$ contains exactly two different critical lines: the vertical line and the tangency line.*

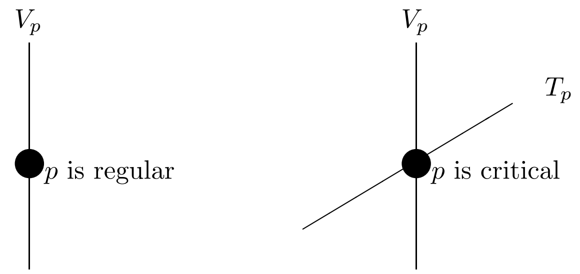

Figure 1. If p is regular, there is only one critical line in $\Delta^k(p)$, the vertical line V_p. If p is critical there are exactly two critical lines in $\Delta^k(p)$, the vertical line V_p and the tangency line T_p.

Proof. Let p be a regular point. Write $p = (m, \ell)$ where $m \in \mathbb{P}^{k-1}\mathbb{R}^2$ and ℓ is a line in $\Delta^{k-1}(m)$. By definition of a regular point, the line ℓ is regular. Let $\Gamma : t \to (m(t), \ell(t))$ be an immersed integral critical curve in the kth level such that $\Gamma(0) = p = (m, \ell)$. To prove the first statement we have to show that Γ is a vertical curve. The projections of critical curves are critical so Γ_1 is critical. We claim Γ_1 is not immersed at $t = 0$. For if it were immersed, then the line ℓ would be tangent to the critical curve Γ_1, contradicting the regularity of this line. It follows from

$\Gamma_1'(0) = 0$ that Γ is tangent at $t = 0$ to the vertical line $ker(d\pi_{k,k-1}(p))$ in $\Delta^k(p)$. Since Γ is critical, by Proposition 2.24 it is a vertical curve.

Consider take p to be a critical point. In view of Proposition 2.24 to prove the second statement of Proposition 2.41 it suffices to construct an immersed critical curve germ $\Gamma : (\mathbb{R}, 0) \to (\mathbb{P}^k\mathbb{R}^2, p)$ which is not vertical. Write $p = (m, \ell)$ as above. Since p is critical, the line $\ell \in \Delta^{k-1}(m)$ is critical and consequently there is an immersed critical integral curve germ $\Psi : (\mathbb{R}, 0) \to (\mathbb{P}^{k-1}\mathbb{R}^2, m)$ such that $\ell = \text{span}(\Psi'(0))$. Let $\Gamma = \Psi^1$ be the one-step-prolongation of Ψ. Then $\Gamma(0) = (m, \ell) = p$, Γ is immersed and critical, and Γ is not vertical since by Proposition 2.6 its one-step-projection is the immersed curve Ψ. □

The decomposition of critical directions into the vertical and tangency directions induces the decomposition of critical points into vertical and tangency points. Recall (Proposition 2.29) that there are no critical points at the first two levels.

Definition 2.42. Let $m \in \mathbb{P}^k\mathbb{R}^2, k \geq 2$ and ℓ is a line in $\Delta^k(m)$. Assume that $p = (m, \ell) \in \mathbb{P}^{1+k}\mathbb{R}^2$ is a critical point. It is called vertical or tangency depending on whether or not the line $\ell \subset \Delta^k(m)$ is a vertical or tangency direction.

The following theorem on the structure of regular, vertical, and tangency points is an immediate corollary of Proposition 2.41.

Theorem 2.43. (see Figure 2). *Let $k \geq 2$ and let $m \in \mathbb{P}^k\mathbb{R}^2$. If m is a regular point then the fiber above m contains no tangency points, i.e. it contains only one critical point which is the vertical point. If m is a critical point then the fiber above m contains exactly two critical points: the vertical point and the tangency point.*

Remark 2.44. Since any point at the second level is regular, there are no tangency points at the third level. Every higher level contains tangency points.

Vertical and tangency points are obtained from vertical curves by prolongation:

Proposition 2.45. *Let $V = V(t)$ be an immersed vertical curve in the kth level of the Monster, $k \geq 2$. Then the following holds for any t in the interval of definition of V:*

1. *The point $V^1(t)$ is vertical.*
2. *If $i \geq 2$ then the point $V^i(t)$ is tangency.*

Proof. The one-step-prolongation of V is an immersed critical integral curve of the form $V^1(t) = (V(t), \ell)$ where $\ell = \text{span}(V'(t))$. Since V is vertical, the line ℓ is vertical, hence $V^1(t)$ is a vertical point.

Now take $i \geq 2$. The i-step-prolongation of V is an immersed critical integral curve of the form $V^i(t) = (V^{i-1}(t), \ell)$ where $\ell = \text{span}(V^{i-1})'(t)$. Since $i - 1 \geq 1$, by Proposition 2.19 the curve V^{i-1} is immersed and critical, but it is not vertical. Being analytic, its restriction to any subinterval of its domain of definition is not vertical. Therefore the line ℓ is critical, but not vertical, so that ℓ is the tangency line within the distribution at $V^{i-1}(t)$. Hence $V^i(t)$ is a tangency point. □

Definition 2.46. A set of points in the same level of the Monster is a singularity class if it is closed with respect to the equivalence of points. A set of integral curves in the same level is a singularity class if it is closed with respect to equivalence of integral curves.

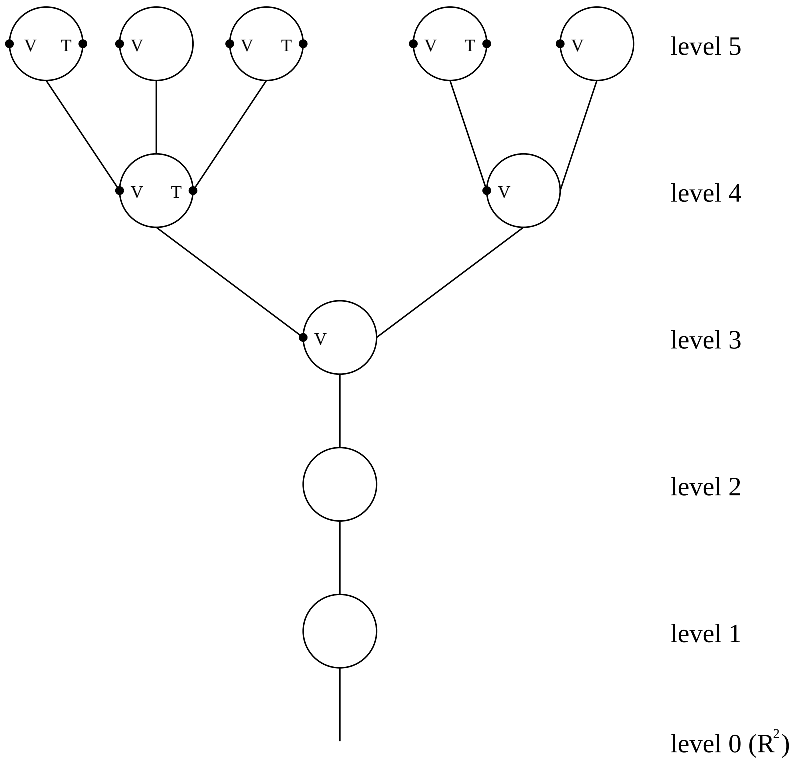

FIGURE 2. The structure of regular and critical points. Critical points are bold, other points are regular. Vertical points are marked by V and tangent points by T.

PROPOSITION 2.47. *The set of regular points, the set of critical points, the set of vertical points and the set of tangency points are all singularity classes in the Monster at level k, $k \geq 3$. The set of regular curves, the set of critical curves, and the set of vertical curves are all singularity classes of curve germs within the Monster at level k, $k \geq 2$.*

PROOF. The first statement follows from the second one, and the second statement follows from Propositions 2.7 and 2.9. $\square$

CHAPTER 3

RVT classes. RVT codes of plane curves. RVT and Puiseux

3.1. Definition of RVT classes

Theorem 2.43 and Figure 2 make clear that a point p of the Monster can have one of several types, R for Regular, V for Vertical, and T for Tangency, provided the level k of that point is greater than or equal to 3. We associate a code to p by recording its type, and the type of its projections to the lower levels.

Definition 3.1. Associate to each point $p \in \mathbb{P}^k\mathbb{R}^2$ the $(k-2)$-tuple

$$(\alpha) = (\alpha_1, ..., \alpha_{k-2}), \qquad \alpha_i = \begin{cases} \mathrm{R} & \text{if} \quad \pi_{k,i+2}(p) \quad \text{is a regular point} \\ \mathrm{V} & \text{if} \quad \pi_{k,i+2}(p) \quad \text{is a vertical point} \\ \mathrm{T} & \text{if} \quad \pi_{k,i+2}(p) \quad \text{is a tangency point} \end{cases}$$

Here $i = 1, ..., k-2$ and $\pi_{k,k}(p) = p$. We call (α) the RVT-code of p. If $k < 3$ the RVT-code is empty. The letters V and T will be called *critical letters*.

Example 3.2. Let $p \in \mathbb{P}^5\mathbb{R}^2$ be a regular point. Assume that the projection $\pi_{5,3}(p) \in \mathbb{P}^3\mathbb{R}^2$ is a vertical point. Then the RVT-code of the point p is either VRR, or VVR, or VTR depending on the type of the point $\pi_{5,4}(p) \in \mathbb{P}^4\mathbb{R}^2$.

Theorem 2.43 implies:

a tuple $(\alpha) = (\alpha_1, ..., \alpha_{k-2})$ *consisting of letters* $\mathrm{R}, \mathrm{V}, \mathrm{T}$ *is the* RVT*-code of some point* $p \in \mathbb{P}^i\mathbb{R}^2$*, if and only if it satisfies both of the following two conditions:*

(1) (α) *does not start with* T*:* $\alpha_1 \neq \mathrm{T}$*;*

(2) T *does not follow* R*: if* $\alpha_j = \mathrm{R}$ *then* $\alpha_{j+1} \neq \mathrm{T}$.

Convention. In what follows we assume that any RVT-code is realizable, i.e. satisfies conditions (1) and (2).

Definition 3.3. The set of points $p \in \mathbb{P}^k\mathbb{R}^2, k \geq 3$ with the RVT-code $(\alpha) = (\alpha_1, ..., \alpha_{k-2})$ will be called the RVT class and will be denoted by the same tuple (α), or, when confusion might arise, class (α).

Proposition 3.4. *Each* RVT *class* (α) *is a singularity class, i.e. if* $p \in (\alpha)$ *then any point equivalent to* p *also belongs to* (α)*.*

Proof. This is a corollary of Proposition 2.9, (ii) which implies that symmetries at level $i \geq 2$ maps critical curves to critical curves and fibers to fibers. □

Claim 3.5. The number of RVT classes for $\mathbb{P}^k\mathbb{R}^2, k \geq 3$ is the $(2k-3)$-d Fibonacci number.

Proof. Follows from (1) and (2) above, and an easy exercise in induction. □

3.2. Two more definitions of a non-singular point

By Cartan's Theorem 1.4 from section 1.6.1 the class of non-singular points in the kth level of the Monster is open. If $k = 1$ or $k = 2$ then these non-singular points form all of the Monster at that level. See Claim 2.14. At higher levels the set of non-singular points coincides with the RVT class $(\mathrm{RR}, ..., \mathrm{R})$, according to the following theorem.

THEOREM 3.6. *Let $p \in \mathbb{P}^k\mathbb{R}^2, k \geq 3$. The following conditions are equivalent:*

(a) *p is non-singular (as per Definition 2.13 from section 2.6);*

(b) *p belongs to the class (RR...R);*

(c) *none of the points $\pi_{k,i}(p) \in \mathbb{P}^i\mathbb{R}^2$ is vertical for $3 \leq i \leq k$.*

PROOF. The equivalence of (b) and (c) follows from the conditions (1) and (2) from the previous section which, as we proved, hold for any RVT-class. The equivalence of (a) and (b) is a direct corollary of Theorem 2.39. □

The equivalence of (a) and (b) in Theorem 3.6 and Theorem 2.15 imply one more reformulation of Cartan's Theorem 1.4:

THEOREM 3.7. *All points of the class* $(\mathrm{RR...R}) \subset \mathbb{P}^k\mathbb{R}^2, k \geq 3$*, are equivalent.*

3.3. Types of RVT classes. Regular and entirely critical prolongations

We will distinguish the following RVT-codes.

- An RVT-code is called regular if it ends with R.
- An RVT-code is called critical if it ends with a critical letter, i.e. a V or a T.

Thus, a critical class consists of critical points and a regular class consists of regular points.

- An RVT code is called entirely critical if it consists entirely of critical letters, V or T, i.e. does not contain letter R.

For example, the RVT-codes VTR and RVR are regular, the RVT-codes RVV and RVT are critical, and the RVT codes VVV and VTT are entirely critical.

- An RVT class (α) is called regular, critical, or entirely critical if its RVT code (α) is regular, critical, or entirely critical.

We will use also exponential notations for RVT classes and codes. R^q, V^q, and T^q will be used for RR...R, VV...V, TT...T (q times). The index q might be zero. For example

$$\mathrm{R}^2\mathrm{VT}^3\mathrm{R}^3 = \mathrm{RRVTTTRRR};\ \mathrm{R}^0\mathrm{VTR}^2 = \mathrm{VTRR};\ \mathrm{R}^2\mathrm{V}^3\mathrm{TR}^0 = \mathrm{RRVVVT}.$$

Given two RVT-codes, $(\alpha) = (\alpha_1, ..., \alpha_k)$ and $(\beta) = (\beta_1, ..., \beta_s)$ we write $(\alpha\beta)$ for the RVT code or RVT class $(\alpha_1, ..., \alpha_k, \beta_1, ..., \beta_s)$.

DEFINITION 3.8. For $s > 0$ we will call $(\alpha\mathrm{R}^s)$ the s-step regular prolongation of (α). For ω an entirely critical class we will call $(\alpha\omega)$ an entirely critical prolongation of (α).

Note that the points of the s-step-regular-prolongation of (α) are s-step-regular-prolongations of the points of (α), according to Definition 2.32.

3.4. Classification problem: reduction to regular RVT classes

A critical RVT class (α) is either entirely critical, or is an entirely critical prolongation of a *unique* regular RVT class, i.e. $(\alpha) = (\widehat{\alpha}\omega)$ with (ω) entirely critical RVT-code and $(\widehat{\alpha})$ regular RVT-code. The following theorem reduces the classification of points within an arbitrary RVT classes to the classification of points within some regular RVT class.

THEOREM 3.9. (Method of Critical sections).

1. *All the points of an entirely critical* RVT *class are equivalent.*

2. *If (α) is an an entirely critical prolongation of $(\widetilde{\alpha})$. then two points $p, \tilde{p}$ of (α) are equivalent if and their projections to $(\widetilde{\alpha})$ are equivalent.*

PROOF. By Theorem 2.43 the vertical point in the fiber over any fixed point of the Monster is unique. The tangency point in the fiber over any critical point is unique. These facts and Proposition 3.4 imply that two points of a critical RVT class are equivalent if and only if their one-step projections are equivalent. Theorem 3.9 follows by iteration. □

REMARK 3.10. We have also called this theorem the "method of critical sections" because V defines a section of the bundle projection $\pi_{i+1,i}$ and T defines a section of the same projection restricted to critical points. As such, V and T define equivariant inverses to the projections induced by restricting $\pi_{i+1,i}$ to the critical RVT classes, $(\alpha\mathrm{V}) \to (\alpha)$ for any class (α), or $(\beta\mathrm{T}) \to (\beta)$ for (β) a critical class.

3.5. RVT classes as subsets of $\mathbb{P}^k\mathbb{R}^2$

In this section we prove the following two theorems.

THEOREM 3.11. *Each* RVT *class $(\alpha) = (\alpha_1, ...\alpha_{k-2})$ is a non-singular analytic submanifold of $\mathbb{P}^k\mathbb{R}^2$ whose codimension is equal to the number of critical letters occurring in (α).*

We will say that an RVT class $(\alpha) \subset \mathbb{P}^k\mathbb{R}^2$ adjoins an RVT class $(\beta) \subset \mathbb{P}^k\mathbb{R}^2$ if the class (α) is in the closure of the class (β). This of course means that (α) and (β) are in the same level of the Monster.

THEOREM 3.12. *An* RVT *class $(\alpha) \subset \mathbb{P}^k\mathbb{R}^2$ adjoins an* RVT *class $(\beta) \subset \mathbb{P}^k\mathbb{R}^2$ if and only if the (β) can be obtained from (α) by replacing some critical letters (*V*'s or* T*'s) by* R*'s. The closure of any* RVT *class is equal to the union of this class together with the* RVT *classes which adjoin it.*

The proof of these theorems is based on the explicit construction of tangency points. This construction is as follows.

Fix a *critical* RVT class $(\alpha) = (\alpha_1, ..., \alpha_k) \subset \mathbb{P}^{k+2}\mathbb{R}^2, k \geq 1$. Theorem 2.43 asserts that the tangency point in the fiber over a critical point $m \in (\alpha)$ exists and unique. Therefore there is a well-defined map

$$\begin{gathered} \mathbb{T} : \text{critical class } (\alpha) \to \mathbb{P}^{k+3}\mathbb{R}^2, \\ (\alpha) \ni p \to (p, \ell) = \text{the tangency point in the fiber over } p. \end{gathered} \tag{3.1}$$

We now construct the map $\mathbb{T}$ more explicitly.

NOTATION 3.13. For a point $m \in \mathbb{P}^i\mathbb{R}^2$ of the Monster denote by V_m one of immersed vertical curve germs $(\mathbb{R},0) \to (\mathbb{P}^i\mathbb{R}^2, m)$ (recall that all such germs are the same up to reparameterization). Denote by $\mathbb{V}^j$ the map

$$\mathbb{V}^j : \mathbb{P}^i\mathbb{R}^2 \to \mathbb{P}^{i+j}\mathbb{R}^2, \quad m \to V^j_m(0) \quad i = 1,2,3,\dots. \tag{3.2}$$

PROPOSITION 3.14. *Let (α) be a critical class whose code has length k (so at level $k+2$). Write $(\alpha) = (\beta \mathrm{VT}^s)$ where $s \geq 0$. Then the map (3.1) is given by*

$$\mathbb{T} = \mathbb{V}^{s+2} \circ \pi_{k+2,k+1-s}.$$

EXAMPLE. Consider the class $(\alpha) = (\mathrm{RVVTT})$ whose code has length 5, in the 7th level of the Monster. For this class $s = 2$. The map (3.1) is the composition of the projection $p \to \pi_{7,4}(p)$ which maps α onto $(\beta) = (\mathrm{RV})$ and the map $\mathbb{V}^4$ from $\mathbb{P}^4\mathbb{R}^2$ to $\mathbb{P}^8\mathbb{R}^2$.

PROOF. Let $p \in (\alpha) \subset \mathbb{P}^{k+2}\mathbb{R}^2$. Since $\mathbb{T}(p)$ is a tangency point, one has $\mathbb{T}(p) = \Gamma^1(0)$ where $\Gamma : (\mathbb{R},0) \to (\mathbb{P}^{k+2}\mathbb{R}^2, p)$ is an immersed critical integral curve germ which is *not* vertical. By Theorem 2.20 there exists and unique $j < k+2$ such that Γ is the $(k+2-j)$-step-prolongation of the vertical curve V_m with $m = \pi_{k+2,j}(p)$, up to reparameterization. By Proposition 2.45 the RVT-code (α) has the form $(\alpha) = (\beta \mathrm{VT}^{k+1-j})$ so that the number s is Proposition 3.14 is equal to $k+1-j$. Now one has

$$\mathbb{T}(p) = \Gamma^1(0) = V_m^{k+3-j}(0) = V^{s+2}_{\pi_{k+2,k+1-s}(p)}(0) = \mathbb{V}^{s+2}\big(\pi_{k+2,k+1-s}(p)\big)$$

as required. □

PROOF OF THEOREM 3.11. We will use induction on the number of letters in the RVT-code. Fix an RVT class $(\widetilde{\alpha}) = (\alpha_1, ..., \alpha_{k+1}) \subset \mathbb{P}^{k+3}\mathbb{R}^2$ and assume that we have already proved Theorem 3.11 for the class $(\alpha) = (\alpha_1, ..., \alpha_k) \subset \mathbb{P}^{k+2}\mathbb{R}^2$. If $\alpha_1 = \cdots = \alpha_k = R$ then we have to prove that the class $(\widetilde{\alpha})$ is open if $\alpha_{k+1} = \mathrm{R}$ and is an analytic submanifold of $\mathbb{P}^{1+k}\mathbb{R}^2$ if α_{k+1} is a critical letter, V or T. Each of these statements is a direct corollary of Theorem 2.43. Consider now the case that at least one of the letters $\alpha_1, ..., \alpha_k$ is critical. Write $(\alpha) = (\beta\mathrm{VT}^s)$, $s \geq 0$, as in Proposition 3.14. By Theorem 2.43 and Proposition 3.14 one has the following relations:

(a) if $\alpha_{k+1} = \mathrm{V}$ then $(\alpha) = \mathbb{V}^1(\widetilde{\alpha})$;

(b) if $\alpha_{k+1} = \mathrm{T}$ (and consequently $\alpha_k \in \{\mathrm{V},\mathrm{T}\}$) then $(\alpha) = \mathbb{V}^{s+2} \circ \pi_{k+2,k-s+1}(\widetilde{\alpha})$;

(c) if $\alpha_{k+1} = \mathrm{R}$ then $(\alpha) = \pi^{-1}_{k+3,k+2}(\widetilde{\alpha}) - (\widetilde{\alpha}, \mathrm{V}) - (\widetilde{\alpha}, \mathrm{T})$.

It is easy to check that for fixed i, j the map (3.2) is an *analytic* section of the torus fibration $\mathbb{P}^{i+j}\mathbb{R}^2 \to \mathbb{P}^i\mathbb{R}^2$. Therefore relations (a)-(c) imply Theorem 3.11 for the class (α).

PROOF OF THEOREM 3.12. Again, we will prove by induction on on the number of letters in the RVT-code, using Theorem 2.43. It suffices to prove that for any RVT codes $(\alpha_1, ..., \alpha_k)$, $(\beta_1, ..., \beta_k)$ none of the RVT classes $(\alpha_1, ..., \alpha_p, \mathrm{T}), (\beta_1, ..., \beta_p, \mathrm{V})$ adjoins the other one. This follows from the continuity of the map $m \to \mathbb{T}(m)$ in (3.1), by Proposition 3.14.

3.6. Why *tangency* points?

We explain the terminology "tangency line".

NOTATION. Given a critical RVT-class $(\alpha) = (\alpha_1, ..., \alpha_k)$ we denote by (α^*) an RVT class obtained from (α) by replacing by R all letters before the **last** letter V.

By Theorem 2.43 any critical RVT-code contains at least one V so (α^*) is well-defined.

EXAMPLE. If the last letter is $\alpha_s = \mathrm{V}$ then $(\alpha^*) = (\mathrm{RR}, ..., \mathrm{RV})$. If the last letter $\alpha_s = \mathrm{T}$ then there exists a largest q, $q < s$ such that $\alpha_q = \mathrm{V}$ while $\alpha_{q+1}, \alpha_{q+2}, ..., \alpha_s = \mathrm{T}$. In this case $(\alpha^*) = (\mathrm{RR}, .., \mathrm{RVT}, \mathrm{T}, .., \mathrm{T})$ with V being in the q-th place. For example

$$(\alpha) = (\mathrm{RVTVRVRVT}) \implies (\alpha^*) = (\mathrm{R}^7\mathrm{VT}),$$
$$(\alpha) = (\mathrm{RVTVRVRVV}) \implies (\alpha^*) = (\mathrm{R}^8\mathrm{V}).$$

THEOREM 3.15. *Let $m \in \mathbb{P}^k\mathbb{R}^2$ be a vertical or tangency point, and let (α) be the* RVT *class of m. Let ℓ_{tan} be the tangency line in $\Delta^k(m)$ and let $L(m)$ be the tangent space at m to the closure $(\bar{\alpha}^*)$ of the class $(\alpha^*) \subset \mathbb{P}^k\mathbb{R}^2$. Then $\ell_{tan} = \Delta^k(m) \cap L(m)$.*

REMARK. Theorem 3.15 includes the statement that $\dim\big(\Delta^k(m) \cap L(m)\big) = 1$. If m is a vertical point then $\mathrm{codim}(\alpha^*) = 1$ and consequently the 2-plane $\Delta^k(m)$ is transversal to the hypersurface $(\bar{\alpha}^*)$. If m is a tangency point then $(\alpha^*) = (\mathrm{R}, ..., \mathrm{RVT}, ..., \mathrm{T})$ with T being repeated $c \geq 1$ times. In this case $\mathrm{codim}(\alpha^*) = \mathrm{codim}L(m) = c + 1 \geq 2$, and the couple $\Delta^k(m), L(m)$ is not generic, since the intersection of a generic codimension ≥ 2 subspace with a 2-plane is $\{0\}$ whereas the intersection of $L(m)$ with $\Delta^k(m)$ is a line.

PROOF. We have to prove two statements:

(a) $\Delta^k(m)$ is not a subset of the tangent space $L(m) = T_m(\bar{\alpha}^*)$, i.e. $\dim\Delta^k(m) \cap L(m) \leq 1$;

(b) the tangency line ℓ_{tan} in $\Delta^k(m)$ does belong to $L(m)$.

PROOF OF (a). Let $(\alpha^*) = (\mathrm{R...RVT}^q) \subset \mathbb{P}^k\mathbb{R}^2, q \geq 0$. Consider the following submanifold $Q \subseteq \mathbb{P}^k\mathbb{R}^2$:

If $q \geq 1$ then Q is equal to the closure of the class $(\mathrm{R}, ..., \mathrm{R}, \mathrm{VT}^{q-1}\mathrm{R})$;

If $q = 0$ then $Q = \mathbb{P}^k\mathbb{R}^2$.

In either case $(\bar{\alpha}^*)$ is a hypersurface in Q, and $m \in (\bar{\alpha}^*) \subset Q$. The immersed vertical curve $V = V(t)$ through m lies in Q, but there are only discrete values of t such that the point $V(t)$ is vertical or tangency, since there are precisely two critical points along a given fiber. Therefore V is not tangent to $(\bar{\alpha}^*)$ but is tangent to $\Delta^k(m)$, showing that the vertical line in $\Delta^i(m)$ does not belong to $L(m)$.

PROOF OF (b). Let $\Gamma : (\mathbb{R}, 0) \to (\mathbb{P}^k\mathbb{R}^2, m)$ be the germ of immersed integral curve which is critical and non-vertical. By Proposition 2.24 Γ is the k-step-prolongation of an immersed vertical curve for some $k \geq 1$. And $\Gamma'(0)$ spans ℓ_{tan}. Note that $(\alpha^*) = (\mathrm{R...RVT}^q)$ and $\Gamma(0) \in (\bar{\alpha}^*)$. Now Proposition 2.45 implies that Γ lies wholly in the closure of the class (α^*): $\Gamma(t) \in (\bar{\alpha}^*)$ for all t. Therefore the line ℓ_{tan} is tangent to $(\bar{\alpha}^*)$, proving (b). □

3.7. RVT code of plane curves

The decomposition of the Monster into RVT classes allows us to define the RVT code of a plane curve germ $c : (\mathbb{R}, 0) \to \mathbb{R}^2$ to be the RVT-code of the point $c^k(0)$ where k is the regularization level of c. The regularization level is defined as follows.

DEFINITION 3.16. Let c be an analytic plane curve germ which admits a regular prolongation. The minimal number k such that the prolongation c^k is regular will be called the regularization level of c. We will say that c regularizes at level k.

By Theorem 2.36 the regularization level is defined for any well-parameterized analytic plane curve germ. Note that if c is immersed then its regularization level is 0. If c is not immersed, but c^1 is immersed then the regularization level of c is 1 (since any immersed curve in the first level is regular). Regularization level 2 is impossible by the first statement of the following proposition.

PROPOSITION 3.17. *Let $c : (\mathbb{R}, 0) \to \mathbb{R}^2$ be an analytic plane curve germ such that c^1 is a non-immersed curve. Assume that c regularizes at level k. Then*

(a) $k \geq 3$;

(b) $c^k(0)$ *is a critical point;*

(c) *the points* $c^{k+1}(0), c^{k+2}(0), \ldots$ *are regular.*

PROOF. The curve c^k is regular, therefore statement (c) follows from Proposition 2.31. To prove (a) and (b) consider the curve c^{k-1}. Since c^1 is not immersed we have $k \geq 2$. Therefore c^{k-1} is an integral curve in the first or higher level. The one-step prolongation of c^{k-1} is a regular curve c^k. By Lemma 2.34 c^{k-1} is immersed. On the other hand the curve c^{k-1} is not regular. Now we see that $k = 2$ is impossible because any immersed curve in the first level is regular. Thus $k \geq 3$ and c^{k-1} is an immersed curve which is tangent to a critical direction. Therefore $c^k(0)$ is a critical point. □

The regularization level of a plane curve germ can also be characterized as follows.

PROPOSITION 3.18. *The regularization level of a plane curve germ c is equal to $k \geq 3$ if and only if c^k is a regular curve and $c^k(0)$ is critical point.*

PROOF. Assume that the regularization level of c is k, $k \geq 3$. Then the curve c^k is regular while c^{k-1} is not. Lemma 2.34 implies that c^{k-1} is an immersed curve. It is not regular, so it must be tangent to a critical direction. The definition of prolongation now implies that $c^k(0)$ is a critical point.

Assume now that c^k is regular, $k \geq 3$, and that the point $c^k(0)$ is critical. We have to show that if $i < k$ then the curve c^i is not regular. This follows from Proposition 2.31: if c^i was a regular curve then the points $c^{i+1}(0), c^{i+2}(0), \ldots$ must be regular, and in particular the point $c^k(0)$ would be regular. □

In view of Proposition 3.17 and the fact that any point in the first or the second level is regular it is natural to define the RVT code of a plane curve as follows.

DEFINITION 3.19. Let $c : (\mathbb{R}^2, 0) \to \mathbb{R}^2$ be an analytic plane curve germ which regularizes at level $k \geq 3$. The RVT code of c is the RVT code of the point $c^k(0)$.

Equivalently, the RVT code of c is the tuple $(\alpha_1, ..., \alpha_{k-2})$ where α_i is the letter, R,V,or T, corresponding to the type of the point $c^{i+2}(0)$.

REMARK 3.20. The RVT code of an analytic plane curve germ c is not defined in either one of the following cases:

(a) the curve c does not admit a regular prolongation. By Theorem 2.36 this is true if and only if c is badly parameterized.

(b) the curve c regularizes at level $k < 3$. By Proposition 3.17 this is true if and only if the one-step-prolongation c^1 is immersed. In particular it is true if c itself is immersed.

It is worth noting that by Proposition 3.17:

the RVT *code of a plane curve is always critical, i.e. ends with* V *or* T.

THEOREM 3.21. *Let* (α) *be the* RVT *code of a plane curve germ* c. *If* $\tilde{c}$ *is a plane curve germ* RL*-equivalent to* c *then* (α) *is also the* RVT *code of* $\tilde{c}$. *Moreover,* (α) *is the* RVT *code of* $\tilde{c}$ *under the weaker assumption that the one-step prolongations* c^1 *and* $\tilde{c}^1$ *are* RL*-contact equivalent.*

PROOF. The RL-equivalence of c and $\tilde{c}$ implies the RL-contact equivalence of c^1 and $\tilde{c}^1$ (Proposition 2.8 with $k = 1$). Assume that c^1 and $\tilde{c}^1$ are RL-contact equivalent Legendrian curve germs at $t = 0$. By Theorem 2.12 the points $c^i(0)$ and $\tilde{c}^i(0)$ are equivalent, for any $i \geq 1$. Now Theorem 3.21 follows from Proposition 3.4 stating that any RVT class is closed with respect to equivalence of points. $\square$

3.8. RVT code and Puiseux characteristic

We have just defined the RVT code of a plane curve germ. It is a discrete invariant with respect to RL-equivalence (Theorem 3.21). Another such discrete invariant is the Puiseux characteristic of a plane curve germ, recalled in subsection 3.8.3. It is natural to expect these invariants to be related. Indeed they are essentially the same! Explaining the equivalence between these two invariants is the main point of this section. The equivalence will be used as a tool in the classification theorems later on, in Chapters 7 and 8.

Recall that the RVT code of a plane curve is always critical. In subsections 3.8.4 and 3.8.5 we establish a bijection between the set of critical RVT codes and the set of Puiseux characteristics of the form

$$\Lambda = [\lambda_0; \lambda_1, ..., \lambda_m], \quad m \geq 1, \quad \lambda_1 > 2\lambda_0. \tag{3.3}$$

The bijection is explicit. We define maps

$$\mathrm{RVT} : \Big\{\text{Puiseux characteristics of form (3.3)}\Big\} \longrightarrow \Big\{\text{critical RVT classes}\Big\} \tag{3.4}$$

$$\mathrm{Pc} : \Big\{\text{critical RVT classes}\Big\} \longrightarrow \Big\{\text{Puiseux characteristics of form (3.3)}\Big\} \tag{3.5}$$

which are inverses to each other:

$$\mathrm{Pc}\,\big(\mathrm{RVT}\,(\Lambda)\big) = \Lambda, \quad \mathrm{RVT}\,\big(\mathrm{Pc}\,(\alpha)\big) = (\alpha). \tag{3.6}$$

Moreover, the following essential theorem holds.

Theorem A. *All plane curve germs with the same Puiseux characteristic* Λ *of the form* (3.3) *regularize at the same level and have the same* RVT *code* RVT(Λ) *where* $\Lambda \to \mathrm{RVT}(\Lambda)$ *is the map constructed in subsection* 3.8.5. *Two plane curve germs with different Puiseux characteristics of the form* (3.3) *have different* RVT *codes.*

The second statement holds due to the existence of the inverse (3.5) to (3.4).

Theorem **A** is not simple. We postpone its proof to Chapter 8. It requires the use of special local coordinate systems, called "KR coordinates" on the Monster (Chapter 7) and depends on relating the prolongations of plane curves to an operation we call directional blow up (the beginning of Chapter 8).

The constraint $\lambda_1 > 2\lambda_0$ in (3.3) is a normalization condition. The two-step prolongation of any plane curve germ $c : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ whose Puiseux characteristic satisfies this constraint yields the same point $c^2(0) \in \mathbb{P}^2\mathbb{R}^2$. See Chapter 7 where in the Notation 7.3 that point is O.

We take a moment to explain why we must impose the constraint $\lambda_1 > 2\lambda_0$ to get our results. The Puiseux characteristic is an invariant of a plane curve germ with respect to the RL-equivalence: plane curve germs with different Puiseux characteristics cannot be RL-equivalent. However, two plane curve germs can have different Puiseux characteristic but their one-step prolongations can be RL-contact equivalent. This happens with the curves (t^2, t^5) and (t^3, t^5) whose Puiseux characteristics are $[2; 5]$ and $[3; 5]$. (See Theorem 3.21.) This non-uniqueness phenomenon of curves with different Puiseux characteristics having isomorphic prolongations is excluded by imposing the restriction on the Puiseux characteristic .

No curves are excluded by the constraint on the Puiseux characteristic . That is to say, the first prolongation of any analytic plane curve germ singularity is RL -contact equivalent to the 1st prolongation of a plane curve germ whose Puiseux characteristic satisfies the constraint $\lambda_1 > 2\lambda_0$. This is spelled out by the following lemma, combined with the definition of the Puiseux characteristic .

Lemma 3.22. *Any well-parameterized non-immersed Legendrian curve germ* $\gamma : (\mathbb{R}, 0) \to \mathbb{P}^1\mathbb{R}^2$ *is* RL*-contact equivalent to the one-step prolongation of a well-parameterized plane curve* c *of the form*

$$(at^q + \text{h.o.t.},\ bt^p + \text{h.o.t.}),\quad q \geq 2,\ p > 2q,\ a, b \neq 0$$

and consequently with the Puiseux characteristic of the form (3.3): $q = \lambda_0,\ p = \lambda_1$.

The lemma is proved in subsection 3.8.2 and is essentially equivalent to the Engel theorem 1.3 that all points in $\mathbb{P}^2\mathbb{R}^2$ are equivalent.

Theorem 3.23. *A plane curve germ* c *has* RVT *code* (α) *if and only if its one-step-prolongation* c^1 *is* RL*-contact equivalent to the one-step-prolongation of a plane curve germ* $\tilde{c}$ *with the Puiseux characteristic* $\mathrm{Pc}(\alpha)$, *where* $(\alpha) \to \mathrm{Pc}(\alpha)$ *is the map constructed in subsection* 3.8.4.

Proof. Assume that c has RVT code (α). Then c^1 is a non-immersed Legendrian curve. (If c^1 is immersed then the RVT code of c is not defined. See Remark 3.20). By Lemma 3.22 there is a Legendrian curve $\tilde{c}$ whose Puiseux characteristic Λ is of the form (3.3) and which is RL-contact equivalent to c^1. By Theorem 3.21

the RVT code of $\tilde{c}$ is also (α). By Theorem A we have that $(\alpha) = \mathrm{RVT}\,(\Lambda)$ and consequently $\Lambda = \mathrm{Pc}\,(\alpha)$.

Assume now that c^1 is RL-contact equivalent to $\tilde{c}^1$, where $\tilde{c}$ is a plane curve germ with Puiseux characteristic $\Lambda = \mathrm{Pc}\,(\alpha)$. Then Λ has the form (3.3) and by Theorem **A** the RVT code of $\tilde{c}$ is $\mathrm{RVT}\,(\Lambda) = \mathrm{RVT} \circ \mathrm{Pc}\,(\alpha) = (\alpha)$. By Theorem 3.21 the RVT code of c is also (α). $\square$

3.8.1. Proof of the regularization Theorem 2.36. In section 2.10 we reduced Theorem 2.36 to the case of a well-parameterized plane curve germ. The proof for such a curve c is as follows. Consider the Legendrian curve $\gamma = c^1$. It is also well-parameterized. If γ is immersed then c regularizes at level 1, so we assume that γ is not immersed. By Lemma 3.22 γ is RL-contact equivalent to $\tilde{c}^1$, where $\tilde{c}$ is a plane curve germs with Puiseux characteristic of the form (3.3). By Theorem **A** the curve $\tilde{c}$ regularizes at some level $(r+1)$. By Theorem 3.21 the curve c regularizes at the same level.

3.8.2. Proof of Lemma 3.22. Recall the standard coordinates x, y, u for $\mathbb{P}^1\mathbb{R}^2$ of section 1.2. By translating and rotating the Cartesian coordinates x, y if necessary, we can assume that $x = y = u = 0$ at $\gamma(0)$ while the contact structure is still described by the vanishing of the 1-form $dy - u dx$. Writing $\gamma(t) = (x(t), y(t), u(t))$ we have $x(0) = y(0) = u(0) = 0$ and $y'(t) = u(t)x'(t)$. Since γ is well-parameterized and not immersed it follows that the plane curve $c(t) = (x(t), y(t))$ is not identically zero (see Lemma 2.38). It follows that $\gamma = c^1$ is the first prolongation of c and that c is a well-parameterized non-immersed plane curve germ. Let $x(t) = at^q + \text{h.o.t.}$, $y(t) = bt^p + \text{h.o.t.}$, $a, b \neq 0$. Then $p > q$ since $u(t) = ct^{p-q} + \text{h.o.t.}, c \neq 0$ and $u(0) = 0$. If $p > 2q$ we are done: the curve c itself has Puiseux characteristic satisfying the constraint of (3.3). The case $p = 2q$ reduces to the case $p > 2q$ by the change of coordinates $(x, y) \to (x, y - (b/a)x^2)$. Finally, we consider the case $p < 2q$. The contactomorphism $(x, y, u) \to (u, y - xu, -x)$ takes γ to the one-step prolongation of a plane curve having the form $\tilde{c}:\ x = -ct^{p-q}, y = O(t^p)$. Since $p < 2q$ we have $p > 2(p-q)$ and the curve $\tilde{c}$ has the required form.

3.8.3. The Puiseux characteristic. The Puiseux characteristic of an analytic plane curve germ is a classical invariant. A number of theorems explaining this invariant can be found in [**W**]. Let us recall its definition and construction.

Any Puiseux characteristic is an integer vector denoted by $[\lambda_0; \lambda_1, ..., \lambda_m]$ satisfying the following conditions, where g.c.d. denotes the greatest common divisor:

1. $1 < \lambda_0 < \lambda_1 < \cdots < \lambda_m$; λ_0 is not a divisor of λ_1;

2. g.c.d.$(\lambda_1, ..., \lambda_m) = 1$;

3. if $m \geq 2$ then for any $i = 1, ..., m-1$:

g.c.d.$(\lambda_0, ...\lambda_i) = d_i > 1$ and d_i is not a divisor of λ_{i+1}.

The Puiseux characteristic of an analytic plane curve germ $c : (\mathbb{R}, 0) \to \mathbb{R}^2$ is defined only if c is not immersed and well-parameterized. Its construction is as follows. (See Chapter 4 of [**W**]). Take any local coordinates x, y centered at the point $c(0)$. Then up to reparameterization and possibly a change of coordinates $(x,y) \to (y,x)$ the curve c has the form

$$x = t^{\lambda_0}, \quad y = a_{\lambda_0}t^{\lambda_0} + a_{\lambda_0+1}t^{\lambda_0+1} + a_{\lambda_0+2}t^{\lambda_0+2} + \cdots, \qquad \lambda_0 \geq 2. \tag{3.7}$$

• Let λ_1 be the minimal integer $\geq \lambda_0 + 1$ such that $a_{\lambda_1} \neq 0$ and λ_0 is not a divisor of λ_1. If λ_0 and λ_1 are relatively prime then $m = 1$ and the Puiseux characteristic of the curve c is $[\lambda_0; \lambda_1]$.

• Assume now that λ_0 and λ_1 are not mutually prime. In this case $m \geq 2$. Let $d_1 = \text{g.c.d.}(\lambda_0, \lambda_1)$. Then λ_2 is the minimal integer $\geq \lambda_1 + 1$ such that $a_{\lambda_2} \neq 0$ and d_1 is not a divisor of λ_2. If $\text{g.c.d.}(\lambda_0, \lambda_1, \lambda_2) = 1$ then $m = 2$ and the Puiseux characteristic is $[\lambda_0; \lambda_1, \lambda_2]$.

• If $\text{g.c.d.}(\lambda_0, \lambda_1, \lambda_2) = d_2 > 1$ then $m \geq 3$ and λ_3 is the minimal integer $\geq \lambda_2$ such that $a_{\lambda_3} \neq 0$ and d_2 is not a divisor of λ_3.

• We continue in the same way till $\text{g.c.d.}(\lambda_0, ..., \lambda_m) = 1$. Such m exists if c is well-parameterized, see [**W**]. Then the Puiseux characteristic of c is $[\lambda_0; \lambda_1, ..., \lambda_m]$.

Example. The Puiseux characteristic of the plane curve germ $(t^8,\ t^{16} + t^{20} + a_{22}t^{22} + a_{26}t^{26} + a_{27}t^{27})$ is $[8; 20, 22, 27]$ provided that $a_{22}, a_{27} \neq 0$.

3.8.4. The map $(\alpha) \to \mathrm{Pc}\,(\alpha)$. In this subsection we give a recursion formulae for the Puiseux characteristic $\Lambda = \mathrm{Pc}\,(\alpha)$ in Theorem 3.23. The map $(\alpha) \to \mathrm{Pc}\,(\alpha)$ is inverse to the map $\Lambda \to \mathrm{RVT}\,(\Lambda)$ in Theorem **A**. The recursion formulae involves the Euclidean algorithm.

3.8.4.1. *The map $\mathbb{E}_\omega : \mathbb{N}^2 \to \mathbb{N}^2$.* For each entirely critical RVT code (ω) we will associate a map $\mathbb{E}_\omega$ on pairs of non-negative integers. This map is built up from the two maps

$$\begin{aligned} \mathbb{E}_{\mathrm{T}}:\ (n_1, n_2) &\to (n_1, n_1 + n_2), \\ \mathbb{E}_{\mathrm{V}}:\ (n_1, n_2) &\to (n_2, n_1 + n_2), \end{aligned}$$

where n_1, n_2 are non-negative integers. Writing out

$$(\omega) = (\omega_1, ..., \omega_m), \quad \omega_i \in \{\mathrm{V}, \mathrm{T}\},$$

we define $\mathbb{E}_\omega$ to be the composition

$$\mathbb{E}_\omega = \mathbb{E}_{\omega_1} \circ \mathbb{E}_{\omega_2} \circ \cdots \circ \mathbb{E}_{\omega_m}.$$

Examples. Let $(\omega) = (\mathrm{VT})$. Since $\mathbb{E}_{\mathrm{T}}(1,2) = (1,3)$ and $\mathbb{E}_{\mathrm{V}}(1,3) = (3,4)$ we have $\mathbb{E}_{\mathrm{VT}}(1,2) = (3,4)$. Next consider $(\omega) = (\mathrm{VV})$. Since $\mathbb{E}_{\mathrm{V}}(1,2) = (2,3)$ and $\mathbb{E}_{\mathrm{V}}(2,3) = (3,5)$ we have $\mathbb{E}_{\mathrm{VV}}(1,2) = (3,5)$. One more example:

$$(\omega) = (\mathrm{VVTVTTV}).$$

In this case one has $\mathbb{E}_\omega(1,2) = (23,39)$ since the successive maps have the effect

$$(1,2) \xrightarrow{\mathbb{E}_{\mathrm{V}}} (2,3) \xrightarrow{\mathbb{E}_{\mathrm{T}}} (2,5) \xrightarrow{\mathbb{E}_{\mathrm{T}}} (2,7) \xrightarrow{\mathbb{E}_{\mathrm{V}}}$$

$$\to (7,9) \xrightarrow{\mathbb{E}_{\mathrm{T}}} (7,16) \xrightarrow{\mathbb{E}_{\mathrm{V}}} (16,23) \xrightarrow{\mathbb{E}_{\mathrm{V}}} (23,39).$$

Note that both $\mathbb{E}_{\mathrm{V}}$ and $\mathbb{E}_{\mathrm{T}}$ map relatively prime pairs to relatively prime pairs. Note also that any entirely critical class starts with V, not with T. These observations imply the following lemma.

LEMMA 3.24. *Let ω be any entirely critical class. Let $(a,b) = \mathbb{E}_\omega(1,2)$. Then a and b are relatively prime numbers and $a < b < 2a$.*

3.8.4.2. *The recursion formulae.* Any critical RVT code (α) either is entirely critical or is an entirely critical prolongation of a regular RVT code, i.e. has one of the following forms:

A. $(\alpha) = (\mathrm{R}^s\omega)$, where $s \geq 0$ and (ω) is an entirely critical RVT code;

B. $(\alpha) = (\beta\mathrm{R}^s\omega)$ where $s \geq 1$, (β) is a critical RVT code and (ω) is an entirely critical RVT code.

The Puiseux characteristic $\mathrm{Pc}\,(\alpha)$ is as follows:

1. In case A set $(a,b) = \mathbb{E}_\omega(1,2)$. Then

$$\mathrm{Pc}\,(\alpha) = [\lambda_0; \lambda_1], \quad \lambda_0 = a,\ \lambda_1 = sa + a + b.$$

2. Let in case B $\mathrm{Pc}\,(\beta) = [\widetilde{\lambda}_0; \widetilde{\lambda}_1, ..., \widetilde{\lambda}_m]$. Set $(a,b) = \mathbb{E}_\omega(1,2)$. Then

$$\mathrm{Pc}\,(\alpha) = [\lambda_0;\ \lambda_1, ..., \lambda_m, \lambda_{m+1}],$$

$$\lambda_i = a\widetilde{\lambda}_i, \quad 0 \leq i \leq m, \qquad \lambda_{m+1} = a \cdot (\widetilde{\lambda}_m + s - 1) + b - a.$$

Items 1. and 2. give a simple recursion formulae for calculating $\mathrm{Pc}\,(\alpha)$ for any critical class (α). Using Lemma 3.24 it is easy to check that the constructed tuple $\mathrm{Pc}\,(\alpha)$ is always Puiseux characteristic and that it always satisfies the requirement $\lambda_1 > 2\lambda_0$, i.e. has the form (3.3).

EXAMPLE 3.25. Let us calculate $\mathrm{Pc}\,(\alpha)$ for

$$(\alpha) = (\mathrm{R}^3\mathrm{VVR}^4\mathrm{VTR}^5\mathrm{VVT}).$$

At first we calculate $\mathrm{Pc}\,(\mathrm{R}^3\mathrm{VV})$ by item 1. with $s = 3$, $(\omega) = \mathrm{VV}$. One has $\mathbb{E}_{\mathrm{VV}}(1,2) = (3,5)$ and we obtain

$$\mathrm{Pc}\,(\mathrm{R}^3\mathrm{VV}) = [3; 17].$$

Now, knowing $\mathrm{Pc}\,(\mathrm{R}^3\mathrm{VV})$ we calculate $\mathrm{Pc}\,(\mathrm{R}^3\mathrm{VVR}^4\mathrm{VT})$ by item 2. with $(\beta) = (\mathrm{R}^3\mathrm{VV})$, $s = 4$, $(\omega) = (\mathrm{VT})$. One has $\mathbb{E}_{\mathrm{VT}}(1,2) = (3,4)$ and we obtain

$$\mathrm{Pc}\,(\mathrm{R}^3\mathrm{VVR}^4\mathrm{VT}) = [9; 51, 61].$$

Knowing the latter associated Puiseux characteristic we calculate $\mathrm{Pc}\,(\alpha)$ by item 2. with $(\beta) = (\mathrm{R}^3\mathrm{VVR}^4\mathrm{VT})$, $s = 5$, $(\omega) = (\mathrm{VVT})$. One has $\mathbb{E}_{\mathrm{VVT}}(1,2) = (4,7)$ and we obtain

$$\mathrm{Pc}\,(\alpha) = [36; 204, 244, 263].$$

3.8.5. The map $\Lambda \to \mathrm{RVT}\,(\Lambda)$. In this subsection we construct a map $\Lambda \to \mathrm{RVT}\,(\Lambda)$ in Theorem **A**. This map is inverse to the map $(\alpha) \to \mathrm{Pc}\,(\alpha)$ in Theorem 3.23 which was constructed in the previous subsection. It sends a Puiseux characteristic Λ of the form (3.3) to a critical RVT code $\mathrm{RVT}\,(\Lambda)$.

3.8.5.1. *The map* $(a, b) \to \omega(a, b)$. At first we need a map which is inverse to the map $(\omega) \to \mathbb{E}_\omega(1, 2)$ constructed in subsection 3.8.4.1.

LEMMA 3.26. *Given positive relatively prime integers* a, b *such that* $a < b < 2a$ *there exists an entirely critical* RVT *code* (ω) *such that* $(a, b) = \mathbb{E}_\omega(1, 2)$.

NOTATION. The RVT code (ω) in Lemma 3.26 will be denoted $\omega(a, b)$.

PROOF. The proof is explicit construction of $\omega(a, b)$. Given two integers $q < p$ such that $q \neq 2p$ denote by $L(q, p)$ the letter

$$\mathrm{L}(q, p) = \begin{cases} \mathrm{T} \text{ if } p > 2q; \\ \mathrm{V} \text{ if } p < 2q. \end{cases}$$

Set $(a_1, b_1) = (b - a, a)$ and define

$$\mathrm{l}_{i+1} = \mathrm{L}(a_i, b_i); \quad (a_{i+1}, b_{i+1}) = \mathbb{E}^{-1}_{\mathrm{l}_{i+1}}(a_i, b_i), \quad i \geq 1.$$

Here $\mathbb{E}^{-1}_{\mathrm{T}}$ and $\mathbb{E}^{-1}_{\mathrm{V}}$ are inverse to the maps $\mathbb{E}_{\mathrm{T}}$ and $\mathbb{E}_{\mathrm{V}}$:

$$\mathbb{E}^{-1}_{\mathrm{T}}(q, p) = (q,\ p - q); \quad \mathbb{E}^{-1}_{\mathrm{V}}(q, p) = (p - q, q).$$

Since a, b are positive, relatively prime, and $a < b < 2a$ it is clear that the letters $\mathrm{L}(a_i, b_i)$ are well-defined (i.e. $b_i \neq 2a_i$) for all $i < r$ where $(a_r, b_r) = (1, 2)$ and that such r exists. Set $(\omega) = \omega(a, b) = (V, \mathrm{l}_2, \mathrm{l}_3, \ldots, \mathrm{l}_r)$. Then $(a, b) = \mathbb{E}_\omega(1, 2)$. □

EXAMPLE 3.27. Let us calculate $\omega(18, 25)$. One has

$$\begin{array}{cccc} \mathrm{L}(18,25) = \mathrm{V} & \mathbb{E}^{-1}_{\mathrm{V}}(18,25) = (7,18) & \mathrm{L}(7,18) = \mathrm{T} & \mathbb{E}^{-1}_{\mathrm{T}}(7,18) = (7,11) \\ \mathrm{L}(7,11) = \mathrm{V} & \mathbb{E}^{-1}_{\mathrm{V}}(7,11) = (4,7) & \mathrm{L}(4,7) = \mathrm{V} & \mathbb{E}^{-1}_{\mathrm{V}}(4,7) = (3,4) \\ \mathrm{L}(3,4) = \mathrm{V} & \mathbb{E}^{-1}_{\mathrm{V}}(3,4) = (1,3) & \mathrm{L}(1,3) = \mathrm{T} & \mathbb{E}^{-1}_{\mathrm{T}}(1,3) = (1,2) \end{array}$$

Therefore $\omega(18, 25) = (\mathrm{VTVVVT})$.

3.8.5.2. *The recursion formulae.* Now we define the map $\Lambda \to \mathrm{RVT}\,(\Lambda)$, where Λ is a Puiseux characteristic of the form (3.3) by the following recursion formulae.

1. If $m = 1$ then we define the integers q and r by the equation

$$\lambda_1 = q\lambda_0 + r, \quad q \geq 2, \quad r < \lambda_0. \tag{3.8}$$

and we set

$$\mathrm{RVT}\,(\Lambda) = \mathrm{R}^{q-2}\omega(\lambda_0,\ \lambda_0 + r).$$

2. Let $m \geq 2$. Define the integers μ, s and μ_1 by the equation

$$\mu = \text{g.c.d. } (\lambda_0, ..., \lambda_{m-1}), \quad \lambda_m = \lambda_{m-1} + s\mu + \mu_1, \quad s \geq 0, \quad \mu_1 < \mu.$$

Consider the Puiseux characteristic

$$\widetilde{\Lambda} = \Big(\frac{\lambda_0}{\mu};\ \frac{\lambda_1}{\mu},\ \cdots\ \frac{\lambda_{m-1}}{\mu}\Big).$$

whose length is smaller than the length of Λ. Set

$$\mathrm{RVT}\,(\Lambda) = \mathrm{RVT}\,(\widetilde{\Lambda})\mathrm{R}^{s+1}\omega(\mu,\ \mu + \mu_1).$$

Note that the entirely critical code $\omega(\lambda_0, \lambda_0 + r)$ and $\omega(\mu, \mu + \mu_1)$ are well defined. In fact, λ_0 and $\lambda_0 + r$ and are relatively prime and $\lambda_0 + r < 2\lambda_0$ since λ_0 and λ_1 are relatively prime and $r < \lambda_0$. The numbers μ and $mu + r$ are relatively prime and $\mu + \mu_1 < 2\mu$ since g.c.d. $(\lambda_0, ..., \lambda_m) = 1$ and $\mu_1 < \mu$. Note also that the tuple $\widetilde{\Lambda}$ is a Puiseux characteristic whatever is the Puiseux characteristic Λ.

Using induction on the length m of a Puiseux characteristic and the observations

$$\mathbb{E}_{\omega(a,b)}(1,2) = (a,b); \quad \omega(a,b) = \omega(\tilde{a},\tilde{b}) \implies (a,b) = (\tilde{a},\tilde{b}) = \mathbb{E}_{\omega}(1,2)$$

it is easy to prove that the map $\Lambda \to \mathrm{RVT}\,(\Lambda)$ is inverse to $(\alpha) \to \mathrm{Pc}\,(\alpha)$, i.e. (3.6) holds.

EXAMPLE 3.28. Let us calculate $\mathrm{RVT}\,([24; 90, 100, 109])$.

Since g.c.d.$(24, 90, 100) = 2$ the calculation of this RVT code reduces to the calculation of $\mathrm{RVT}\,([12; 45, 50])$. Since g.c.d. $(12, 45) = 3$ then the calculation of $\mathrm{RVT}\,([12; 45, 50])$ reduces to the calculation of $\mathrm{RVT}\,([4; 15])$. One has

$$\mathrm{RVT}\,([24; 90, 100, 109]) = \mathrm{RVT}\,([12; 45, 50])\mathrm{R}^5\omega(2,3) =$$

$$\mathrm{RVT}\,([4; 15])\mathrm{R}^2\omega(3,5)\mathrm{R}^5\omega(2,3) = \mathrm{R}\omega(4,7)\mathrm{R}^2\omega(3,5)\mathrm{R}^5\omega(2,3).$$

Now we calculate $\omega(4,7) = \mathrm{VVT},\ \omega(3,5) = \mathrm{VV},\ \omega(2,3) = V$. We obtain

$$\mathrm{RVT}\,([24; 90, 100, 109]) = \mathrm{RVVTR}^2\mathrm{VVR}^5\mathrm{V}.$$

CHAPTER 4

Monsterization and Legendrization. Reduction theorems

4.1. Definitions and basic properties

Definition 4.1. Let $S \subset \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ be a singularity class (see Notation 2.10 and Definition 2.46). The k-step-Monsterization of S is the operation:

$$S \to \mathrm{Monster}^k(S) = \{\gamma^k(0),\ \gamma \in S\} \ \subset\ \mathbb{P}^{1+k}\mathbb{R}^2.$$

Example 4.2. According to Definition 2.13 the set of non-singular points at any level $k > 0$ is the "Monsterization" to that step k of the class of immersed Legendrian germs.

Proposition 4.3. *Let $S \subset \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ be any singularity class containing no constant curve germs. Then $\mathrm{Monster}^k(S)$ is a singularity class in $\mathbb{P}^{1+k}\mathbb{R}^2$.*

Proof. We must prove that if $p \in \mathrm{Monster}^k(S)$ and $\tilde{p}$ is equivalent to p then $\tilde{p} \in \mathrm{Monster}^k(S)$. Let $p = \gamma^k(0), \gamma \in \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ and let Φ be a local symmetry at level $(1+k)$ bringing p to $\tilde{p}$. By Proposition 2.9

$$\tilde{p} = \Phi(p) = \Phi\big(\gamma^k(0)\big) = \big(\Phi \circ \gamma^k\big)(0) = \big((\Phi_k \circ \gamma)^k\big)(0).$$

The projection Φ_k is a local contactomorphism of $\mathbb{P}^1\mathbb{R}^2$ and S is a singularity class in $\mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ so that $\Phi_k \circ \gamma \in S$ and $\tilde{p} \in \mathrm{Monster}^k(S)$. □

Next we define a partial inverse to Monsterization called Legendrization. This operation associates to each point p of the higher Monsters $\mathbb{P}^i\mathbb{R}^2$, $i > 1$ a singularity class $\mathrm{Leg}(p) \subset \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$. Our first attempt at Legendrization was to project all immersed integral curves Γ through p. Call the resulting singularity class $\mathrm{LEG}(p)$. It is an invariant of p, but it is too rough of an invariant in that it contains singularities which are too "deep" , such as the germ of the constant curve which has infinite codimension within $\mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$. To get rid of these "deep" singularities we will project only a certain invariant subset of the integral curves, namely the regular integral curve. Recall that these are immersed integral curves tangent to regular directions. See Definition 2.30.

Notation. For $p \in \mathbb{P}^{1+k}\mathbb{R}^2$ we denote by $\mathrm{Int}^{\mathrm{reg}}(p)$ the set of all *regular* curve germs $\Gamma : (\mathbb{R}, 0) \to (\mathbb{P}^{1+k}\mathbb{R}^2, p)$.

Definition 4.4. The Legendrization of a point $p \in \mathbb{P}^{1+k}\mathbb{R}^2$ is the operation

$$p \to \mathrm{Leg}(p) = \{\Gamma_k,\ \ \Gamma \in \mathrm{Int}^{\mathrm{reg}}(p)\} \ \subset\ \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$$

(Recall that Γ_k is the k-step-projection of the curve Γ.) The Legendrization of a singularity class $Q \subset \mathbb{P}^{1+k}\mathbb{R}^2$ is the union of the Legendrizations of its points, i.e.

it is the operation

$$Q \to \mathrm{Leg}(Q) = \{\mathrm{Leg}(p), \quad p \in Q\} \ \subset \ \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2).$$

PROPOSITION 4.5. *The set* $\mathrm{Int}^{\mathrm{reg}}(p)$ *and consequently the set* $\mathrm{Leg}(p)$ *are never empty. The set* $\mathrm{Leg}(p)$ *is closed with respect to reparameterization of the curves. All curves in* $\mathrm{Leg}(p)$ *are well-parameterized.*

PROOF. The first statement follows, for example, from Proposition 2.29. The second statement is a direct corollary of Proposition 2.7. The fact that any curve $\gamma \in \mathrm{Leg}(p)$ is well-parameterized follows from Propositions 2.6 and 2.37. Indeed, if $\gamma \in \mathrm{Leg}(p)$ then by Proposition 2.6 the prolongation of γ to the level of the point p is a regular and consequently immersed curve. But if γ is badly parameterized then by Proposition 2.37 any of its prolongation is badly parameterized and consequently not immersed. □

PROPOSITION 4.6. *The Legendrization of a singularity class in* $\mathbb{P}^{1+k}\mathbb{R}^2$ *is a singularity class in* $\mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$*. Monsterization is a left inverse to Legendrization:* $\mathrm{Monster}^k\big(\mathrm{Leg}(Q)\big) = Q$ *for any singularity class* $Q \subset \mathbb{P}^{1+k}\mathbb{R}^2$*.*

PROOF. The second statement follows from Proposition 2.6. To prove the first statement, let Q be a singularity class at level $1+k$, let $p \in Q$ and $\gamma \in \mathrm{Leg}(p)$, so that $\gamma = \Gamma_k$, where $\Gamma \in \mathrm{Int}^{\mathrm{reg}}(p)$. Let $\widetilde{\gamma} \in \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ be any another Legendrian curve germ equivalent to γ, so that $\widetilde{\gamma} = \Psi \circ \gamma \circ \phi$ for some local difffeomorphism $\phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ and a local contactomorphism Ψ of $\mathbb{P}^1\mathbb{R}^2$. We must show that $\widetilde{\gamma} \in \mathrm{Leg}(Q)$. By Proposition 2.9 one has

$$\widetilde{\gamma} = \Psi \circ \Gamma_k \circ \phi = (\Psi^k)_k \circ \Gamma_k \circ \phi = (\Psi^k \circ \Gamma \circ \phi)_k = (\Phi \circ \Gamma \circ \phi)_k = \widetilde{\Gamma}_k.$$

Here $\Phi = \Psi^k$ is a local symmetry at level $(1+k)$ and we have set $\widetilde{\Gamma} = \Phi \circ \Gamma \circ \phi$. Now $\tilde{p} = \Phi(p) \in Q$ since Q is a singularity class. It remains to show that $\widetilde{\Gamma}$ is a regular integral curve through $\tilde{p}$ in order to conclude that $\widetilde{\gamma} \in \mathrm{Leg}(Q)$. Since Γ is regular, $\widetilde{\Gamma}$ is regular by Proposition 2.47. □

PROPOSITION 4.7. *If* $p \in \mathbb{P}^{1+k}\mathbb{R}^2$ *is a non-singular point then* $\mathrm{Leg}(p)$ *is the set of immersed Legendrian curve germs* $\gamma : (\mathbb{R}, 0) \to \mathbb{P}^1\mathbb{R}^2$ *such that* $\gamma^k(0) = p$*.*

PROOF. Assume that p is a non-singular point and $\gamma \in \mathrm{Leg}(p)$. Let $\Gamma : (\mathbb{R}, 0) \to (\mathbb{P}^{1+k}\mathbb{R}^2, p)$ be a regular integral curve germ such that $\gamma = \Gamma_k$. By Theorem 2.39 p is a regular prolongation of a point in $\mathbb{R}^2$. Then by Proposition 2.33 γ is a regular and consequently immersed curve. By Proposition 2.6 one has $\gamma^k(0) = p$.

Assume now that $\gamma : (\mathbb{R}, 0) \to \mathbb{P}^{1+k}\mathbb{R}^2$ is an immersed Legendrian curve germ such that $\gamma^k(0) = p$. Any immersed Legendrian curve in the first level is a regular integral curve. By Proposition 2.31 the curve $\Gamma = \gamma^k$ is a regular integral curve. By Proposition 2.6 $\gamma = \Gamma_k$, therefore $\gamma \in \mathrm{Leg}(p)$. □

PROPOSITION 4.8. *For* $k \geq 2$ *the* k*-step-Monsterization of the class of immersed Legendrian curve germs is the class* $\mathrm{R}^{k-1} \subset \mathbb{P}^{k+1}\mathbb{R}^2$*. The Legendrization of the class* R^{k-1} *is the class of immersed Legendrian curve germs.*

PROOF. The first statement is a direct corollary of Theorem 3.6 and Definition 2.13. The second statement follows from Proposition 2.33 according to which the k-step-projection of any regular integral curve through a point of the class R^{k-1} is a regular and consequently immersed Legendrian curve in $\mathbb{P}^1\mathbb{R}^2$. □

REMARK 4.9. If $S \subset \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ is a singularity class then $\mathrm{Leg}\big(\mathrm{Monster}^k(S)\big) \subseteq S$, but it can happen that $\mathrm{Leg}(\mathrm{Monster}^k(S))$ does not coincide with S, i.e. Monsterization is not right inverse to Legendrization. This happens when S contains Legendrian germs whose k-step-prolongations are not regular, i.e. either not immersed or immersed but tangent to a critical direction.

THEOREM 4.10. *The Legendrization of a regular prolongation of an* RVT *class (see Definition* 3.8*) coincides with the Legendrization of the original class.*

PROOF. Follows from Propositions 2.31 and 2.33. Let (α) be an RVT class in $\mathbb{P}^{1+k}\mathbb{R}^2$ and let $\gamma \in \mathrm{Leg}(\alpha)$. Then γ is the k-step projection of a regular integral curve germ

$$\Gamma : (\mathbb{R}, 0) \to (\mathbb{P}^{1+k}\mathbb{R}^2, p), \quad p \in (\alpha).$$

Let $s \geq 1$. By Propositions 2.31 the s-step-prolongation Γ^s is a regular integral curve, and by the same theorem the point $\Gamma^s(0)$ belongs to the class $(\alpha\mathrm{R}^s)$. Therefore the $(k+s)$-step projection $(\Gamma^s)_{k+s}$ of Γ^s is a Legendrian curve in the class $\mathrm{Leg}(\alpha\mathrm{R}^s)$. But by Proposition 2.6 $\Gamma^s_{k+s} = \Gamma_k = \gamma$, hence $\gamma \in \mathrm{Leg}(\alpha\mathrm{R}^s)$.

We have proved that $\mathrm{Leg}(\alpha) \subseteq \mathrm{Leg}(\alpha\mathrm{R}^s)$. Let us now prove that $\mathrm{Leg}(\alpha\mathrm{R}^s) \subseteq \mathrm{Leg}(\alpha)$. Let $\gamma \in \mathrm{Leg}(\alpha\mathrm{R}^s)$. Then $\tilde{\gamma}$ is the $(k+s)$-step projection of a regular integral curve germ $\widetilde{\Gamma} : (\mathbb{R}, 0) \to (\mathbb{P}^{1+k}\mathbb{R}^2, \tilde{p})$ where $\tilde{p} \in (\alpha\mathrm{R}^s)$. By Proposition 2.33 the s-step projection $\widetilde{\Gamma}_s$ of $\widetilde{\Gamma}$ is a regular curve. Since $\tilde{p} \in (\alpha\mathrm{R}^s)$, one has $\widetilde{\Gamma}_s(0) \in (\alpha)$, therefore the k-step projection of Γ_s, which is exactly the curve γ, belongs to the class $\mathrm{Leg}(\alpha)$. □

Finally, we relate the Legendrization of an RVT classes with RVT-codes of plane curves (section 3.7). Recall that the RVT-code of any plane curve germ is critical.

THEOREM 4.11. *Let (α) be a critical* RVT *class and let $c : (\mathbb{R}, 0) \to \mathbb{R}^2$ be a plane curve germ. Then $c^1 \in \mathrm{Leg}(\alpha)$ if and only if (α) is the* RVT *code of c.*

PROOF. Let k be the level of (α). Assume that $c^1 \in \mathrm{Leg}(\alpha)$. Then $c^k = (c^1)^{k-1}$ is a regular integral curve and $c^k(0) \in (\alpha)$. Since (α) is critical, the point $c^k(0)$ is critical. By Proposition 3.18 k is the regularization level of c. And now (α) is the RVT-code of c since $c^k(0) \in (\alpha)$.

Assume now that (α) is the RVT-code of c. Then c^k is a regular curve and $c^k(0) \in (\alpha)$. To prove that $c^1 \in \mathrm{Leg}(\alpha)$ it suffices to note that c^1 is the $(k-1)$-step-projection of c^k. □

4.2. Explicit calculation of the Legendrization of RVT classes

The Legendrization of the open RVT class $\mathrm{R}^k \subset \mathbb{P}^k\mathbb{R}^2$ is the class of immersed Legendrian curve germs (Theorem 4.8). Obviously, any other regular RVT class is a regular prolongation of a critical RVT class. Therefore Theorem 4.10 reduces the calculation of the Legendrization of any RVT class to that of critical RVT classes. Our Theorem 3.23 from section 3.8 and Theorem 4.11 imply the validity of the following calculation of the Legendrization of any critical RVT class in terms of its RVT-code.

THEOREM 4.12. *Let* (α) *be a critical* RVT *class. Consider its Puiseux characteristic* $\mathrm{Pc}\,(\alpha) = [\lambda_0; \lambda_1, ..., \lambda_m]$ *as constructed in section* 3.8.4. *The Legendrization of* (α) *consists of Legendrian curve germs* RL*-contact equivalent to the one-step-prolongations of plane curve germs of the form*

$$x = t^{\lambda_0}, \quad y = t^{\lambda_1} f_1(t^{d_1}) + \cdots + t^{\lambda_{m-1}} f_{m-1}(t^{d_{m-1}}) + t^{\lambda_m} f_m(t),$$
$$d_i = \text{g.c.d.}(\lambda_0, \lambda_1, ..., \lambda_i), \quad f_1(0), f_2(0), ..., f_m(0) \neq 0. \tag{4.1}$$

PROOF. Let $\gamma \in \mathrm{Leg}(\alpha)$. The one-step-projection γ_1 is not a constant curve. Indeed, if γ_1 was a constant curve then by Lemma 2.38 γ must either be immersed or badly parameterized. Propositions 4.6 and 4.8 imply that immersed Legendrian curves cannot belong to the Legendrization of any critical class. Badly parameterized curves cannot belong to the Legendrization of any RVT class. See Proposition 4.5. Thus γ_1 is a non-constant curve so that $\gamma = c^1$ where $c = \gamma_1$ is a plane curve germ. By Theorem 4.11 $\gamma \in \mathrm{Leg}(\alpha)$ if and only if (α) is the RVT-code of c. By Theorem 3.23 this is so if and only if c^1 is RL-contact equivalent to $\tilde{c}^1$ where the plane curve germ $\tilde{c}$ has Puiseux characteristic $[\lambda_0; \lambda_1, ..., \lambda_m] = \mathrm{Pc}\,(\alpha)$. It remains to note that any plane curve germ with the Puiseux characteristic $[\lambda_0; \lambda_1, ..., \lambda_m]$ is RL-contact equivalent to a plane curve germ of the form (4.1). See [**W**]. (To obtain (4.1) it suffices to take local coordinates in which the curve has the form (3.7) and apply a change of coordinates of the form $(x, y) \to (x, y - f(x))$ with a certain function $f(x)$).) □

EXAMPLE 4.13. In Example 3.25 we calculated $\mathrm{Pc}\,(\mathrm{R}^3\mathrm{VVR}^4\mathrm{VTR}^5\mathrm{VVT}) = [36; 204, 244, 263]$. One has

$$\text{g.c.d.}(36, 204) = 12, \quad \text{g.c.d.}(36, 204, 244) = 4, \quad \text{g.c.d.}(36, 204, 244, 263) = 1.$$

By Theorem 4.12 the Legendrization of the class $(\mathrm{R}^3\mathrm{VVR}^4\mathrm{VTR}^5\mathrm{VVT})$ is the class of Legendrian curve germs RL-contact equivalent to the one-step-prolongations of plane curve germs of the form

$$x = t^{36}, \quad y = t^{204} f_1(t^{12}) + t^{244} f_2(t^4) + t^{263} f_3(t), \quad f_1(0), f_2(0), f_3(0) \neq 0.$$

4.3. From points to Legendrian curves

The following theorem includes Theorem 2.12 and gives a necessary condition for the equivalence of points of the Monster tower.

THEOREM 4.14. *Let* $p, \tilde{p} \in \mathbb{P}^{1+k}\mathbb{R}^2$.

1. *If the Legendrizations of the points* p *and* $\tilde{p}$ *contain RL-contact equivalent germs* $\gamma \in \mathrm{Leg}(p)$ *and* $\widetilde{\gamma} \in \mathrm{Leg}(\tilde{p})$ *then the points* p *and* $\tilde{p}$ *are equivalent.*

2. *If the points* p *and* $\tilde{p}$ *are equivalent then their Legendrizations* $\mathrm{Leg}(p)$ *and* $\mathrm{Leg}(\tilde{p})$ *are contactomorphic.*

To say that $\mathrm{Leg}(p)$ and $\mathrm{Leg}(\tilde{p})$ are contactomorphic means that there exists a local contactomorphism $\Psi : (\mathbb{P}^1\mathbb{R}^2, p_1) \to (\mathbb{P}^1\mathbb{R}^2, \tilde{p}_1)$, where p_1 and $\tilde{p}_1$ are the projections of p and $\tilde{p}$ to $\mathbb{P}^1\mathbb{R}^2$, such that for any $\gamma \in \mathrm{Leg}(p), \widetilde{\gamma} \in \mathrm{Leg}(\tilde{p})$ we have that $\Psi \circ \gamma \in \mathrm{Leg}(\tilde{p})$ and $\Psi^{-1} \circ \widetilde{\gamma} \in \mathrm{Leg}(p)$.

PROOF. The first statement is a corollary of the second statement of Theorem 2.12. To prove the second statement we follow the notation and lines of the proof of Proposition 4.6. If the symmetry Φ at level $(1 + k)$ sends p to $\tilde{p}$ and if $\Gamma \in \mathrm{Int}^{\mathrm{reg}}(p)$

then $\widetilde{\Gamma} = \Phi \circ \Gamma$ belongs to $\mathrm{Int}^{\mathrm{reg}}(\tilde{p})$. Then $\Psi = \Phi_k$ is a local contactomorphism of $\mathbb{P}^1\mathbb{R}^2$. By Proposition 2.9, (ii) $\widetilde{\Gamma}_k = (\Phi\circ\Gamma)_k = \Psi\circ\Gamma_k$. It follows that $\Psi\circ\Gamma_k \in \mathrm{Leg}(\tilde{p})$. The same argument yields that $\Psi^{-1} \circ \widetilde{\Gamma}_k \in \mathrm{Leg}(p)$ for any curve $\widetilde{\Gamma} \in \mathrm{Int}^{\mathrm{reg}}(\tilde{p})$. $\square$

The following statement is also a direct corollary of Theorem 2.12.

THEOREM 4.15. *Let $Q \subset \mathbb{P}^{1+k}\mathbb{R}^2$ be a singularity class whose Legendrization $\mathrm{Leg}(Q) \subset \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ consists of a finite number s of orbits with respect to the* RL-*contact equivalence, these orbits being represented by the Legendrian curve germs $\gamma_1, \dots, \gamma_s$. Then any point of the class Q is equivalent to one of the points $\gamma_1^k(0), \dots, \gamma_s^k(0)$ and consequently the class Q consists of $\tilde{s} \le s$ orbits. In particular, if $s = 1$ then all points of Q are equivalent.*

4.4. Simplest classification results

Theorems 4.10 and 4.12 allow us to obtain explicit Legendrizations of RVT classes. In this section we illustrate these theorems and Theorems 4.14 and 4.15 by examples of series of RVT classes of low codimension consisting of a single orbit or of a finite number $s > 1$ of orbits.

4.4.1. Classes of the form $\mathrm{R}^s\mathrm{V}\mathrm{R}^q$ and A-singularities. Consider the RVT classes $(\alpha) = \mathrm{R}^s\mathrm{V}\mathrm{R}^q$. Calculate $\mathbb{E}_{\mathrm{V}}(1,2) = (2,3)$ and $\mathrm{Pc}\,(\mathrm{R}^s\mathrm{V}) = [2; 2s+5]$ using section 3.8.4. By Theorem 4.12 the Legendrization of the class $(\mathrm{R}^s\mathrm{V})$ consists of Legendrian curve germs RL-contact equivalent to the one-step prolongations of the plane curve germs $c : (t^2, t^{2s+5}f(t))$, where $f(0) \neq 0$. The class $\mathrm{R}^s\mathrm{V}$ and $\mathrm{R}^s\mathrm{V}\mathrm{R}^q$ have the same Legendrization by Theorem 4.10. The curve c represents the the classical A-singularity and is RL-equivalent to (t^2, t^{2s+5}). Therefore c^1 is RL-contact equivalent to the one-step prolongation of (t^2, t^{2s+5}) (Proposition 2.8). Consequently the Legendrization of (α) consists of a single orbit with respect to the RL-contact equivalence. By Theorem 4.14 (or Theorem 4.15) we see that *all points of the class $\mathrm{R}^s\mathrm{V}\mathrm{R}^q$ are equivalent.* We also obtain the following statement which is equivalent to the classification of codimension one singularities of Goursat 2-distributions discussed earlier and found in [**Mor4**]:

THEOREM 4.16. *A generic singular point in $\mathbb{P}^k\mathbb{R}^2, k \ge 3$ is equivalent to one of the points realized as the k-step prolongations of one of the $k-2$ plane curves*

$$c_s(t) = (t^2, t^{2s+5}), \quad s = 0, 1, \dots, k-3$$

("A_{s+2}-singularities"). "Generic" here means that the orbits of these points $c_s^k(0)$ form a dense open subset O within the subvariety of all singular points $S \subset \mathbb{P}^k\mathbb{R}^2$. The set S is a codimension one subvariety of $\mathbb{P}^k\mathbb{R}^2$ and is the closure of the union of the classes $\mathrm{R}^s\mathrm{V}\mathrm{R}^q$, $q + s = k-3$, these being the orbits of the above points. The variety $S \setminus O$ has codimension one in S, and hence codimension 2 within $\mathbb{P}^k\mathbb{R}^2$.

4.4.2. Classes of the form VTR^q and VVR^q. Calculate

$$\mathbb{E}_{\mathrm{VT}}(1,2) = (3,4),\ \mathbb{E}_{\mathrm{VV}}(1,2) = (3,5),\ \mathrm{Pc}\,(\mathrm{VT}) = [3; 7],\ \mathrm{Pc}\,(\mathrm{VV}) = [3, 8]$$

using section 3.8.4. By Theorem 4.12 and 4.10 the Legendrization of VTR^q, and VVR^q are the classes of Legendrian curve germs RL-contact equivalent to the one-step prolongations of the plane curve germs $(t^3, t^7 f(t))$, and $(t^3, t^8 f(t)), f(0) \neq 0$. Though not all such plane curve germs are RL-equivalent, their one-step prolongations are RL-contact equivalent. See Example B.5 in Appendix B. Therefore the Legendrization of each of the classes $\mathrm{VTR}^q, \mathrm{VVR}^q$ consists of a single orbit. Theorems 4.14 and 4.15 now imply:

THEOREM 4.17. *The classes* (VVR^q), *and* (VTR^q) *consist of a single orbit. A point in one of these classes is equivalent to the* $(q+4)$*-step-prolongation of the plane curve germ* (t^3, t^7), *respectively* (t^3, t^8), *evaluated at* $t = 0$.

4.4.3. Classes $\mathrm{VR}^m\mathrm{VR}^q$. In the same way we obtain

THEOREM 4.18. *The class* $\mathrm{VR}^m\mathrm{VR}^q$ $(m \geq 1, q \geq 0)$ *consists of a single orbit. Any one of its points is equivalent to the* $(m+q+4)$*-step-prolongation of the plane curve* $(t^4, t^{10} + t^{9+2m})$ *evaluated at* $t = 0$.

To prove this we calculate $\mathrm{Pc}\,(\mathrm{VR}^m\mathrm{V}) = [4; 10, 9+2m]$ using section 3.8.4. By Theorem 4.12 and 4.10 the Legendrization of the class $\mathrm{VR}^m\mathrm{VR}^q$ consists of Legendrian curve germs RL-contact equivalent to the one-step prolongations of plane curve germs $\big(t^4, t^{10}f_1(t^2) + t^{9+2m}f_2(t)\big), f_1(0), f_2(0) \neq 0$. The one-step prolongations of such plane curve germs are all RL-contact equivalent (see Example B.6 in Appendix B). As in the previous examples, the equivalence of all points of the class $\mathrm{VR}^m\mathrm{VR}^q$ follows from Theorems 4.14 and 4.15.

4.4.4. Classes of the form $\mathrm{R}^m\mathrm{VTR}^q$ and $\mathrm{R}^m\mathrm{VVR}^q$, $m \geq 1$. Calculate

$$\mathrm{Pc}\,(\mathrm{R}^m\mathrm{VTR}^q) = [3; 3m+7], \quad \mathrm{Pc}\,(\mathrm{R}^m\mathrm{VVR}^q) = [3; 3m+8]$$

using section 3.8.4. By Theorem 4.12 and 4.10 the Legendrization of the class $(\mathrm{R}^m\mathrm{VTR}^q)$, respectively $(\mathrm{R}^m\mathrm{VVR}^q)$ consists of Legendrian curve germs RL-contact equivalent to the one-step prolongations of plane curve germs of the form $(t^3, t^{3m+7}f(t))$, respectively $(t^3, t^{3m+8}f(t))$, $f(0) \neq 0$. The set of one-step prolongations of such plane curve germs intersects a finite number of orbits with respect to the RL-contact equivalence. See Example B.5 in Appendix B. The number of orbits depends on m and q and can be bigger than 1. Theorem 4.15 implies:

PROPOSITION 4.19. *The* RVT *classes* $(\mathrm{R}^m\mathrm{VTR}^q)$ *and* $(\mathrm{R}^m\mathrm{VVR}^q)$ *consist of a finite number of orbits, for any* $m, q \geq 0$.

Unlike the previous examples, this theorem does not give a final classification of the classes $(\mathrm{R}^m\mathrm{VTR}^q)$ and $(\mathrm{R}^m\mathrm{VVR}^q)$. To obtain a final classification Theorems 4.14 and 4.15 are not enough, see the next section. The final classification of these classes can be found in Table 5.4, along with a final classification of many other RVT classes. It requires further reduction theorems. See sections 4.6 - 4.8.

4.5. On the Implications and Shortfalls of Theorems 4.14 and 4.15

As we showed in the previous section, for some of the most important RVT classes Theorems 4.14 and 4.15 alone imply significant classification results. But these Theorems by themselves provide far less than is needed for a complete reduction of the problem of classification of points in the Monster to that of the RL-contact classification of Legendrian curve germs. Such a reduction is not possible, even in principle. Several closely related things go wrong when trying to force such a reduction. The number $\tilde{s}$ of orbits in a given Monster class which appears in Theorem 4.15 might be much smaller than the number s; there of orbits in the corresponding Legendrian singularity class. One might even have $\tilde{s} = 1$ with s arbitrarily large. The root of the problem which prevents such a reduction, even in principle, is that the data needed to describe a curve germ (as opposed to a singularity class) is by its very nature, infinite dimensional, involving the full power series expansion, while the data needed to describe a point of the Monster at a fixed level is finite-dimensional. To make the Legendrian data finite, we will use finite jets of Legendrian curves. *In what follows we will show that the problem of classification of points in the Monster can be completely reduced to that of the* RL-*contact classification of Legendrian curve jets.* This reduction, given below, is based on Theorem 4.14 and the analysis of the structure of the set $\mathrm{Leg}(p)$ for a fixed point p in the Monster.

4.6. From points to Legendrian curve jets. The jet-identification number

Introduce the following notation and definitions.

(1) Given a Legendrian curve germ $\gamma = \gamma(t) \in \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ we write $j^r\gamma$ for its r-jet at $t = 0$.

(2) Given a set $S \subset \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ we denote by j^rS the set $\{j^r\gamma,\ \gamma \in S\}$.

(3) A reparameterization of an r-jet $\xi \in j^r\,\mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ is the r-jet $j^r(\gamma(\phi(t)))$, where $\gamma(t)$ is a Legendrian curve germ representing the jet ξ and $\phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ is a local diffeomorphism.

(4) Two r-jets in $j^r\,\mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ are RL-contact equivalent if the following holds for some (and then any) Legendrian curve germs $\gamma, \tilde{\gamma}$ representing these jets: $\tilde{\gamma}$ is RL-contact equivalent to a curve whose r-jet coincides with $j^r\gamma$.

To check that definition (3) is good, i.e. that the choice of representative γ is irrelevant, and also to check the validity of the parenthetical claim in (4) that "for some (and then any)" , it suffices to note that any local diffeomorphism Φ and any reparameterization ϕ "respect" the filtration by jets, i.e. $j^r(\Phi\circ\gamma\circ\phi) = j^r(\Phi\circ\tilde{\gamma}\circ\phi)$ if $j^r\gamma = j^r\tilde{\gamma}$.

Fix a point p in the Monster. What is the structure of the set $\mathrm{Leg}(p) \subset \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$? We know (Proposition 4.5) that it is closed with respect to the reparameterization of the curves. What else? One can hope that along with any Legendrian curve germ $\mathrm{Leg}(p)$ also contains all Legendrian curve germs sharing the same r-jet, for r is sufficiently big. Assume that this is so for all $r \geq r_0$, but not for $r < r_0$. What can be said about the set $j^{r_0}\,\mathrm{Leg}(p)$? The best one could hope for is that it consists of a single r_0-jet up to reparameterization! In this case p can be identified with this r_0-jet.

Definition 4.20 (the jet-identification number). Fix a point p in the Monster. Assume that there exists an integer r satisfying the following two conditions:

(1) If $\gamma \in \mathrm{Leg}(p)$ then $\tilde{\gamma} \in \mathrm{Leg}(p)$ for any germ $\tilde{\gamma} \in \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ such that $j^r\tilde{\gamma} = j^r\gamma$.

(2) The set $j^r\,\mathrm{Leg}(p)$ consists of a single r-jet up to reparameterization.

In this case we will say that

- the point p can be identified with a single r-jet (the r-jet in item (2))
- r is the *jet-identification number* of the point p.

We will see later that not all points have jet identification numbers. But first, we show that the number, when defined, is well-defined.

Proposition 4.21. *When a point has a jet-identification number, that number is unique.*

Proof. Assume, by way of contradiction, that $r_1 < r_2$ are jet-identification numbers for the same point p. Let $\gamma \in \mathrm{Leg}(p)$. Since $r_1 < r_2$ we can find another curve $\tilde{\gamma}$ such that $j^{r_1}\tilde{\gamma} = j^{r_1}\gamma$ but with $j^{r_2}\tilde{\gamma} \neq j^{r_2}(\gamma \circ \phi)$ for any reparameterization ϕ. Indeed, for any positive integer r, the space of Legendrian $(r+1)$-jets having a fixed r-jet is 2-dimensional while the space of reparameterizations of an $(r+1)$-jet which do not change its r-jet is 1-dimensional. By item (1) in Definition 4.20 for $r = r_1$ one has $\tilde{\gamma} \in \mathrm{Leg}(p)$. This contradicts item (2) in Definition 4.20 for $r = r_2$. □

The next property asserts that the jet-identification number (when defined) is an invariant. This statement is not obvious. It uses the second statement (2) of Theorem 4.14, a statement which we have not used so far.

Proposition 4.22. *If p has jet-identification number r and $\tilde{p}$ is equivalent to p then $\tilde{p}$ also has jet-identification number r.*

Proof. We have to check that (1) and (2) of Definition 4.20 hold. By Theorem 4.14, part (2), the equivalence of p and $\tilde{p}$ implies the existence of a local contactomorphism Φ such that $\Phi \circ \gamma \in \mathrm{Leg}(p)$ whenever $\gamma \in \mathrm{Leg}(\tilde{p})$. Fix such a contactomorphism Φ.

Proof of (1). Let $\gamma \in \mathrm{Leg}(\tilde{p})$, and $\tilde{\gamma} \in \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ with $j^r\tilde{\gamma} = j^r\gamma$. We have to show that $\tilde{\gamma} \in \mathrm{Leg}(\tilde{p})$. We have $\Phi \circ \tilde{\gamma} \in \mathrm{Leg}(p)$. We also have $j^r(\Phi \circ \tilde{\gamma}) = j^r(\Phi \circ \gamma)$ since $j^r\tilde{\gamma} = j^r\gamma$. By definition of jet identification number, $\Phi \circ \tilde{\gamma} \in \mathrm{Leg}(p)$. Applying Theorem 4.14, part (2), again we obtain $\tilde{\gamma} \in \mathrm{Leg}(\tilde{p})$.

Proof of (2). Let $\gamma, \tilde{\gamma} \in \mathrm{Leg}(\tilde{p})$. We must show that $j^r\tilde{\gamma}$ is a reparameterization of $j^r\gamma$. Then, again by Theorem 4.14, part (2), we have $\Phi \circ \gamma,\ \Phi \circ \tilde{\gamma} \in \mathrm{Leg}(p)$. Since r is the jet-identification number for p then the r-jets $j^r(\Phi \circ \gamma),\ j^r(\Phi \circ \tilde{\gamma})$ are the same up to reparameterization. Consequently the r-jets $j^r\gamma,\ j^r\tilde{\gamma}$ are also the same up to reparameterization. □

Combining the *two* statements of Theorem 4.14 we obtain the following crucial property:

the equivalence problem for points of the Monster which have jet-identification numbers reduces to the RL*-contact equivalence problem for Legendrian curve jets.*

Namely, the following theorem holds.

Theorem 4.23. *Let p and $\tilde{p}$ be points in $\mathbb{P}^i\mathbb{R}^2$ whose jet-identification numbers r and $\tilde{r}$ are defined. Take* **any** *germ $\gamma \in \mathrm{Leg}(p)$ and* **any** *germ $\tilde{\gamma} \in \mathrm{Leg}(\tilde{p})$. The points p and $\tilde{p}$ are equivalent if and only if $r = \tilde{r}$ and the r-jets $j^r\gamma, j^r\tilde{\gamma}$ are* RL-*contact equivalent.*

Proof. Assume that $r = \tilde{r}$ and the r-jets $j^r\gamma, j^r\tilde{\gamma}$ are RL-contact equivalent. This means that $\tilde{\gamma}$ is RL-equivalent to a Legendrian curve germ μ such that $j^r\mu = j^r\gamma$. By (1) in Definition 4.20 one has $\mu \in \mathrm{Leg}(p)$. By Theorem 4.14, (1) the points p and $\tilde{p}$ are equivalent.

Assume now that p and $\tilde{p}$ are equivalent. By Proposition 4.22 one has $r = \tilde{r}$. By Theorem 4.14, part (2), the germ $\tilde{\gamma}$ is RL-contact equivalent to some germ $\mu \in \mathrm{Leg}(p)$. The equivalence of curve germs implies the equivalence of their r-jets, therefore the r-jets $j^r\tilde{\gamma}$ and $j^r\mu$ are RL-contact equivalent. By (2) of Definition 4.20 the r-jets $j^r\mu$ and $j^r\gamma$ are the same up to reparameterization. Therefore the r-jets $j^r\tilde{\gamma}$ and $j^r\gamma$ are RL-contact equivalent. $\square$

For which points of the Monster are the jet-identification numbers defined? The next theorem answers this question.

THEOREM 4.24. *At level* 3 *or higher a point p of the Monster has a jet-identification number if and only if it is regular. All points of a fixed regular* RVT *class have the same jet-identification number.*

DEFINITION 4.25. The jet-identification number of a regular RVT class is the jet-identification number of some (and hence any) point of this class.

For completeness' sake we record what happens at levels 1 and 2.

PROPOSITION 4.26. *Points at the first level do not have a jet-identification number. Every point at the second level has jet-identification number* 1.

The assertions of Theorem 4.24 stating that regular points have jet-identification numbers and that all points of a fixed regular RVT class have the same jet-identification number follow immediately from Theorem 4.40 in section 4.8. The other assertion of the theorem, that critical points do not have a jet-identification number, is proved in section 4.11. Proposition 4.26 is proved in section 4.12.

4.7. The parameterization number

Regular points have jet identification numbers by Theorem 4.24. How are these numbers to be computed? The answer, which is spelled out precisely in Theorem 4.40 of section 4.8 requires another number attached to points, the parameterization number of a point. This number requires the definition of well-parameterized curves, which can be found back in Definition 2.35, and the following definition of the order of good parameterization.

DEFINITION 4.27. Let $c : (\mathbb{R}, 0) \to \mathbb{M}^n$ be the germ at $t = 0 \in \mathbb{R}$ of a well-parameterized analytic curve in an n-manifold M^n (see section 2.10). The order of good parameterization of c is the minimal integer d satisfying the following condition: any curve germ $\tilde{c} : (\mathbb{R}, 0) \to \mathbb{M}^n$ such that $j^d\tilde{c} = j^d c$ is also well-parameterized.

Any well-parameterized analytic curve germ c has a finite order of good parameterization. See [**W**]. This order is an RL-invariant: c and $\Phi \circ c \circ \phi$ have the same order of good parameterization, where Φ and ϕ are diffeomorphisms of the underlying manifold, and of the real line. This RL invariance can be checked with routine logic, using that $j^d(\Phi \circ c \circ \phi) = j^d\Phi \circ j^d c \circ j^d\phi$.

EXAMPLE 4.28. If $c(t) = (x(t), y(t)) = (t^r, f(t))$ is a well-parameterized plane curve germ and the function $f(t)$ has vanishing r-jet then its order of good parameterization is the smallest exponent λ relatively prime to r for which the coefficient of t^λ in the power series expansion of f is non-zero. In particular, if c is a plane curve with Puiseux characteristic $[\lambda_0; \lambda_1, \ldots, \lambda_m]$ (see equations (3.7) and (4.1)) then its parameterization number is λ_m. Indeed, by truncating the Taylor expansion of c to any degree less than λ_m we obtain a curve which is not well-parameterized, while any curve whose Taylor expansion agrees with c at order λ_m or greater is necessarily well-parameterized.

EXAMPLE 4.29. A polynomial curve of degree k can be well-parameterized, while its order of good parameterization can be greater than k. For example the well-parameterized curve $(x(t), y(t)) = (t^4, t^2+t^5)$ has degree 5 but its order of good parameterization is 7. To see this, complete the square, to realize that $t^2 + t^5 = (t+(1/2)t^4)^2 \bmod t^6$. Setting $f(t) = (t+(1/2)t^4)$ we see that $c(t)$ and $(f(t)^4, f(t)^2)$ share the same 5-jet, but the latter curve is not well -parameterized. Indeed the two curves have the same 6-jets, but the 7-jets of their x-components differ, so the order of good parameterization of c must be at least 7. To see that its order equals 7 either reparameterize c with a parameter τ satisfying $\tau^2 = t^2 + t^5$ (an "R" transformation), or change variables by $(x, y) \mapsto (x-y^2, y)$ (an "L" transformation). Either change puts c into the standard preliminary form of a curve of the class A_3, whose representative is (t^2, t^7) and which has order of good parameterization 7 according to the previous example.

Recall that any curve in the Legendrization of any point of the Monster is well-parameterized. See Proposition 4.5.

PROPOSITION 4.30. *The order of good parameterization of any non-singular point, and in particular of any point in the first or the second level of the Monster, is equal to* 1.

THEOREM 4.31. *Let p be a point in the third or higher level of the Monster. The order of good parameterization is the same for all $\gamma \in \mathrm{Leg}(p)$. Moreover, it depends only on the* RVT *code of p, i.e. the order of good parameterization is the same for all curves in the Legendrization of any fixed* RVT *class.*

In view of this theorem we define the parameterization number of a point and of an RVT class as follows.

DEFINITION 4.32 (parameterization number). Let p be any point at any level. The parameterization number of p is the order of good parameterization of some (and hence any) curve germ $\gamma \in \mathrm{Leg}(p)$. The parameterization number of an RVT class (α) is the parameterization number of some (and hence any) point of (α). Equivalently, it is the order of good parameterization of some (and hence any) Legendrian curve germ $\gamma \in \mathrm{Leg}(\alpha)$.

It is worth noting that *the parameterization number is defined for any point of the Monster and any* RVT *class while the jet-identification number is defined only for regular points and regular* RVT *classes.* The following theorems gives a simple way to calculate the parameterization number of any RVT class.

PROPOSITION 4.33. *The parameterization number of the* RVT *class* (R...R) *is equal to* 1. *If (α) is a regular* RVT *class and $(\alpha) \neq$* (R...R) *so that (α) is a regular prolongation of a critical* RVT *class $(\widehat{\alpha}) : (\alpha) = (\widehat{\alpha}\mathrm{R}^q), q \geq 1$, then the parameterization number of (α) coincides with the parameterization number of $(\widehat{\alpha})$.*

This proposition reduces the calculation of the parameterization number of RVT classes to the case of critical RVT classes.

THEOREM 4.34. *Let* (α) *be a critical* RVT *class. Let* $\mathrm{Pc}\,(\alpha) = [\lambda_0; \lambda_1, ..., \lambda_m]$ *be the Puiseux characteristic constructed in section* 3.8.4. *The parameterization number of* (α) *is equal to* $\lambda_m - \lambda_0$.

EXAMPLE 4.35. The parameterization numbers of all critical RVT classes (α) at levels $3, 4$ and 5 are given in Table 4.1. Since the parameterization number of (α) coincides with the parameterization number of any regular prolongation of (α) (Proposition 4.33), the table contains the parameterization numbers of *all* RVT classes at levels 3,4, and 5.

TABLE 4.1. Parameterization numbers of critical RVT classes (α) at levels 3,4, and 5 (= parameterization number of any regular prolongation of (α), see Proposition 4.33).

Class (α)	level	$\mathrm{Pc}\,(\alpha)$	param. number
V	3	$[2; 5]$	3
RV	4	$[2; 7]$	5
VT	4	$[3; 7]$	4
VV	4	$[3; 8]$	5
RRV	5	$[2; 9]$	7
RVV	5	$[3; 11]$	8
RVT	5	$[3; 10]$	7
VRV	5	$[4; 10, 11]$	7
VVV	5	$[5; 13]$	8

Table 4.1 shows that the exact upper bound for the parameterization number of a point at level 3,4, or 5 is equal to 3,5, or 8 respectively, and that this upper bound is realized by points of the RVT classes $\mathrm{V}, \mathrm{VV}, \mathrm{VVV}$ respectively. This observation can be generalized:

CLAIM 4.36. *The exact upper bound for the parameterization number of a point in the* k*th level of the Monster,* $k \geq 3$*, is the* $(k+1)$*-st Fibonacci number. This upper bound is realized by points of the* RVT *class* V^{k-2}.

PROOF. In view of Theorem 4.34, Proposition 4.33 and the explicit recursion formulae for computing $\mathrm{Pc}\,(\alpha)$ in section 3.8.4, this claim is an arithmetical statement and its proof is an exercise (rather involved) on induction. We leave the proof to the reader. $\square$

PROOF OF PROPOSITIONS 4.30 AND 4.33. Theorem 4.30 and the first statement of Theorem 4.33 are direct corollaries of Propositions 4.7 and 4.8. The second statement of Theorem 4.33 is a direct corollary of Theorem 4.10 stating that the Legendrization of any regular prolongation of an RVT class coincides with the Legendrization of this class.

Proof of Theorems 4.31 and 4.34. Theorem 4.31 is a part of Theorem 4.34. This theorem will be deduced from Theorem 4.12 which states that the Legendrization of a critical RVT class consists of the RL-contact equivalence classes generated by the one-step-prolongations of plane curve germs of the form (4.1). Since the order of good parameterization is an invariant with respect to RL-equivalence (in particular RL-contact equivalence), it suffices to prove the following statement.

Proposition 4.37. *Let* $[\lambda_0; \lambda_1, ..., \lambda_m]$ *be any Puiseux characteristic satisfying the condition* $\lambda_1 > 2\lambda_0$. *Let* c *be a plane curve germ of the form* (4.1). *The order of good parameterization of the space curve* c^1 *is equal to* $\lambda_m - \lambda_0$.

Remarks 4.38.

1. We *may* assume $\lambda_1 > 2\lambda_0$ since the Puiseux characteristic $[\lambda_0; \lambda_1, ..., \lambda_m]$ in Theorem 4.12 belongs to the image of the map Pc which consists of Puiseux characteristics satisfying this condition.

2. The assumption $\lambda_1 > 2\lambda_0$ cannot be taken away from Proposition 4.37. For example, the curve $c = (t^3, t^5)$ has first prolongation $(t^3, t^5, (5/3)t^2)$ in the standard coordinates of section 1.2 and consequently its order of good parameterization is 3. But $3 \neq 5 - 3$.

Proof of Proposition 4.37. Use the coordinates (x, y, u) described in section 1.2. In these coordinates c^1 has the form

$$x = t^{\lambda_0},\ y = f(t^r) + t^{\lambda_m} h_1(t),\ u = g(t^r) + t^{\lambda_m - \lambda_0} h_2(t),\quad h_1(0) \neq 0, h_2(0) \neq 0, \tag{4.2}$$

where r is a divisor of λ_0, r and λ_m are relatively prime, and f and g are polynomials of one variable such that $f(t^r) = at^{\lambda_1} + \text{h.o.t.}, g(t^r) = bt^{\lambda_1 - \lambda_0} + \text{h.o.t.}, a, b \neq 0$. It is clear that the order of good parameterization of (4.2) is not smaller than $\lambda_m - \lambda_0$. To prove show that the order is equal to $\lambda_m - \lambda_0$ we must show that any space curve sharing this $(\lambda_m - \lambda_0)$-jet is well-parameterized. Notice that $y(t) = o(x(t))$ and $u(t) = o(x(t))$ as follows from our assumption $\lambda_1 > 2\lambda_0$, and from $\lambda_m \geq \lambda_1$. It follows that any curve with the same $(\lambda_m - \lambda_0)$-jet as curve (4.2) can be reparameterized to be put in the same form as (4.2) (with new f, g, h_1, h_2 , but satisfying the same constraints) and consequently is well-parameterized.

4.8. Evaluating the jet-identification number

Theorem 4.39. *The jet-identification number of the class* R^k *is equal to* $k+1$.

We give two proofs momentarily.

Any regular RVT class which is not the open class (R...R) is a regular prolongation of a critical RVT class, i.e. has the form $(\alpha\mathrm{R}^q)$ where (α) is a critical RVT-code and $q \geq 1$.

Theorem 4.40. *Let* (α) *be a critical* RVT *class and* d *its parameterization number. Then the jet-identification number of the regular* RVT *class* $(\alpha\mathrm{R}^q)$, $q \geq 1$, *equals*

$$r = d + q - 1.$$

Combining this theorem with Theorem 4.34 we obtain the following explicit formulae for the jet-identification number of any regular RVT class.

THEOREM 4.41. (Theorem 4.40 combined with Theorem 4.34). *Let* (α) *be any critical* RVT *class, i.e. the* RVT-code (α) *ends with* V *or* T. *Let* $\mathrm{Pc}(\alpha) = [\lambda_0; \lambda_1, ..., \lambda_m]$ *be the Puiseux characteristic constructed in section* 3.8.4. *Let* $q \geq 1$. *The jet-identification number* r *of the class* $(\alpha\mathrm{R}^q)$ *is equal to*

$$r = \lambda_m - \lambda_0 + q - 1.$$

EXAMPLE 4.42. Using Table 4.1 (which contains the parameterization numbers of critical RVT classes at levels 3, 4, and 5) and Theorem 4.40 we obtain the jet-identification numbers of all regular classes in level ≤ 6, see Table 4.2.

TABLE 4.2. Jet-determination numbers of regular RVT classes in levels 3,4,5,6

Regular class	level	Jet-identif. number	Regular class	level	Jet-identif. number
R	3	2	VRRR	6	5
RR	4	3	RVRR	6	6
VR	4	3	RRVR	6	7
RRR	5	4	VVRR	6	6
VRR	5	4	VTRR	6	5
RVR	5	5	RVVR	6	8
VVR	5	5	RVTR	6	7
VTR	5	4	VRVR	6	7
RRRR	6	5	VVVR	6	8

Table 4.2 shows that the exact upper bound for the jet-identification number of a point in level 3,4,5,6 is equal to 2,3,5,8 respectively, and this upper bound is realized by points of the RVT classes R, VR, VVR, VVVR respectively. This observation can be generalized:

CLAIM 4.43. *The exact upper bound for the jet-identification number of a regular point in the* k*th level of the Monster,* $k \geq 3$, *is the* k*th Fibonacci number. This upper bound is realized by points of the* RVT *class* $\mathrm{V}^{k-3}\mathrm{R}$.

PROOF. Denote by f_i the Fibonacci numbers. We have to prove that the identification number $r(\alpha)$ of any regular RVT class (α) in the kth level does not exceed f_k and that this bound is exact. If $(\alpha) = \mathrm{R}^{k-2}$ then $r(\alpha) = k-1$ (Theorem 4.39) and obviously $k - 1 \leq f_k$. If $(\alpha) \neq \mathrm{R}^{k-2}$ then (α) has the form $(\alpha) = (\beta\mathrm{R}^q)$ where $q \geq 1$ and β is a critical class in level $(k-q)$. By Claim 4.36 the parameterization number $d(\beta)$ of (β) does not exceed f_{k-q+1} and this upper bound is realized by the class $(\beta) = \mathrm{V}^{k-q-2}$. By Theorem 4.40 $r(\alpha) = d(\beta) + q - 1$ and consequently $r(\alpha) \leq f_{k-q+1} + q - 1$. Note that $f_j - f_i \geq j - i$ for any $j \geq i \geq 2$. Therefore $r(\alpha) \leq f_k$. The jet-identification number of the class $\mathrm{V}^{k-3}\mathrm{R}$ is equal to f_k (again, by Theorem 4.40 and Claim 4.36). $\square$

Coordinate proof of Theorem 4.39. The class R^k is in level $k+2$. A point p in this class is represented by the $(k+2)$-fold prolongation of an immersed plane curve c. Use coordinates $x, y, u_1, \ldots, u_{k+2}$ as per Example 2.3. Then p is represented in evaluating the $(k+2)$-jet of a function $y = f(t)$ at $t = 0$:

$$x = 0,\ y = f(0),\ u_1 = f'(0),\ \ldots,\ u_{k+2} = f^{(k+2)}(0).$$

The corresponding immersed Legendrian curve $\gamma = c^1 \in \mathrm{Leg}(p)$ with $\gamma^{k+1}(0) = p$ is expressed as $\gamma(t) = (t, f(t), f'(t))$ in the x, y, u_1 coordinates. From this coordinate expression, we see that p is uniquely determined by the $(k+1)$-jet of any Legendrian curve in $\mathrm{Leg}(p)$. Hence the jet-identification number of the class R^k is $k+1$.

Coordinate-free proof of Theorem 4.39. In view of Proposition 4.8, to prove Theorem 4.39 we have to prove the following two statements:

1. If $\gamma, \tilde{\gamma} : (\mathbb{R}, 0) \to \mathbb{P}^1\mathbb{R}^2$ are immersed Legendrian curve germs with the same i-jets and $\gamma^i(0) = p$ then $\tilde{\gamma}^i(0) = p$.

2. If $\gamma, \tilde{\gamma} : (\mathbb{R}, 0) \to \mathbb{P}^1\mathbb{R}^2$ are immersed Legendrian curve germs such that $\gamma^i(0) = \tilde{\gamma}^i(0) = p$ then their i-jets of γ and $\tilde{\gamma}$ are the same up to reparameterization.

These two claims are the particular case $k = 1$ of the Proposition 4.44 which follows immediately, and requires the

Notation. The sign "= (repar.) =" between two curves or two curve jets means that these curves or jets are the same up to reparameterization.

Proposition 4.44. *Let* $\Gamma, \widetilde{\Gamma} : (\mathbb{R}, 0) \to (\mathbb{P}^k\mathbb{R}^2, p)$ *be immersed integral curve germs. Let* $i \geq 1$. *Then*

$$\Gamma^i(0) = \widetilde{\Gamma}^i(0) \iff j^i\Gamma = (\text{repar.}) = j^i\widetilde{\Gamma}. \tag{4.3}$$

This proposition is proved in section 4.9. It will be used in section 4.10 where we reduce Theorem 4.40 to the following statement on prolongation of plane curves.

Theorem B. *Let c^* be a plane curve germ whose order of good parameterization is m and whose regularization level is k, $k \geq 3$. Let c be another plane curve germ. Let $q \geq 1$. Then the k-step-prolongations $(c^*)^k$ and c^k have the same q-jet up to reparameterization if and only if the curves c^* and c have the same $(m+q-1)$-jets up to reparameterization.*

The proof of Theorem **B** requires, like that of Theorem **A** (section 3.8), a special local coordinate system (Chapter 7) and the directional blow up operation described at the beginning of Chapter 8. Therefore both proofs are postponed to Chapter 8.

4.9. Proof of Proposition 4.44

The proof proceeds by showing that, roughly speaking, a neighborhood of Γ^i look like a neighborhood of Γ times a Euclidean factor $\mathbb{R}^i$ whose coordinates are jet coordinates for the deviation of $\widetilde{\Gamma}'$ from Γ'.

Since a small piece of Γ is an embedded curve, we can find a a non-vanishing vector field X tangent to Γ and to Δ^k, defined in some neighborhood N of Γ. Along Γ we have $\Gamma'(t) = X(\Gamma(t))$, Choose any other analytic vector field Y on N so that $\{X, Y\}$ span Δ^k within N. (Shrink N if necessary.) For $p \in N$ we can write any line $\ell \subset \Delta(p)$ as the span of $aX(p) + bY(p)$. Then $[a, b]$ form projective coordinates for the fibers $\mathbb{RP}^1$ over N, and $u = b/a$ forms the corresponding affine coordinates, good as long as the line is not the span of Y alone. We have by this means trivialized a neighborhood N_1 of Γ^1 in $\mathbb{P}^{1+k}\mathbb{R}^2$, so that $N_1 = N \times \mathbb{R}$ with u coordinatizing the $\mathbb{R}$ factor and Γ^1 given by $(\Gamma(t), 0)$ in this trivialization. In this trivialization, the distribution Δ^{k+1} is spanned by $\partial/\partial u_1$ and $X_u = X + u_1 Y$.

We claim that a neighborhood N_i of Γ^i is diffeomorphic to $N \times \mathbb{R}^i$ where coordinates on the $\mathbb{R}^i$ are $u_1, \ldots, u_i$ and the distribution Δ^{k+i} is spanned by

$$\partial/\partial u_i \text{ and } X_{u(i)} = X + u_1 Y + u_2 \partial/\partial u_1 + u_3 \partial/\partial u_2 + \ldots + u_i \partial/\partial u_{i-1}.$$

This claim is established by induction. Suppose that the claim holds for i. To establish the claim for $i+1$, take the neighborhood N_{i+1} to consist of pairs (m, ℓ) with $m \in N_i$ such that ℓ not spanned by $\partial/\partial u_i$. Any such line can be uniquely written as the span of $X_{u(i)} + u_{i+1}\partial/\partial u_i$, thus providing the needed new coordinate u_{i+1} for N_{i+1}. We leave it to the reader to show that over N_{i+1} the distribution Δ^{k+i+1} is spanned by $\partial/\partial u_{i+1}$ and $X_{u(i+1)}$. In these trivializing coordinates, we have $u_1 = u_2 = \ldots = u_i = 0$ identically along Γ^i.

Consider now another immersed integral curve $\widetilde{\Gamma}$ with $\Gamma(0) = \widetilde{\Gamma}(0)$. We have $\widetilde{\Gamma}' = a(t)X + b(t)Y$ along $\widetilde{\Gamma}$. Clearly, $\widetilde{\Gamma}^1(0) = \Gamma^1(0)$ if and only if $b(0) = 0$ which in turn holds if and only if $j^1\widetilde{\Gamma}(0)$ is a reparameterization of $j^1\Gamma(0)$: i.e if and only if their first derivatives agree up to scale at $t = 0$. In the trivialization for N_1 we have $\widetilde{\Gamma}^1(t) = (\widetilde{\Gamma}(t), u_1(t))$, with $u(t) = b(t)/a(t)$.

Let us assume that $\widetilde{\Gamma}^1 = \Gamma^1(0)$ now. Then $a(0) \neq 0$ and we can reparameterize $\widetilde{\Gamma}$ so that $\widetilde{\Gamma}'(t) = X + u(t)Y$. It follows that in the N_1 trivialization we have that $u_1 = u(t)$ along $\widetilde{\Gamma}^1$. Now $\widetilde{\Gamma}^i$ is an integral curve for Δ^{k+i} projecting onto $\widetilde{\Gamma}$. We have, in our trivialzation $\widetilde{\Gamma}^i = (\widetilde{\Gamma}(t), u_1(t), \ldots, u_i(t))$, so that on the one hand $\widetilde{\Gamma}^{i\prime} = (\widetilde{\Gamma}' + \Sigma_j (du_j/dt)\partial/\partial u_j$ while on the other hand $\widetilde{\Gamma}^{i\prime} = \alpha(t)X_{u(i)} + \beta(t)\partial/\partial u_i$ since $\widetilde{\Gamma}^i$ is an integral curve. Comparing these two expressions for $\widetilde{\Gamma}'$ we see that

$$\alpha(t) = 1, \quad \beta(t) = u_{i+1}(t), \quad u_{j+1}(t) = u_j'(t), \quad j = 1, \ldots, i-1.$$

Since $u_1(t) = u(t)$, we see that $u_{j+1}(t) = d^j u/dt^j$, $j = 1, 2, \ldots$. But for Γ^i we have $u_1 = u_2 = \ldots u_i = 0$ identically. Consequently $\widetilde{\Gamma}^i(0) = \Gamma^i(0)$ if and only if the i jets of the two curves agree at $t = 0$, which establishes the claim.

4.10. From Theorem B to Theorem 4.40

Fix a point $p \in (\alpha \mathrm{R}^q)$ where (α) is a critical RVT code and $q \geq 1$. Fix a Legendrian curve $\gamma^* \in \mathrm{Leg}(p)$. Let d be the parameterization number of the class (α). To prove Theorem 4.40 we have to prove the following two statements:

(a) If $\gamma \in \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ and $j^{d+q-1}\gamma = j^{d+q-1}\gamma^*$ then $\gamma \in \mathrm{Leg}(p)$.

(b) If $\gamma \in \mathrm{Leg}(p)$ then $j^{d+q-1}\gamma = (\text{repar.}) = j^{d+q-1}\gamma^*$.

Using Theorem 4.14 and Proposition 4.22 it is easy to see that in proving (a) and (b) we may replace γ^* by any Legendrian curve germ RL-contact equivalent to γ^*. The curve γ^* is well-parameterized (see Proposition 4.5) and since (α) is a critical RVT class, the curve γ^* is not immersed (see Proposition 4.8). By Lemma 3.22 γ^* is RL-contact equivalent to $(c^*)^1$ where c^* is a plane curve germ of the form

$$c^* : \quad x^*(t) = t^{\lambda_0}, \quad y^*(t) = t^{\lambda_1} + \text{h.o.t.}, \quad \lambda_1 > 2\lambda_0. \tag{4.4}$$

In what follows we assume that $\gamma^* = (c^*)^1$.

NOTATION. By M we denote the order of good parameterization of c^*. (It is the last integer λ_m in the Puiseux characteristic of c^*).

The order of good parameterization of $\gamma^* = (c^*)^1$ is the parameterization number d of the class (α). By Proposition 4.37 one has

$$d = M - \lambda_0. \tag{4.5}$$

Let γ be a curve as in statements (a) and (b). Since $q \geq 1$, it is, like γ^*, not immersed but it is well-parameterized. Therefore the one-step-projection of γ is not a constant curve, hence $\gamma = c^1$, where c is some well-parameterized plane curve germ.

CLAIM 4.45. If s is any integer with $s - \lambda_0 > \lambda_0$ then

$$j^{s-\lambda_0}\gamma = (\text{repar.}) = j^{s-\lambda_0}\gamma^* \iff j^s c = (\text{repar.}) = j^s c^*.$$

PROOF. We prove the claim using coordinates as in section 1.2. In these coordinates

$$c(t) = (x(t), y(t)), \quad \gamma(t) = (x(t), y(t), u(t)), \quad u(t) = y'(t)/x'(t)$$

$$c^*(t) = (x^*(t), y^*(t)), \quad \gamma^*(t) = (x^*(t), y^*(t), u^*(t)), \quad u^*(t) = (y^*)'(t)/(x^*)'(t).$$

First, suppose that $j^s c$ and $j^s c^*$ are equal up to reparameterization. Since $s > 2\lambda_0$ we have that the lowest order coordinate of both c and c^* is the first coordinate, namely $x(t)$ and $x^*(t)$. Being equal up to reparameterization, we can reparameterize c so that $x(t) = t^{\lambda_0}$. Moreover, since $s > \lambda_0$ and the order s jets agree, the order of both y and y^* are greater than λ_0. Now

$$u(t) = \frac{y'(t)}{\lambda_0 t^{\lambda_0 - 1}}, \quad u^*(t) = \frac{(y^*)'(t)}{\lambda_0 t^{\lambda_0 - 1}}.$$

From these expressions we see that the s-jet of y determines the $s - \lambda_0$ jet of u and similarly for y^*. This proves that $j^{s-\lambda_0}\gamma = (\text{repar.}) = j^{s-\lambda_0}\gamma^*$. The reverse implication of the claim is proved likewise. □

Since $\lambda_1 > 2\lambda_0$ in (4.4), we have $M > 2\lambda_0$. Since $q \geq 1$, we have $M+q-1 > 2\lambda_0$. Therefore we can take $s = M + q - 1$ in Claim 4.45. With such s we get that

$$j^{d+q-1}\gamma = (\text{repar.}) = j^{d+q-1}\gamma^* \iff j^{M+q-1}c = (\text{repar.}) = j^{M+q-1}c^*. \tag{4.6}$$

Now we express the conditions $\gamma^* \in \mathrm{Leg}(p), \gamma \in \mathrm{Leg}(p)$ in terms of prolongations of c^* and c. Let k be the level of the class (α). Then $\gamma^* \in \mathrm{Leg}(p)$ if and only if $(c^*)^{k+q}(0) = p$ and $(c^*)^{k+q}$ is a regular curve. By Propositions 2.31 and 2.33 the curve c^{k+q} is regular if and only if the curve c^k is regular. We obtain:

CLAIM 4.46. $\gamma^* \in \mathrm{Leg}(p)$ if and only if $(c^*)^k$ is a regular curve and $(c^*)^{k+q}(0) = p$. Similarly, $\gamma \in \mathrm{Leg}(p)$ if and only if c^k is a regular curve and $c^{k+q}(0) = p$.

This claim and (4.6) allow to reformulate statements (a) and (b) as follows:

(a)$_1$ If $j^{M+q-1}c = j^{M+q-1}c^*$ then c^k is a regular curve and $c^{k+q}(0) = (c^*)^{k+q}(0)$.

(b)$_1$ If c^k is a regular curve and $c^{k+q}(0) = (c^*)^{k+q}(0)$ then $j^{M+q-1}c = (\text{repar.}) = j^{M+q-1}c^*$.

Now we will prove (a)$_1$ and (b)$_1$ using Theorem **B** and Proposition 4.44. We also will use the following: since (α) is a critical RVT class in level k and since

$(c^*)^k$ is a regular curve, by Proposition 3.18 *the level k of the class (α) is the regularization level of the plane curve germ c^*.*

Proof of statement $(a)_1$. By Theorem **B** $j^q c^k = \text{(repar.)} = j^q(c^*)^k$. The curve $(c^*)^k$ is regular, so its 1st jet is non-vanishing and tangent to a regular direction. Since $q \geq 1$, it follows that the same holds for c^k's 1st derivative, and consequently c^k is also a regular. Now $(a)_1$ follows from Proposition 4.44 with $\Gamma = c^k, \widetilde{\Gamma} = (c^*)^k$.

Proof of statement $(b)_1$. By Proposition 4.44 with $\Gamma = c^k, \widetilde{\Gamma} = (c^*)^k$ one has $j^q c^k = \text{(repar.)} = j^q c^{k*}$. Now $(b)_1$ follows from Theorem **B**.

4.11. Proof that critical points do not have a jet-identification number

We must show that if p is a critical point then it does not have a jet-identification number. We use Theorems 4.40 and 4.31 and argue by contradiction. Assume that r is the jet-identification number of p. Let d be the parameterization number of p. (Recall every point has a parameterization number.) We first show that r cannot be less than d, and then show it cannot be greater than or equal to d.

Fix a curve $\gamma \in \mathrm{Leg}(p)$. If $r < d$ then there is a Legendrian curve germ $\widetilde{\gamma}$ such that $j^r\widetilde{\gamma} = j^r\gamma$ and such that the order of good parameterization of $\widetilde{\gamma}$ is less than d. By Theorem 4.31 $\widetilde{\gamma} \notin \mathrm{Leg}(p)$. This shows that r cannot be smaller that d.

Assume now that $r \geq d$. Take any point $\widehat{p}$ which is a one-step-regular-prolongation of p. There are an $\mathbb{RP}^1$'s worth of such points. But we will show, by way of contradiction, that $\widehat{p} = \gamma^{k+1}(0)$ where k is the level of p. Let (α) be the RVT-code of p. Then $\widehat{p} \in (\alpha\mathrm{R})$. Since (α) is a critical RVT class, by Proposition 4.33 the parameterization number of $(\alpha\mathrm{R})$ coincides with that of (α) and equals d. By Theorem 4.40 the jet-identification number of the class (αR) and consequently of the point $\widehat{p}$ is $d = d+1-1$. Take a curve $\widehat{\gamma} \in \mathrm{Leg}(\widehat{p})$. By Theorem 4.10 $\widehat{\gamma} \in \mathrm{Leg}(p)$. By definition of the jet identification number $j^r\gamma = \text{(repar.)} = j^r\widehat{\gamma}$. Since $r \geq d$, one automatically has $j^d\gamma = \text{(repar.)} = j^d\widehat{\gamma}$. But d is the jet-identification number of $\widehat{p}$, so we find, by definition of the jet identification number (of $\widehat{p}$), that $\gamma \in \mathrm{Leg}(\widehat{p})$ and that $\widehat{p} = \gamma^{k+1}(0)$ where k is the level of p. But $\widehat{p}$ is an *arbitrary* one-step-regular prolongation of p and there are a continuum of such. We have our contradiction, since we started out with a single curve γ.

4.12. Proof of Proposition 4.26

A point at level 2 is a pair (p, ℓ), ℓ a line, in $\Delta^1(p)$. Such a line ℓ is realized as the tangent line to an immersed Legendrian curve γ. The line determines, and is determined by, the 1st jet of the curve, up to reparameterization. This establishes that all points at level 2 have jet identification number equal to 1.

Now we move to the case of level 1. We are to show that $p \in \mathbb{P}^1\mathbb{R}^2$ has no jet identification number. The key is to realize that $\mathrm{Leg}(p)$ consists of all *immersed* Legendrian curves through p. First we show that the jet identification number of p cannot be 0. Let $\gamma \in \mathrm{Leg}(p)$ so that γ is an immersed Legendrian curve. We can find a non-immersed Legendrian curve germ $\widetilde{\gamma}$ having the same 0-jet as γ, i.e. with $\widetilde{\gamma}(0) = p$. Being non-immersed, $\widetilde{\gamma} \notin \mathrm{Leg}(p)$ establishing that the jet-identification number of p cannot be 0. On the other hand the jet-identification number of p

cannot be $r \geq 1$: for this would imply that all curves $\gamma \in \mathrm{Leg}(p)$ have the same up-to-reparameterization r-jet whereas $\mathrm{Leg}(p)$ consists of *all* immersed Legendrian curve germs through p, so there r-jets are basically arbitrary.

4.13. Conclusions. Things to Come

Theorems 4.23 and 4.41 reduce the equivalence problem for regular points in the Monster to the RL-contact equivalence problem for finite jets of Legendrian curves, and the classification problem (constructing normal forms) for any RVT class to the RL-contact classification problem for certain classes of Legendrian curve jets. Theorem 3.9 from section 3.4 reduces the equivalence problem for critical points or RVT classes to the equivalence problem for regular points or RVT classes. Consequently the equivalence and classification problems for **arbitrary** points or RVT classes of the Monster have been reduced to these same problems for finite jets of Legendrian curves or classes of such jets. It follows (see section 1.6) that both problems for Goursat flags also reduce to these same problems for finite jets of Legendrian curves. The reductions just described are presented in the form of an explicit easily programmed algorithms in the next chapter, Chapter 5. There we apply the algorithms to prove a number of classification results. In the chapter following, Chapter 6, we use the algorithm to determine all simple points of the Monster.

All theorems and statements formulated thus far have either been proved or have been reduced to one of the Theorems **A** (section 3.8) or **B** (section 4.8). As mentioned above, the proofs of Theorems **A** and **B** must await certain classes of canonical local coordinates (the “KR coordinates”) on the Monster (Chapter 7) and the directional blow-up operation (Chapter 8), and so we withhold these proofs until the end of Chapter 8.

CHAPTER 5

Reduction algorithm. Examples of classification results

We present explicit algorithms for reducing the equivalence problem for points in the Monster to the equivalence problems for jets of Legendrian curves (section 5.2) and for reducing the classification problem for an RVT class to the classification problems for classes of jets of Legendrian curves (section 5.3). Each algorithm has a sub-algorithm for calculating the Legendrization and the parameterization number of an RVT class. This sub-algorithm is given in section 5.1. A number of classification results obtained using the algorithms are given in sections 5.4 - 5.7.

The algorithms in sections 5.1 - 5.3 are direct corollaries of our results in the previous chapters. In describing the algorithms we refer to these results. The main results used are:

(1) Theorem 1.3 stating that all points at the first level are equivalent and all points at the second level are equivalent.

(2) Theorem 3.7 stating that all points of the open class (R^k) in level $(k+2)$ are equivalent.

(3) Theorem 3.9 reducing the classification of points within a critical RVT class to the classification of points within a regular RVT class.

(4) Theorem 4.23 reducing the classification of points of a regular RVT class (α) to the classification of a certain class of Legendrian curve jets.

    Using theorem 4.23 for this reduction requires knowing:

    (a) the Legendrization of the class (α);

    (b) the jet-identification number of the class (α).

(5) Theorem 4.10 which reduces computing the Legendrization of a regular class $(\alpha) \neq (\mathrm{RR...R})$ to computing the Legendrization of the critical class $(\widehat{\alpha})$ where $(\alpha) = (\widehat{\alpha}\mathrm{R}^q)$.

(6) Theorem 4.12 which gives a formula for the Legendrization of any critical RVT class.

(7) Theorem 4.40 which computes the jet-identification number of any regular RVT class. Applying this theorem requires the ability to calculate the parameterization number of any critical RVT class.

(8) Theorem 4.34 which gives a formula for the parameterization number of any critical RVT class.

5.1. Algorithm for calculating the Legendrization and the parameterization number

STEP 1. The Legendrization of the open class R^q consists of immersed Legendrian curve germs (Proposition 4.8) and is generated under RL-contact equivalence by a single germ: the one-step-prolongation of the plane curve germ $(t, 0)$. The parameterization number of this class is equal to 1 (Proposition 4.33). Any class except R^q is a regular prolongation of a critical class (α) and by Theorem 4.10 its Legendrization coincides with that of (α). Consequently its parameterization number also coincides with that of (α). To calculate the Legendrization and the parameterization number of (α) express it in the form

$$(\alpha) = \mathrm{R}^{s_1}\omega^{(1)}\mathrm{R}^{s_2}\omega^{(2)}\mathrm{R}^{s_3}\omega^{(3)}\cdots\mathrm{R}^{s_m}\omega^{(m)}, \tag{5.1}$$

where $s_1 \geq 0,\ s_2, ..., s_m \geq 1$ and $(\omega^{(i)})$ are entirely critical RVT codes.

STEP 2. Calculate $(a_i, b_i) = \mathbb{E}_{\omega^{(i)}}(1, 2)$, as per section 3.8.4.1.

We recall the calculations of this section. We write out $\omega^{(i)} = (l_1, ..., l_r)$ with $l_1, ..., l_r$ being letters, V or T. The length r depend on i. Then

$$(a_i, b_i) = \mathbb{E}_{l_1} \circ\ \cdots\ \circ \mathbb{E}_{l_r}(1, 2),\ \text{where}$$
$$E_{\mathrm{T}}:\ (n_1, n_2)\ \to\ (n_1, n_1 + n_2),\quad E_{\mathrm{V}}:\ (n_1, n_2)\ \to\ (n_2, n_1 + n_2).$$

STEP 3. Calculate the Puiseux characteristic associated with the classes

$$(\alpha^{(j)}) = (\mathrm{R}^{s_1}\omega^{(1)}R^{s_2}\omega^{(2)}\cdots R^{s_j}\omega^{(j)}),\ j = 1, ..., m.$$

According to section 3.8.4

$$\mathrm{Pc}\,(\alpha^{(1)}) = [a_1;\ s_1 a_1 + a_1 + b_1]$$
$$\mathrm{Pc}\,(\alpha^{(j)}) = [\lambda_0; \lambda_1, ..., \lambda_j],\quad j \geq 2,$$

where the integers $\lambda_1, .., \lambda_j$ can be calculated using the recursion formulae

$$\lambda_i = a_j \cdot \widetilde{\lambda}_i,\quad i = 0, 1, ..., j-1;$$
$$\lambda_j = a_j \cdot (\widetilde{\lambda}_{j-1} + s_j - 1) + b_j - a_j,$$

in which

$$[\widetilde{\lambda}_0; \widetilde{\lambda}_1, ..., \widetilde{\lambda}_{j-1}] = \mathrm{Pc}\,(\alpha^{(j-1)}).$$

STEP 4. Since $(\alpha) = (\alpha^{(m})$ on the last step $(j = m)$ we obtain the Puiseux characteristic $\mathrm{Pc}\,(\alpha) = [\lambda_0; \lambda_1, ..., \lambda_m]$ associated with the class (α). Let

$$d_i = \text{g.c.d.}(\lambda_0, \ldots, \lambda_i), i = 1, ..., m-1.$$

By Theorem 4.12 the Legendrization of (α) consists of Legendrian curve germs RL-contact equivalent to the one-step prolongations of plane curve germs of the form

$$x = t^{\lambda_0},\quad y = t^{\lambda_1} f_1(t^{d_1}) + \cdots + t^{\lambda_{m-1}} f_{m-1}(t^{d_{m-1}}) + t^{\lambda_m} f_m(t),$$

where f_i are arbitrary functions of one variable such that $f_1(0), f_2(0), ..., f_m(0) \neq 0$.

By Theorem 4.34 the paramaterization number of (α) is equal to $\lambda_m - \lambda_0$.

The given algorithm was already used to calculate the Legendrization of the RVT class (R^3VVR^4VTR^5VVT), see Examples 3.25 and 4.13. The parameterization number of this class is equal to 263-36 = 227.

Here is another illustrative example.

EXAMPLE 5.1. Consider the class (RVTTRRV). Express this RVT code in the form (5.1) with

$$m=2,\quad s_1=1,\quad s_2=2,\quad \omega^{(1)}=(\mathrm{VTT}),\quad \omega^{(2)}=(\mathrm{V}).$$

Calculate

$$\mathbb{E}_{\omega^{(1)}}(1,2)=\mathbb{E}_{\mathrm{V}}\circ\mathbb{E}_{\mathrm{T}}\circ\mathbb{E}_{\mathrm{T}}(1,2)=\mathbb{E}_{\mathrm{V}}\circ\mathbb{E}_{\mathrm{T}}(1,3)=\mathbb{E}_{\mathrm{V}}(1,4)=(4,5);$$
$$\mathbb{E}_{\omega^{(2)}}(1,2)=\mathbb{E}_{\mathrm{V}}(1,2)=(2,3)$$

so that $(a_1,b_1)=(4,5),\quad (a_2,b_2)=(2,3)$. Calculate now

$$\mathrm{Pc}\,(\mathrm{RVTT})=[4;13];\ \text{ and then }\ \mathrm{Pc}\,(\mathrm{RVTTRRV})=[8;26,29].$$

We obtain that the Legendrization of the class (RVTTRRV) consists of Legendrian curve germs RL-contact equivalent to the one-step prolongations of plane curve germs of the form

$$x=t^8,\ y=t^{26}f_1(t^2)+t^{29}f_2(t),\quad f_1(0),f_2(0)\neq 0,$$

and the parameterization number of this class is equal to $29-8=21$.

5.2. Reduction algorithm for the equivalence problem

Let p and $\tilde{p}$ be points at the same level k of the Monster. We are to determine if p and $\tilde{p}$ are equivalent. We show here how to reduce this equivalence problem to the well-studied problem of determining whether or not two r-jets of Legendrian curves are RL-contact equivalent.

STEP 1. Is $k\le 2$ or is $k>2$? If $k\le 2$ then p and $\tilde{p}$ are equivalent. (See Theorem 1.3. Or Claim 2.14 and Theorem 2.15). If $k\ge 3$ and $p,\tilde{p}$ belong to different RVT classes then they are not equivalent. See Proposition 3.4.

In what follows we assume that p and $\tilde{p}$ belong to the same RVT class (α) in the kth level, $k\ge 3$.

STEP 2. Is the class critical or regular? If the class (α) is critical (i.e. it ends with a V or T) then, as in section 3.4 we have two mutually exclusive alternatives:

CASE 2.1. The code is entirely critical. In this case the points p and $\tilde{p}$ are equivalent by the first statement Theorem 3.9.

CASE 2.2. The code contains at least one letter R. In this case we can express the code (α) in the form

$$(\alpha)=(\widehat{\alpha}\omega),\quad (\widehat{\alpha})\text{ regular},\quad (\omega)\text{ entirely critical.}\tag{5.2}$$

The class $(\widehat{\alpha})$ belongs to the level k_1 where $k_1<k$. Consider the projections

$$\pi_{k,k_1}(p),\ \pi_{k,k_1}(\tilde{p})\in(\widehat{\alpha}).\tag{5.3}$$

The points $p,\tilde{p}$ are equivalent if and only if their projections (5.3) are equivalent, according to the second statement of Theorem 3.9. If $k_1\le 2$ then the projections are always equivalent (see Step 1).

Cases 2.1 and 2.2 above reduce us to the case where the points $p,\tilde{p}$ are in the same *regular* RVT class $(\widehat{\alpha})\subset\mathbb{P}^k\mathbb{R}^2$, $k\ge 3$. If the RVT code of $(\widehat{\alpha})$ consists of letters R only then by Theorem 3.6 p and $\tilde{p}$ are non-singular points and then they are equivalent (Theorem 2.15). Therefore in what follows we consider the case that $(\widehat{\alpha}) \subset \mathbb{P}^k\mathbb{R}^2, k \geq 3$ is a regular RVT class whose code contains at least one letter V or T. In this case $(\widehat{\alpha})$ has the form

$$(\widehat{\alpha}) = (\beta R^q), \quad \beta \text{ critical}, \quad q \geq 1. \tag{5.4}$$

Step 3. Find regular curve germs $\Gamma, \widetilde{\Gamma} : (\mathbb{R}, 0) \to \mathbb{P}^k\mathbb{R}^2$ such that $\Gamma(0) = p$ and $\widetilde{\Gamma}(0) = \tilde{p}$. Such germs exist by Proposition 4.5. Project Γ and $\widetilde{\Gamma}$ to $\mathbb{P}^1\mathbb{R}^2$ so as to obtain Legendrian curve germs

$$\gamma = \pi_{k,1}\Gamma, \quad \tilde{\gamma} = \pi_{k,1}\widetilde{\Gamma}.$$

Step 4. Calculate the parameterization number d of the class (β) according to the algorithm in section 5.1, via of Puiseux characteristic of (β).

Step 5. By Theorem 4.40 the jet-identification number of $(\widehat{\alpha})$ is $d + q - 1$. Therefore each of the points p and $\tilde{p}$ can be identified with a unique $(d + q - 1)$-jet of a Legendrian curve, say γ and $\tilde{\gamma}$. By Theorem 4.23 the points are equivalent if and only if these $(d + q - 1)$-jets are RL-contact equivalent.

5.3. Reduction algorithm for the classification problem

The classification problem for a singularity class S is the problem of constructing a subset $N \subset S$ such that any point in S is equivalent to one and only one point in N. In this case N is called an exact normal form for S.

Fix an RVT class (α) at the kth level, $k \geq 3$. The following algorithm reduces the problem of finding an exact normal form for (α) to the problem of finding an exact normal form for a certain class of Legendrian curve jets.

Step 1. Is the code (α) entirely critical (no Rs) or not? If it is entirely critical then all the class (α) are equivalent. (See Theorem 3.9). Any single point of class (α) provides an exact normal form for class (α).

Step 2. Having dispensed with the case of an entirely critical class, is the class (α) critical or regular? If it is critical then it has the form (5.2). The class $(\widehat{\alpha})$ is a regular class in level $k_1, k_1 < k$. Assume that we know an exact normal form $N_{\widehat{\alpha}}$ for the class $(\widehat{\alpha})$. The second statement of Theorem 3.9 implies that the set

$$N_\alpha = \{p \in (\alpha) : \quad \pi_{k,k_1}(p) \in N_{\widehat{\alpha}}\}$$

is an exact normal form for the class (α).

Steps 1 and 2 reduce the classification problem for the class (α) to the case of (α) regular. If (α) is the Cartan class (RR...R) then all its points are equivalent. If $(\alpha) \neq$ (RR...R) then (α) has form of equation (5.4): $(\alpha) = (\beta \mathrm{R}^q)$ with β critical, $q \geq 1$. In what follows we assume that (α) is of that form.

Step 3. Calculate a finite-dimensional set P of plane curve germs describing the Legendrization of the class (β) so that Leg(β) consists of Legendrian curve germs RL-contact equivalent to one-step-prolongations of curves $c \in P$. Calculate the parameterization number d of (β). An algorithm for these calculations is given in section 5.1.

Step 4. By Theorem 4.40 the class (α) has the jet-identification number $d+q-1$. Find an exact normal form $\mathcal{N}$ for the class $j^{d+q-1}\,\mathrm{Leg}(\beta)$ with respect to the RL-contact equivalence. Express this form as

$$\mathcal{N} = \{j^{d+q-1}c^1,\ c \in P_{\mathcal{N}},\quad P_{\mathcal{N}} \subset P\}.$$

Theorem 4.10, stating that $\mathrm{Leg}(\alpha) = \mathrm{Leg}(\beta)$, Theorem 4.23, and the basic properties of integral curves in sections 2.2 and 2.3 imply that the set

$$\mathrm{Monster}^k(P) = \{c^k(0),\ c \in P_{\mathcal{N}}\}$$

is an exact normal form for the class $(\alpha) = (\beta\mathrm{R}^q)$.

Remark. Finding the $\mathcal{N}$ of step 4 is a finite-dimensional classification problem. Moduli can occur here.

5.4. Classes of small codimension consisting of a finite number of orbits

The open class R^s consists of a single orbit. Any RVT class of codimension 1 also consists of a single orbit (Theorem 4.16). In this section we find all classes of codimension ≤ 3 consisting of a finite number of orbits. Also, we will determine which of the regular prolongations of the codimension 4 classes, VTTT and $\mathrm{VTTR}^m\mathrm{V}$ $(m \geq 1)$ consist of a finite number of orbits. These results will be used in Chapter 6 for the determination of simple points.

Any RVT class of codimension ≤ 3 is a regular prolongation of one of the critical RVT classes given in the first column of Table 5.1. The classes VTTT and $\mathrm{VTTR}^m\mathrm{V}$ are contained in Table 5.2.

Theorem 5.2. *Let (α) be one of the* RVT *classes in the first column of Table* 5.1 *or Table* 5.2. *Let q^* be the number in the last column, the row of (α). If $q^* = \infty$ then the class $(\alpha\mathrm{R}^q)$ consists of a finite number of orbits for any $q \geq 0$. If $q^* < \infty$ then the class $(\alpha\mathrm{R}^q)$ consists of a finite number of orbits if and only if $q \leq q^*$.*

Remark 5.3. For $\alpha = \mathrm{R}^s\mathrm{VR}^m\mathrm{VR}^n\mathrm{V}$ (the last row of Table 5.1) and $s \geq 1$, $m \geq 1, n \geq 3$ there are infinitely many orbits already in the class (α). To see this, look at the 5th row of the same Table. From Theorem 5.2 it follows that there are infinitely many orbits in the class $\mathrm{R}^s\mathrm{VR}^m\mathrm{VR}^n$, with s, m, n in the same range. Now apply Theorem 3.9, i.e the method of critical sections.

The proof of Theorem 5.2 is based on the algorithm for calculating the Legendrization and the parameterization number given in section 5.1. We also need the following corollary of Theorem 4.40 and Theorem 4.23.

Theorem 5.4 (Corollary of Theorems 4.23 and 4.40). *Let d be the parameterization number of a critical* RVT *class (α). Let $q \geq 1$. The class $(\alpha\mathrm{R}^q)$ consists of a finite number of orbits if and only if the class of Legendrian jets $j^{d+q-1}\,\mathrm{Leg}(\alpha)$ consists of a finite number of orbits with respect to the* RL-*contact equivalence.*

Proof. By Theorem 4.10 the classes (α) and $(\alpha\mathrm{R}^q)$ have the same Legendrization. By Theorem 4.40 the jet-identification number of the class $(\alpha\mathrm{R}^q)$ is equal to $d+q-1$. Now Theorem 5.4 follows from Theorem 4.23. $\square$

TABLE 5.1. RVT classes of codimension ≤ 3.

Class	codim	A set of plane curves describing $\mathrm{Leg}(\alpha)$. $f_i(0) \neq 0$.	param. numb. d	q^* (see Th. 5.2)
R	0	immersed curves	1	∞
$\mathrm{R}^s\mathrm{V}$, $s \geq 0$	1	$\big(t^2,\ \ t^{2s+5} f_1(t)\big)$	$2s+3$	∞
$\mathrm{R}^s\mathrm{VT}$, $s \geq 0$	2	$t^3,\ \ \big(t^{3s+7} f_1(t)\big)$	$3s+4$	∞
$\mathrm{R}^s\mathrm{VV}$, $s \geq 0$	2	$\big(t^3,\ \ t^{3s+8} f_1(t)\big)$	$3s+5$	∞
$\mathrm{R}^s\mathrm{VR}^m\mathrm{V}$ $s \geq 0,\ m \geq 1$	2	$\Big(t^4,\ \ t^{4s+10} f_1(t^2)+$ $t^{4s+9+2m} f_2(t)\Big)$	$4s+5+$ $+2m$	$s=0:\ \ \infty$ $s \geq 1:\ \ 2$
$\mathrm{R}^s\mathrm{VTT}$, $s \geq 0$	3	$\big(t^4,\ \ t^{9+4s} f_1(t)\big)$	$4s+5$	$s=0:\ \ \infty$ $s \geq 1:\ \ 2$
$\mathrm{R}^s\mathrm{VVT}$, $s \geq 0$	3	$\big(t^4,\ \ t^{11+4s} f_1(t)\big)$	$4s+7$	$s=0:\ \ \infty$ $s \geq 1:\ \ 3$
$\mathrm{R}^s\mathrm{VTV}$, $s \geq 0$	3	$\big(t^5,\ \ t^{12+5s} f_1(t)\big)$	$5s+7$	$s=0:\ \ 4$ $s \geq 1:\ \ 2$
$\mathrm{R}^s\mathrm{VVV}$, $s \geq 0$	3	$\big(t^5,\ \ t^{13+5s} f_1(t)\big)$	$5s+8$	$s=0:\ \ 4$ $s \geq 1:\ \ 3$
$\mathrm{R}^s\mathrm{VTR}^m\mathrm{V}$ $s \geq 0,\ m \geq 1$	3	$\Big(t^6,\ \ t^{6s+14} f_1(t^2)+$ $t^{6s+13+2m} f_2(t)\Big)$	$6s+7+$ $+2m$	$s=0:\ 2$ $s \geq 1, m=1:\ 1$ $s \geq 1, m \geq 2:\ 0$
$\mathrm{R}^s\mathrm{VR}^m\mathrm{VT}$ $s \geq 0, m \geq 1$	3	$\Big(t^6,\ \ t^{6s+15} f_1(t^3)+$ $t^{6s+13+3m} f_2(t)\Big)$	$6s+7+$ $+3m$	1
$\mathrm{R}^s\mathrm{VR}^m\mathrm{VV}$ $s \geq 0,\ m \geq 1$	3	$\Big(t^6,\ \ t^{6s+15} f_1(t^3)+$ $t^{6s+14+3m} f_2(t)\Big)$	$6s+8+$ $+3m$	2
$\mathrm{R}^s\mathrm{VVR}^m\mathrm{V}$ $s \geq 0,\ m \geq 1$	3	$\Big(t^6,\ \ t^{6s+16} f_1(t^2)+$ $t^{6s+15+2m} f_2(t)\Big)$	$6s+9+$ $+2m$	$s=0:\ 2$ $s \geq 1, m=1:\ 2$ $s \geq 1, m=2:\ 1$ $s \geq 1, m \geq 3:\ 0$
$\mathrm{R}^s\mathrm{VR}^m\mathrm{VR}^n\mathrm{V}$ $s \geq 0,$ $m, n \geq 1$	3	$\Big(t^8,\ \ t^{8s+20} f_1(t^4)+$ $t^{8s+18+4m} f_2(t^4)+$ $t^{8s+17+4m+2n} f_3(t)\Big)$	$8s+9+$ $+4m+$ $+2n$	0 unless $s \geq 1, n \geq 3$ see Remark 5.3

To prove Theorem 5.2 from Theorem 5.4 we apply the algorithm of section 5.1. We obtain that the Legendrization of a critical class (α) in Tables 5.1 or 5.2 is the

TABLE 5.2. Some RVT classes of codimension 4

Class	codim.	A set of plane curves describing $\mathrm{Leg}(\alpha)$. $f_i(0) \neq 0$.	param. number d	q^* (see Th. 5.2)
VTTT	4	$\big(t^5,\ \ t^{11} f_1(t)\big)$	6	3
$\mathrm{VTTR}^m V$ $m \geq 1$	4	$\Big(t^8,\ \ t^{18} f_1(t^2) + t^{17+2m} f_2(t)\Big)$	$9 + 2m$	$m = 1:\ 2$ $m = 2:\ 1$ $m \geq 3:\ 0$

class of Legendrian curves RL-contact equivalent to the one-step prolongations of plane curves in the third column, and the parameterization number of (α) is the number d in the fourth column, the row of (α). The number q^* in the last column comes from the following proposition proved in Appendix B, section B.5.

PROPOSITION 5.5. *Fix a row in Table* 5.1 *or Table* 5.2. *Let P be the set of plane curve germs given in the third column and let P^1 be the one-step prolongation of P. Let d and q^* be the integers in the last two columns. Let $q \geq 0$ and $r = d + q - 1$.*

1. If $q^* = \infty$ or if $q^* < \infty$ and $q \leq q^*$ then the set $j^r P^1$ is covered by a finite number of orbits with respect to RL*-contact equivalence.*

2. Let $q^* < \infty$ and $q > q^*$. Then the set $j^r P^1$ is not covered by a finite number of orbits with respect to RL*-contact equivalence. Moreover, for any $\xi \in j^r P^1$ and any neighborhood U of ξ in the space of r-jets of Legendrian curves the set $U \cap j^r P^1$ is not covered by a finite number of orbits with respect to* RL*-contact equivalence.*

PROOF OF THEOREM 5.2. For $q \geq 1$ Theorem 5.2 is a direct corollary of Proposition 5.5 and Theorem 5.4. In the case $q = 0$ we apply Theorem 3.9. This theorem reduces the case $q = 0$ to the following statement: the classes

$$\mathrm{R}^s,\ \mathrm{R}^s\mathrm{VR}^m,\ \mathrm{R}^s\mathrm{VTR}^m,\ \mathrm{R}^s\mathrm{VVR}^m,\ \mathrm{R}^s\mathrm{VR}^m\mathrm{VR}^n,\ \mathrm{VTTR}^m$$

consist of a finite number of orbits for any $s \geq 0, m \geq 1, n \in \{1, 2\}$. This statement is a part of Theorem 5.2 with $q \geq 1$.

5.5. Classification of tower-simple points

The classes in Table 5.1 with $q^* = \infty$ in the last column are the classes

$$\mathrm{R},\ \mathrm{R}^s\mathrm{V},\ \mathrm{R}^s\mathrm{VT},\ \mathrm{R}^s\mathrm{VV},\ \mathrm{VR}^m\mathrm{V},\ \mathrm{VTT},\ \mathrm{VVT},\quad s \geq 0,\ m \geq 1. \tag{5.5}$$

These classes can be characterized as being those classes whose regular prolongations consist of "tower-simple" points. A "tower neighborhood " of a point p at level i is a collection of open sets U_q at level $q + i$, $q \geq 0$ with $p \in U_0$ and U_q projecting onto U_r for $r < q$. We say that a point is "tower-simple" if it admits a tower neighborhood with each U_q consisting of a finite number of equivalence classes. See Chapter 6 for the more formal definition of "tower-simple". These neighborhoods

U_q will contain the regular prolongations of the classes (5.5), which is to say, the RVT classes

$$\mathrm{R}^q,\ \mathrm{R}^s\mathrm{VR}^q,\ \mathrm{R}^s\mathrm{VTR}^q,\ \mathrm{R}^s\mathrm{VVR}^q,\ \mathrm{VR}^m\mathrm{VR}^q,\ \mathrm{VTTR}^q,\ \mathrm{VVTR}^q, \qquad s\geq 0,\ m\geq 1, q\geq 0. \tag{5.6}$$

By Theorem 5.2 any of the classes (5.6) consist of a finite number of different orbits. In Chapter 6 we prove that these RVT classes are precisely the classes whose points are tower-simple. (For trivial reasons, every point at level 1 or 2 is tower-simple.)

The classification of points of classes (5.6) requires Proposition 5.6 immediately below on the RL-contact classification of certain Legendrian curves and their finite jets, proved in Appendix B, section B.3, B.4. Consider Table 5.3. Its first three columns form a part of Table 5.1. The first column consists of all RVT classes (5.5). The second column, $\widetilde{\mathrm{Leg}}(\alpha)$, is the Legendrization of the class (α) of the first column, up to RL-contact equivalence. The third column is the parameterization number of (α). The last column $\widehat{\mathrm{Leg}}(\alpha)$ is an exact normal form for the set of Legendrian curve germs in the second column, with respect to RL-contact equivalence. The notation $E_{s,j}$ and $E'_{s,j}$ found in rows 3 and 4 of that table stand for the plane curves:

- $E_{s,j} = (t^3, t^{3s+7} \pm t^{3s+8+3j}),\ \ \pm \hookrightarrow +$ if j is even,
- $E'_{s,j} = (t^3, t^{3s+8} \pm t^{3s+10+3j}),\ \ \pm \hookrightarrow +$ if j is odd.

Table 5.3. Classification of the Legendrizations of classes (5.5).

Class (α) $s, m \geq 1$	$\widetilde{\mathrm{Leg}}(\alpha)$ $f(0), g(0) \neq 0$	Param. number	Normal form $\widehat{\mathrm{Leg}}(\alpha)$
R	immersed Leg. curves	1	$(t,0)^1$
$\mathrm{R}^s\mathrm{V}$	$(t^2, t^{2s+5}f(t))^1$	$2s+3$	$(t^2, t^{2s+5})^1$
VT	$(t^3, t^7 f(t))^1$	4	$(t^3, t^7)^1$
VV	$(t^3, t^8 f(t))^1$	5	$(t^3, t^8)^1$
$\mathrm{R}^s\mathrm{VT}$	$(t^3, t^{3s+7}f(t))^1$	$3s+4$	$(E_{s,0})^1; \ldots; (E_{s,s-1})^1;$ $(t^3, t^{3s+7})^1$
$\mathrm{R}^s\mathrm{VV}$	$(t^3, t^{3s+8}f(t))^1$	$3s+5$	$(E'_{s,0})^1; \ldots; (E'_{s,s-1})^1;$ $(t^3, t^{3s+8})^1$
$\mathrm{VR}^m\mathrm{V}$	$(t^4, t^{10}f(t^2) + t^{9+2m}g(t))^1$	$2m+5$	$(t^4, t^{10} + t^{9+2m})^1$
VTT	$(t^4, t^9 f(t))^1$	5	$(t^4, t^9 \pm t^{11})^1;$ $(t^4, t^9)^1$
VVT	$(t^4, t^{11} f(t))^1$	7	$(t^4, t^{11} \pm t^{13})^1;$ $(t^4, t^{11} \pm t^{17})^1;$ $(t^4, t^{11})^1$

PROPOSITION 5.6. *In Table* 5.3 *consider the row for the class* (α).

(i) *Any Legendrian curve from the second column (denoted there as* $\widetilde{\mathrm{Leg}}(\alpha)$ *) is RL-contact equivalent to one of the finite number of curves coming from the fourth column (denoted there as* $\widehat{\mathrm{Leg}}(\alpha)$*).*

(ii) *The* i*-jets of two Legendrian curves in the fourth column are RL-contact equivalent if and only if these* i*-jets are equal.*

EXAMPLE 5.7. An example is in order for explaining part (ii) of the proposition, and how it applies to the classification problem. Consider the class VVT which makes up the subject of the last row of Table 5.3. The 4th column of the Table contains 5 curves:

A. $((t^4, t^{11} \pm t^{13})^1$

B. $((t^4, t^{11} \pm t^{17})^1$

C. $(t^4, t^{11})^1$

These Legendrian curves have the same 8-jet, therefore for $i \leq 8$ Proposition 5.6 (ii) is empty: these curves represent a single i- jet. For $i \in \{9, 10, 11, 12\}$. the set of i-jets of these five Legendrian curves consists of three elements because A and B have the same i-jet and A and C have different i-jets. Proposition 5.6, (ii) asserts that in this range these three i-jets are mutually inequivalent. For $i \geq 13$ the i-jets of these five Legendrian curves are all different, so they represent five distinct RL-equivalence classes of i-jets.

The implications for the classification of the regular prolongations of VVT is as follows. According to the third column of Table 5.3 the parameterization number of VVT is $d = 7$, so that, according to Theorem C of section 4.8, the jet-determination number of the class VVTR^q, $q > 0$, is $d+(q-1) = 6+q$. Thus VVTR and VVTRR consist of a single class. For $3 \leq q \leq 6$ the classes VVTR^q consists of precisely three equivalence classes. And for $q > 6$ the classes VVTR^q consists of 5 equivalence classes.

Following the lines of the previous example, by using Proposition 5.6 and our reduction algorithm in section 5.3 we obtain Table 5.4 which gives a complete classification of the points within the RVT classes (5.6). Recall that by Theorem 2.11 any point in the k-th level of the Monster can be represented as the evaluation of the k-fold prolongation of a non-constant plane analytic curve. That curve can be thought of as a "normal form" for the point. In Table 5.4 we summarize in these terms the normal forms for points of any regular prolongation of classes (5.5). For example, as per the example immediately above, the last entry of the last column specifies a single normal form representative if $q \leq 2$, three normal form representatives if $q \in \{3, 4, 5, 6\}$, and five normal form representatives if $q \geq 7$.

THEOREM 5.8. *Let* $(\widehat{\alpha})$ *be one of the RVT classes (5.6). Any point of* $(\widehat{\alpha})$ *is equivalent to one and only one of the points* $c^k(0)$*, where* c *is a plane curve germ from the last column of Table 5.4, and* k *is the level of* $(\widehat{\alpha})$ *as given in the third column.*

PROOF. If $(\widehat{\alpha}) = (\mathrm{R}^q)$ then Theorem 5.8 is a direct corollary of Theorem 3.6 and the well-known theorem on local contact equivalence of all immersed Legendrian curve germs (see, for example, [**AG**]).

TABLE 5.4. Classification of tower-simple points.

RVT class	codim	level k	Jet-identif. number r (if $q \neq 0$)	Exact normal form
R^q, $q \geq 1$	0	$q+2$	1	$(t, 0)$
$\mathrm{R}^s\mathrm{V}\mathrm{R}^q$ $s, q \geq 0$	1	$s+q+3$	$2s+2+q$	(t^2, t^{2s+5})
$\mathrm{R}^s\mathrm{VT}\mathrm{R}^q$ $s, q \geq 0$	2	$s+q+4$	$3s+3+q$	$s=0$ or $q \in \{0,1\}$: (t^3, t^7) $s \geq 1, q \geq 2$: $E_{s,j}$ and (t^3, t^{3s+7}), $j = 0, ..., \min(j^*, s-1)$ $j^* = [(q-2)/3]$
$\mathrm{R}^s\mathrm{VV}\mathrm{R}^q$ $s, q \geq 0$	2	$s+q+4$	$3s+4+q$	$s=0$ or $q \in \{0,1,2\}$: (t^3, t^8) $s \geq 1, q \geq 3$: $E'_{s,j}$ and (t^3, t^{3s+8}), $j = 0, ..., \min(j^*, s-1)$ $j^* = [(q-3)/3]$
$\mathrm{V}\mathrm{R}^m\mathrm{V}\mathrm{R}^q$ $m \geq 1, q \geq 0$	2	$m+q+4$	$6+3m+q$	$(t^4, t^{10} + t^{9+2m})$
$\mathrm{VTT}\mathrm{R}^q$ $q \geq 0$	3	$q+5$	$q+4$	$q \in \{0,1,2\}$: (t^4, t^9) $q \geq 3$: $(t^4, t^9 \pm t^{11})$; (t^4, t^9)
$\mathrm{VVT}\mathrm{R}^q$ $q \geq 0$	3	$q+5$	$q+6$	$q \in \{0,1,2\}$: (t^4, t^{11}) $q \in \{3,4,5,6\}$: $(t^4, t^{11} \pm t^{13})$; (t^4, t^{11}) $q \geq 7$: $(t^3, t^{11} \pm t^{13})$; $(t^4, t^{11} \pm t^{17})$; (t^4, t^{11})

Consider now the case that $(\widehat{\alpha})$ is a critical RVT class, i.e. the case $q = 0$ in Table 5.4. In this case the normal form in the last column of Table 5.4 consists of a single curve, i.e. we have to prove that all points of $(\widehat{\alpha})$ are equivalent. The equivalence of all points of each of the classes $\mathrm{R}^s\mathrm{V}, \mathrm{R}^s\mathrm{VT}, \mathrm{R}^s\mathrm{VV}, \mathrm{VTT}, \mathrm{VVT}$ follows from Theorem 3.9 and the equivalence of all points of the open class R^s. It remains to prove that all points of the class $\mathrm{V}\mathrm{R}^m\mathrm{V}$ are equivalent. This follows from Theorem 3.9 and the equivalence of all points of the class VR^m which was proved in section 4.4.1.

Now consider the case that $(\widehat{\alpha})$ is a regular class and $(\widehat{\alpha}) \neq (\mathrm{R}^q)$. Then $(\widehat{\alpha}) = (\alpha R^q)$, where (α) is a critical RVT-code of Table 5.3 and $q \geq 1$. We know the parameterization number d of (α) from the third column of Table 5.3. By Theorem 4.40 the jet-identification number r of $(\widehat{\alpha})$ is equal to $r = d + q - 1$. This jet-identification number is given in the fourth column of Table 5.4. The Legendraization $\mathrm{Leg}(\widehat{\alpha})$ coincides with $\mathrm{Leg}(\alpha)$ (see Theorem 4.10). Now we use Theorem 4.23. It reduces Theorem 5.8 to the following statement:

the r-jet of any Legendrian curve in the second column of Table 5.3, the row of (α), is RL-contact equivalent to the r-jet of the Legendrian curve c^1, where c is one and only one of the plane curve germs given in the last column of Table 5.4, the row of the class $(\widehat{\alpha}) = (\alpha \mathrm{R}^q)$.

This statement is a direct corollary of Proposition 5.6 and the following claim:

Claim. The one-step-prolongations of plane curve germs in the last column of Table 5.4, the row of $(\widehat{\alpha}) = (\alpha R^q)$ have different r-jets and the set of these r-jets coincides with the set of r-jets of the one-step-prolongations of plane curve germs in the last column of Table 5.3, the row of $(\widehat{\alpha}) = (\alpha R^q)$.

This claim can be easily checked straightforwardly. Consider for example the RVT class $(\widehat{\alpha}) = (\mathrm{VTTR}^q), q \geq 1$, whose jet-identification number is equal to $r = q + 4$. The normal form in Table 5.3 , the row of (VTT), consists of three curves: $(t^4, t^9 \pm t^{11})^1$ and $(t^4, t^9)^1$. If $q \in \{0, 1, 2\}$ then $r \in \{4, 5, 6\}$ and the r-jets of these Legendrian curves are all the same; they coincide with the r-jet of the curve $(t^4, t^9)^1$. If $q \geq 3$ then $r \geq 7$ and the r-jets of the given three curves are different. This corresponds to the normal form for the class (VTTR^q) given in the last column of Table 5.4 : it consists of a single plane curve (t^4, t^9) for $q \leq 2$ and of three plane curves $(t^4, t^9 \pm t^{11})$ and (t^4, t^9) for $q \geq 3$. □

5.6. Classes of high codimension consisting of one or two orbits

By Theorem 3.9 any RVT class of the form $(\mathrm{R}^s\omega)$ where (ω) is an entirely critical RVT code consists of one orbit. The codimension of $(\mathrm{R}^s\omega)$ is the length of (ω) which can be arbitrarily large. In this section we present a few more types of classes of arbitrary high codimension which consist of either one or two orbits.

Proposition 5.9. *Let (ω) be any critical* RVT *code and let $s \geq 0$. The class* $(\mathrm{R}^s\omega\mathrm{R})$ *consists of a single orbit. The class* $(\mathrm{R}^s\omega\mathrm{RR})$ *consists of one or two orbits.*

Proof. The Puiseux characteristic $\mathrm{Pc}\,(\mathrm{R}^s\omega)$ has the form $[\lambda_0; \lambda_1]$, where λ_0 and λ_1 are relatively prime and $\lambda_1 > 2\lambda_0$, see section 3.8.4.2. By Theorems 4.10 and 4.12 the Legendrization of the classes $\mathrm{R}^s\omega\mathrm{R}$ and $\mathrm{R}^s\omega\mathrm{RR}$ coincides with that of $(\mathrm{R}^s\omega)$ and consists of Legendrian curve germs RL-contact equivalent to the one-step prolongation of plane curve germs of the form

$$(t^{\lambda_0},\ t^{\lambda_1} f(t)), \quad f(0) \neq 0. \tag{5.7}$$

By Theorem 4.34 the parameterization number of the class $\mathrm{R}^s\omega$ is equal to $\lambda_1 - \lambda_0$. By Theorem 4.40 the jet-identification number of the classes $\mathrm{R}^s\omega\mathrm{R}$ and $\mathrm{R}^s\omega\mathrm{RR}$ are equal to $\lambda_1 - \lambda_0$ and to $\lambda_1 - \lambda_0 + 1$ respectively. The $(\lambda_1 - \lambda_0)$-jets and the

$(\lambda_1 - \lambda_0 + 1)$-jets of the one-step prolongations of curves of the form (5.7) are represented by the space curve of the form

$$x(t) = t^{\lambda_0},\ y(t) = \kappa t^{\lambda_1},\ u(t) = y'(t)/x'(t),\quad \kappa \in \mathbb{R},\ \kappa \neq 0, \tag{5.8}$$

$$\begin{gathered} x(t) = t^{\lambda_0},\quad y(t) = \kappa_1 t^{\lambda_1} + \kappa_2 t^{\lambda_1+1},\quad u(t) = y'(t)/x'(t), \\ \kappa_1, \kappa_2 \in \mathbb{R},\ \kappa_1 \neq 0 \end{gathered} \tag{5.9}$$

respectively, where the contact structure on $\mathbb{P}^1\mathbb{R}^2$ is described by the 1-form $dy - udx$. Now Proposition 5.9 follows from Theorem 4.23 and the use of a scaling transformation to transform the parameter κ in (5.8) to 1 and the pair of parameters (κ_1, κ_2) in (5.9) to $(1, 1)$ or to $(1, 0)$. The scaling transform we use is a contactomorphism Ψ, on (x, y, u) combined with a time scaling:

$$\Psi : x \to k_1 x,\quad y \to k_2 y,\quad u \to (k_2/k_1)u, \qquad \phi : t \to k_3 t \tag{5.10}$$

with suitable constants k_1, k_2, k_3. Recall that the group of pairs (Ψ, ϕ) of contactomorphisms Ψ (fixing $p = (0, 0, 0)$) and diffeomorphisms $\phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ acts on the space of r-jets of Legendrian curves through p at $t = 0$ by $j^r\gamma \mapsto j^r(\Psi \circ \gamma \circ \phi^{-1})$ and that this action generates the RL-contact equivalence relation. To transform κ to 1 take $k_1 = k_3 = 1,\ k_2 = \kappa$ in (5.10). If in (5.9) one has $\kappa_2 = 0$ then the pair $(\kappa_1, \kappa_2) = (\kappa_1, 0)$ reduces to $(1, 0)$ by the same scale contactomorphism, with κ_1 in place of κ. If $\kappa_2 \neq 0$ then the pair (κ_1, κ_2) reduces to $(1, 1)$ by taking

$$k_1 = (\kappa_1/\kappa_2)^{\lambda_0},\quad k_2 = \kappa_1 \cdot (\kappa_1/\kappa_2)^{\lambda_1},\quad k_3 = \kappa_1/\kappa_2.$$

□

We present one more family of RVT classes of arbitrarily high codimension which consist of not more than two orbits.

Proposition 5.10. *Let* (ω) *and* $(\widetilde{\omega})$ *be entirely critical classes and* $s \geq 0$, $q \geq 1$ *integers. If the class* $(\mathrm{R}^s\omega\mathrm{R}^q)$ *consists of a single orbit then the class* $(\widehat{\alpha}) = \mathrm{R}^s\omega\mathrm{R}^q\widetilde{\omega}\mathrm{R}$ *consists of either one or two orbits.*

Proof. According to the algorithm in section 5.1 the Legendrizations of the class $\mathrm{R}^s\omega$ consists of Legendrian curve germs RL-contact equivalent to the one-step prolongations of plane curve germs of the form (5.7) with certain relatively prime integers λ_0, λ_1 such that $\lambda_1 > 2\lambda_0$, and the Legendrization of the class $\mathrm{R}^s\omega\mathrm{R}^q\widetilde{\omega}$ is described in the same way by plane curve germs of the form

$$\begin{gathered} x = t^{a\lambda_0},\quad y = t^{a\lambda_1} f_1(t^a) + t^{\lambda_2} f_2(t),\quad f_1(0) \neq 0, f_2(0) \neq 0, \\ \lambda_2 = a(\lambda_1 + q - 1) + b - a,\quad (a, b) = E_{\widetilde{\omega}}(1, 2). \end{gathered} \tag{5.11}$$

According to the same algorithm the parameterization number of the classes $(\mathrm{R}^s\omega)$ and $(\mathrm{R}^s\omega\mathrm{R}^q\widetilde{\omega})$ are equal to $\lambda_1 - \lambda_0$ and $\lambda_2 - a\lambda_0$ respectively. By Theorem 4.40 the jet-identification number of the class $(\mathrm{R}^s\omega\mathrm{R}^q)$ is equal to $\lambda_1 - \lambda_0 + q - 1$. Since all points of the class $(\mathrm{R}^s\omega\mathrm{R}^q)$ are equivalent, by Theorem 4.23 the set of $(\lambda_1 - \lambda_0 + q - 1)$-jets of one step prolongations of plane curves (5.7) is covered by a single orbit with respect to RL-contact equivalence. Therefore the one-step prolongation of any of the curves (5.7) is RL-contact equivalent to the one-step prolongation of a plane curve germ of the form $\big(t^{\lambda_0}, t^{\lambda_1} + o(t^{\lambda_1+q-1})\big)$. It follows that the one-step prolongation of any of the curves (5.11) is RL-contact equivalent to the one-step prolongation of a plane curve germ of the form

$$x = t^{a\lambda_0},\quad y = t^{a\lambda_1} + t^{\lambda_2} f(t) + o(t^{a(\lambda_1+q)-1}). \tag{5.12}$$

Recall that the parameterization number of the class $(\mathrm{R}^s\omega\mathrm{R}^q\widehat{\omega})$ is $\lambda_2 - a\lambda_0$. By Theorem 4.40 the jet-identification number of the class $(\mathrm{R}^s\omega\mathrm{R}^q\widehat{\omega}\mathrm{R})$ also equals $\lambda_2 - a\lambda_0$. The Legendrization of the class $(\mathrm{R}^s\omega\mathrm{R}^q\widehat{\omega}\mathrm{R})$ coincides with that of the class $(\mathrm{R}^s\omega\mathrm{R}^q\widehat{\omega})$ (see Theorem 4.10), i.e. can be described by plane curve germs (5.12). Now Theorem 4.23 reduces Proposition 5.10 to the following statement:

CLAIM. The set of $(\lambda_2 - a\lambda_0)$-jets of the one-step prolongations of curves (5.12) intersects at most two orbits with respect to RL-contact equivalence.

To prove this statement observe that since $b < 2a$ (see Lemma 3.24) then $a(\lambda_1 + q) - 1 \geq \lambda_2$ and the part $o(t^{a(\lambda_1+q)-1})$ does not play role in calculation the set of $(\lambda_2 - a\lambda_0)$-jets of the one-step prolongations of curves (5.12). It follows that this set of jets is represented by by the Legendrian curves of the form

$$x(t) = t^{a\lambda_0}, \quad y(t) = t^{a\lambda_1} + \kappa t^{\lambda_2}, \quad u(t) = y'(t)/x'(t), \quad \kappa \in \mathbb{R},\ \kappa \neq 0,$$

where the contact structure on $\mathbb{P}^1\mathbb{R}^2$ is described by the 1-form $dy - udx$. It remains to note that κ can be reduced to 1 or to -1 by a scale contactomorphism (5.10) with suitable k_1, k_2, k_3. □

Finally Proposition 5.10 and the first statement of Proposition 5.9 imply:

PROPOSITION 5.11. *Any class of the form* $\mathrm{R}^{\geq 0}\omega\mathrm{R}\widehat{\omega}\mathrm{R}$ *where* ω *and* $\widehat{\omega}$ *are entirely critical* RVT *codes consists of one or two orbits.*

5.7. Further examples of classification results; Moduli

By the same method we can classify the points of **any** RVT class. Tables 5.5 and 5.6 summarize several several examples in which moduli appear.

For completeness' sake, let's give a definition of "moduli". We suppose we have a space X, endowed with an equivalence relation.

DEFINITION 5.12. Let $\mathcal{F}_{\mathbf{b}} \subset X; \mathbf{b} \in U \subset \mathbb{R}^N$ be a family of subsets of X parameterized by the parameter $\mathbf{b}$. Here U is an open subset of $\mathbb{R}^N$, $N \geq 1$. We say that $\mathbf{b}$ is a modulus for the family if, every $\mathbf{b}^* \in U$ is contained in a neighborhood $V \subset U$ such that for $\mathbf{b} \in V$, $\mathbf{b} \neq \mathbf{b}^*$ we have that no point of $\mathcal{F}_{\mathbf{b}^*}$ is equivalent to any point of $\mathcal{F}_{\mathbf{b}}$.

Note that if $\mathcal{F}_{\mathbf{b}}$ is an exact normal form for X (i.e. $\mathcal{F}^*_{\mathbf{b}}$ is a single point of X for any $\mathbf{b}^* \in U$ and any $x \in X$ is equivalent to $F^*_{\mathbf{b}}$ with one and only one $\mathbf{b}^* \in U$) then $\mathbf{b}$ is a modulus, but the converse is not true.

In our applications, X will be a space of germs of curves or functions or an RVT class in the Monster. Table 5.6 contains exact normal forms for certain RVT classes.

The RVT classes in Tables 5.5, 5.6 have the form $(\alpha\mathrm{R}^q)$ where α is a critical class. Applying our algorithm in section 5.1 we obtain that up to the RL-contact equivalence the Legendrization of (α) is the set $\widehat{\mathrm{Leg}}(\alpha)$ given in the third column of Table 5.5, and the parameterization number d of (α) is the number given in the fourth column of this table. By Theorem 4.40 the jet-identification number r of the class $(\alpha\mathrm{R}^q)$ is equal to $r = d + q - 1$. It is given in the last column of Table 5.5. It remains to find an exact normal form for the set $j^r\widehat{\mathrm{Leg}}(\alpha)$ and to express it in the form $\{c^1, c \in \widehat{P}\}$, where $\widehat{P}$ is a certain set of plane curve germs. The set $\widehat{P}$ is given

TABLE 5.5. Legendrization of some RVT classes of codimension 3

RVT class(α)	level k	$\widehat{\mathrm{Leg}}(\alpha)$ $f_i(0) \neq 0$	Param. number d	Jet-identif. number r (if $q \neq 0$)
VTVR^q	$q+5$	$\big(t^5, t^{12} f_1(t)\big)$	7	$6+q$
RVRVTR^q	$q+7$	$\big(t^6, t^{21} f_1(t^3) + t^{22} f_2(t)\big)$	16	$15+q$
RVRVVR^q	$q+7$	$\big(t^6, t^{21} f_1(t^3) + t^{23} f_2(t)\big)$	17	$16+q$
VVRVR^q	$q+6$	$\big(t^6, t^{16} f_1(t^2) + t^{17} f_2(t)\big)$	11	$10+q$

TABLE 5.6. Exact normal forms for the classes of Table 5.5.

RVT class	level k	Exact normal form $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{r} \in \mathbb{R}, \quad \mathbf{r} \geq 0$
VTVR^q	$q+5$	$0 \leq q \leq 4: \quad \big(t^5,\ t^{12} + t^{13}\big)$ $q = 5, 6: \quad \big(t^5,\ t^{12} + t^{13} + \mathbf{a}t^{16}\big)$, $q \geq 7: \quad \big(t^5,\ t^{12} + t^{13} + \mathbf{a}t^{16} + \mathbf{b}t^{18}\big)$
RVRVTR^q	$q+7$	$0 \leq q \leq 1: \quad \big(t^6,\ t^{21} + t^{22}\big)$ $q = 2, 3: \quad \big(t^6,\ t^{21} + t^{22} + \mathbf{a}t^{23}\big)$ $q = 4: \quad \big(t^6,\ t^{21} + t^{22} + \mathbf{a}t^{23} + \mathbf{b}t^{25}\big)$ $q = 5, 6: \quad \big(t^6,\ t^{21} + t^{22} + \mathbf{a}t^{23} + \mathbf{b}t^{25} + \mathbf{c}t^{26}\big)$
RVRVVR^q	$q+7$	$0 \leq q \leq 2: \quad \big(t^6,\ t^{21} \pm t^{23}\big)$ $q = 3: \quad \big(t^6,\ t^{21} \pm t^{23} + \mathbf{a}t^{25}\big)$ $q = 4, 5: \quad \big(t^6,\ t^{21} \pm t^{23} + \mathbf{a}t^{25} + \mathbf{r}t^{26}\big)$
VVRVR^q	$q+6$	$0 \leq q \leq 2: \quad \big(t^6,\ t^{16} + t^{17}\big)$ $q = 3, 4: \quad \big(t^6,\ t^{16} + t^{17} + \mathbf{a}t^{19}\big)$

(for certain r) in the last column of Table 5.6. It is obtained using theorems on the classification of Legendrian curves given in Appendix B, sections B.1 and B.2. Now Theorem 4.23 implies:

Fix an RVT class (α) in Table 5.6 in the kth level of the Monster (the number k is given in the second column). Any point of (α) is equivalent to one and only one of the points $c^k(0)$, where c is a plane curve germ in the last column of Table 5.6, with the row of that entry stating the class (α). The parameters $\mathbf{a}, \mathbf{b}, \mathbf{c}$ take any real values, and the parameter $\mathbf{r}$ takes any non-negative real values.

CHAPTER 6

Determination of simple points

6.1. Tower-simple and stage-simple points

In singularity theory, a point is called *simple* if it is contained in a neighborhood covered by a finite number of equivalence classes. For a point of the Monster, the notion of "neighborhood" can be taken either as an open set within the level of the Monster containing that point, or within the entire Monster tower over that point. These two choices lead to two definitions of "simple".

DEFINITION 6.1. A stage-neighborhood of a point $p \in \mathbb{P}^i\mathbb{R}^2$ is an open set in $\mathbb{P}^i\mathbb{R}^2$ containing p. A tower-neighborhood of the point p is an infinite sequence $U_0, U_1, ...,$ where U_j is an open set in $\mathbb{P}^j\mathbb{R}^2$, $\pi_{j+1,j}U_{j+1} = U_j$ and $p \in U_i$.

DEFINITION 6.2. A point $p \in \mathbb{P}^i\mathbb{R}^2$ is called stage-simple if there is a stage neighborhood U of p in $\mathbb{P}^i\mathbb{R}^2$ covered by a finite number of orbits. The point p is called tower-simple if there is a tower neighborhood $(U_0, U_1, ...)$ of p such that each of the open sets $U_j, j = 0, 1, ...$ is covered by a finite number of orbits.

Any tower-simple point is simple, but the converse is not true.

Simplicity is a property of RVT classes:

THEOREM 6.3. *A point is tower-simple simple (resp. stage-simple) if and only if every point of its* RVT *class is tower-simple (resp. stage-simple).*

This theorem is a direct corollary of Theorems 6.4 and 6.6 on the determination of tower-simple and stage-simple points. These theorems and their corollaries are formulated in section 6.2 and 6.3 and proved in sections 6.4 - 6.7. The proofs are based on the classification results of Chapter 5.

6.2. Determination theorems

The Darboux and Engel Theorems (Theorem 1.3) assert that all points at level 1 are equivalent, and that all points at level 2 are equivalent. Therefore *all points at level 1 and 2 are simple.* These points are also tower simple as is seen by noting that the regular prolongation of any such point lies in the dense open RVT class RR...R which forms a single open orbit. (See Theorem 3.6 and the Cartan Theorem, Theorem 2.15). Therefore points at level 1 and 2 are tower-simple. The simple and tower-simple points at the higher levels can be characterized as follows.

THEOREM 6.4. *A point of* $\mathbb{P}^i\mathbb{R}^2, i > 2$ *is tower simple if and only if its* RVT *class is either one of the classes*

$$\text{R},\ \text{R}^s\text{V},\ \text{R}^s\text{VT},\ \text{R}^s\text{VV},\ \text{VR}^m\text{V},\ \text{VTT},\ \text{VVT},\quad s \geq 0,\ m \geq 1. \tag{6.1}$$

or a regular prolongation of one of these classes.

REMARK 6.5. The explanation of the list (6.1) is as follows. A critical RVT class (α) belongs to this list if and only if it has codimension ≤ 3 and one has $q^* = \infty$ in the last column of Table 5.1, the row of (α). See Theorem 5.2.

A complete classification of tower-simple points was already given in sect. 5.5.

THEOREM 6.6. *A point of $\mathbb{P}^i\mathbb{R}^2, i > 2$ is stage-simple if and only if its* RVT *class does* **not** *adjoin any of the classes listed in Tables 6.1 and 6.2. (We call the classes of these Tables "fencing classes".)*

REMARK 6.7. An RVT class adjoins itself. Accordingly, the classes in Tables 6.1, 6.2 are not stage simple.

TABLE 6.1. Fencing RVT classes starting with R

Class	codim	level
$\mathrm{R}^{\geq 1}\mathrm{VR}^{\geq 1}\mathrm{VR}^{\geq 3}$	2	≥ 9
$\mathrm{R}^{\geq 1}\mathrm{VTTR}^{\geq 3}$	3	≥ 9
$\mathrm{R}^{\geq 1}\mathrm{VVTR}^{\geq 4}$	3	≥ 10
$\mathrm{R}^{\geq 1}\mathrm{VTR}^{\geq 2}\mathrm{VR}^{\geq 1}$	3	≥ 9
$\mathrm{R}^{\geq 1}\mathrm{VTRVR}^{\geq 2}$	3	≥ 9
$\mathrm{R}^{\geq 1}\mathrm{VVR}^{\geq 3}\mathrm{VR}^{\geq 1}$	3	≥ 10
$\mathrm{R}^{\geq 1}\mathrm{VVR}^{2}\mathrm{VR}^{\geq 2}$	3	≥ 10

TABLE 6.2. Fencing RVT classes starting with V

Class	codim	level
$\mathrm{VR}^{\geq 1}\mathrm{VR}^{\geq 1}\mathrm{VR}^{\geq 1}$	3	≥ 8
$\mathrm{VR}^{\geq 1}\mathrm{VTR}^{\geq 2}$	3	≥ 8
$\mathrm{VR}^{\geq 1}\mathrm{VVR}^{\geq 3}$	3	≥ 9
$\mathrm{VTR}^{\geq 1}\mathrm{VR}^{\geq 3}$	3	≥ 9
$\mathrm{VTVR}^{\geq 5}$	3	≥ 10
$\mathrm{VVVR}^{\geq 5}$	3	≥ 10
$\mathrm{VTTTR}^{\geq 4}$	4	≥ 10
$\mathrm{VTTR}^{\geq 3}\mathrm{VR}^{\geq 1}$	4	≥ 10
$\mathrm{VTTR}^{2}\mathrm{VR}^{\geq 2}$	4	≥ 10

Theorems 6.4 and 6.6 imply Theorem 6.3. In view of Theorem 6.3 the following definition makes sense.

DEFINITION 6.8. We will say that an RVT class is stage-simple, respectively tower-simple if some (and then any) point of this class is stage-simple, respectively tower-simple.

Theorem 6.6 and Theorems 6.29 and 6.30 below (which imply Theorem 6.6) can be expressed as follows: the collection of RVT classes in Tables 6.1, 6.2 is a "fence" for stage-simple points.

DEFINITION 6.9. We will say that the RVT classes in Tables 6.1 and 6.2 are fencing classes.

REMARK 6.10. A partial explanation of the list of fencing classes is as follows.
1. All fencing classes except classes of the form

$$\mathrm{VTTTR}^{\geq 4}, \mathrm{VTTR}^{\geq 3}\mathrm{VR}^{\geq 1}, \mathrm{VTTR}^2\mathrm{VR}^{\geq 2} \tag{6.2}$$

(the last 3 rows of Table 6.2) have codimension ≤ 3.

2. Any fencing class (α) of codimension ≤ 3 has the form
(*) $(\alpha) = (\beta\mathrm{R}^q)$, where (β) is a critical RVT code from Table 5.1 and q is bigger than the number q^* in the last column Table 5.1, the row of (β). By Theorem 5.2 any such class is *not* covered by a finite number of orbits.
3. Consider all RVT classes of codimension ≤ 3 having the form (*). Such a class is either fencing or has one of the forms

$$\mathrm{R}^{\geq 1}\mathrm{VTVR}^{\geq 3}, \ \ \mathrm{R}^{\geq 1}\mathrm{VVVR}^{\geq 4}, \mathrm{R}^{\geq 1}\mathrm{VR}^{\geq 1}\mathrm{VTR}^{\geq 2},$$

$$\mathrm{R}^{\geq 1}\mathrm{VR}^{\geq 1}\mathrm{VVR}^{\geq 3}, \ \mathrm{R}^{\geq 1}\mathrm{VR}^{\geq 1}\mathrm{VR}^{\geq 1}\mathrm{VR}^{\geq 1}, \ \mathrm{VVR}^{\geq 1}\mathrm{VVR}^{\geq 3}.$$

The reason that the classes just listed are not fencing classes is that each one adjoins some fencing class of the form $\mathrm{R}^{\geq 1}\mathrm{VR}^{\geq 1}\mathrm{VR}^{\geq 3}$ (see the first row of Table 6.1).
4. The fencing RVT classes of the form (6.2) have codimension 4. They have the form $(\beta\mathrm{R}^q)$, where (β) is a critical RVT code in Table 5.2 and q is bigger than the number q^* in the last column of Table 5.2, the row of (β). By Theorem 5.2 any such class is *not* covered by a finite number of orbits.

Remarks 6.5 and 6.10 give a partial explanation of Theorems 6.4 and 6.6. Theorem 6.4 is proved in section 6.5 and Theorem 6.6 is proved in sections 6.6 and 6.7. The proofs are based on our classification results from Chapter 5 and the notion of local simplicity of a class from section 6.4.

EXAMPLE 6.11.
(a). The RVT classes of the form $\mathrm{R}^{\geq 1}\mathrm{VT}^3\mathrm{R}$ or $\mathrm{R}^{\geq 1}\mathrm{VT}^4$ are stage-simple. These classes start with R and therefore cannot adjoin any class of Table 6.2. It is easy to check that these classes do not adjoin any of the classes of Table 6.1.
(b). The RVT classes of the form $\mathrm{R}^{\geq 1}\mathrm{VT}^3\mathrm{RR}$ or $\mathrm{R}^{\geq 1}\mathrm{VT}^5$ are not stage-simple since they adjoin the class of the form $\mathrm{R}^{\geq 1}\mathrm{VTTR}^3$ in the second row of Table 6.1.
(c). The RVT classes VT^i for $0 \leq i < 7$ are stage-simple, since it is easily checked that none of these classes adjoin any of the classes of Table 6.1 or of Table 6.2). The class VT^7 is not stage-simple because it adjoins the fencing class $\mathrm{VT}^3\mathrm{R}^4$ in Table 6.2.

The set of simple points of the Monster is much larger than the set of tower-simple points. By Theorem 6.4 there are no tower-simple RVT classes of codimension greater than 3. On the other hand, there are stage-simple RVT classes of codimension ≤ 7, see Example 6.11, (c).

Proposition 6.12. *There are no stage-simple* RVT *classes of codimension* ≥ 8. *There are stage-simple* RVT *classes of every codimension* ≤ 7. *There are no regular stage-simple* RVT *classes of codimension* ≥ 7. *There are regular* RVT *classes of every codimension* ≤ 6.

Proof. The last two statements are direct corollaries of Theorem 6.15 in the next section which gives an explicit description of all stage-simple regular RVT classes. The absence of critical RVT classes (α) of codimension ≥ 8 follows from the absence of regular RVT classes of codimension ≤ 7 because (α) has the form $(\alpha) = (\beta\mathrm{C})$, $\mathrm{C} \in \{\mathrm{V}, \mathrm{T}\}$ and (α) adjoins the regular class $(\beta\mathrm{R})$ of codimension ≤ 7. Finally, the classes $\mathrm{R}, \mathrm{V}, \mathrm{VT}, \mathrm{VT}^2, ..., \mathrm{VT}^6$ are examples of stage-simple classes of codimension $0, 1, 2, ..., 7$ respectively, see Example 6.11, (c). □

The third column in Tables 6.1 and 6.2 is the minimal level of the Monster containing the corresponding RVT classes. Note that an RVT class coded by r letters belongs to level $(r+2)$, i.e. lives in the Monster manifold of dimension $r+4$. The minimal number in the third column of Tables 6.1 and 6.2 is 8. Therefore Theorem 6.6 implies the following corollary.

Corollary 6.13 (results obtained in the works described in Section 1.6.3). *All points in any level* ≤ 7 (*i.e. of the Monster manifolds of dimension* ≤ 9) *are stage-simple.*

Theorem 6.6 allows us to determine stage-simple points in the Monster manifold of any dimension ≥ 10.

Example 6.14. The only fencing classes in the 8th level of the Monster are VRVRVR and VRVTRR. Therefore a point in the Monster manifold of dimension 10 is stage-simple if and only if its RVT code does not have the form $\mathrm{VL}_1\mathrm{VL}_2\mathrm{VL}_3$ or $\mathrm{VL}_1\mathrm{VTL}_2\mathrm{L}_3$, where each L_i denotes any of the letters $\mathrm{R}, \mathrm{V}, \mathrm{T}$.

6.3. Explicit description of stage-simple RVT classes

Theorem 6.6 and a straightforward RVT analysis allows us to describe all stage-simple RVT classes explicitly. In this section we present an explicit description of all the *regular* stage-simple RVT classes.

Theorem 6.15. *Any regular stage-simple* RVT *class* $(\alpha) \neq (\mathrm{RR...R})$ *has one of the forms*

$$\mathrm{R}^{\geq 1}\omega\mathrm{R}^{q\geq 1} \tag{6.3}$$

$$\omega\mathrm{R}^{q\geq 1} \tag{6.4}$$

$$\mathrm{R}^{\geq 1}\omega^{(1)}\mathrm{R}^{m\geq 1}\omega^{(2)}\mathrm{R}^{q\geq 1} \tag{6.5}$$

$$\omega^{(1)}\mathrm{R}^{m\geq 1}\omega^{(2)}\mathrm{R}^{q\geq 1} \tag{6.6}$$

where $\omega, \omega^{(1)}, \omega^{(2)}$ *are entirely critical* RVT *codes. An* RVT *class of the form* (6.3), *respectively* (6.4) *is stage-simple if and only if* ω *is an entirely critical* RVT *code appearing in the first column of Table* 6.3, *respectively Table* 6.4 *and* q *satisfies the constraint given in its row. An* RVT *class* (6.5), *respectively* (6.6) *is stage-simple if and only if* $(\omega^{(1)}, \omega^{(2)})$ *is one of the pairs of entirely critical* RVT *codes in the first two columns and the same row of Table* 6.5, *respectively Table* 6.6, *and* m, q *satisfy the constraint given in its row.*

Notation and Meaning in the Tables. In the following four tables C, C_i denotes *any* critical letter, i.e. either V or T. The entries of the final column are fencing classes adjoined by the class $\alpha = \alpha(q, \omega)$ (or $\alpha = \alpha(m, q, \omega)$) with ω as per the first column, and with the positive integer q (or pair (m, q)) *not* satisfying the constraint of the 2nd column.

Table 6.3. Stage-simple regular RVT classes of the form $\mathrm{R}^{\geq 1}\omega\mathrm{R}^{q\geq 1}$.

ω	q	codim	for other q adjoins a fencing class of the form
V, VC	$\forall\, q$	1, 2	
VCV	$q \in \{1,2\}$	3	$\mathrm{R}^{\geq 1}\mathrm{VRVR}^{\geq 3}$
VTT	$q \in \{1,2\}$	3	$\mathrm{R}^{\geq 1}\mathrm{VTTR}^{\geq 3}$
VVT	$q \in \{1,2,3\}$	3	$\mathrm{R}^{\geq 1}\mathrm{VVTR}^{\geq 4}$
$\mathrm{VC_1VC_2}$	$q = 1$	4	$\mathrm{R}^{\geq 1}\mathrm{VRVR}^{\geq 3}$
VTTC	$q = 1$	4	$\mathrm{R}^{\geq 1}\mathrm{VTTR}^{\geq 3}$
VVTC	$q \in \{1,2\}$	4	$\mathrm{R}^{\geq 1}\mathrm{VVTR}^{\geq 4}$
$\mathrm{VVTC_1C_2}$	$q = 1$	5	$\mathrm{R}^{\geq 1}\mathrm{VVTR}^{\geq 4}$

Table 6.4. Stage-simple regular RVT classes of the form $\omega\mathrm{R}^{q\geq 1}$.

ω	q	codim	for other q adjoins a fencing class of the form
V, VC	$\forall\, q$	1, 2	
VCT	$\forall\, q$	3	
VTTT	$q \in \{1,2,3\}$	4	$\mathrm{VTTTR}^{\geq 4}$
VVCV	$q \in \{1,2\}$	4	$\mathrm{RVRVR}^{\geq 3}$
VTVV	$q \in \{1,2\}$	4	$\mathrm{VRVVR}^{\geq 3}$
VTTV	$q \in \{1,2\}$	4	$\mathrm{VTRVR}^{\geq 3}$
VCVT	$q = 1$	4	$\mathrm{VRVTR}^{\geq 2}$
VTTTC	$q \in \{1,2\}$	5	$\mathrm{VTTTR}^{\geq 4}$
VCVVT	$q = 1$	5	$\mathrm{VRVVR}^{\geq 3}$
VVTTC	$q = 1$	5	$\mathrm{RVTTR}^{\geq 3}$
VVTVC	$q = 1$	5	$\mathrm{RVRVR}^{\geq 3}$
$\mathrm{VTTVC_2}$	$q = 1$	5	$\mathrm{VTRVR}^{\geq 3}$
$\mathrm{VTTTC_1C_2}$	$q = 1$	6	$\mathrm{VTTTR}^{\geq 4}$

TABLE 6.5. Stage-simple regular RVT classes of the form $R^{\geq 1}\omega^{(1)}R^{m\geq 1}\omega^{(2)}R^{q\geq 1}$.

$\omega^{(1)}$	$\omega^{(2)}$	m, q	codim	for other m, q adjoins a fencing class of the form
V	V	$\forall\ m,\ q \in \{1,2\}$	2	$R^{\geq 1}VR^{\geq 1}VR^{\geq 3}$
V	VC	$\forall m,\ q = 1$	3	$R^{\geq 1}VR^{\geq 1}VR^{\geq 3}$
VV	V	$m = 1,\ q \in \{1,2\}$ or $m = 2, q = 1$	3	$R^{\geq 1}VVR^{\geq 3}VR^{\geq 1}$ or $R^{\geq 1}VVR^{2}VR^{\geq 2}$ or $R^{\geq 1}VR^{\geq 1}VR^{\geq 3}$
VT	V	$m = q = 1$	3	$R^{\geq 1}VTR^{\geq 2}VR^{\geq 1}$ or $R^{\geq 1}VTR^{\geq 1}VR^{\geq 2}$
VV	VC	$m = q = 1$	4	$R^{\geq 1}VVR^{\geq 2}VR^{\geq 2}$ or $R^{\geq 1}VR^{\geq 1}VR^{\geq 3}$
VVT	V	$m = q = 1$	4	$R^{\geq 1}VVTR^{\geq 4}$

TABLE 6.6. Stage-simple regular RVT classes of the form $\omega^{(1)}R^{m\geq 1}\omega^{(2)}R^{q\geq 1}$.

$\omega^{(1)}$	$\omega^{(2)}$	m, q	codim	for other m, q adjoins a fencing class of the form
V	V	$\forall\ m, \forall\ q$	2	
V	VV	$\forall m,\ q \in \{1,2\}$	3	$VR^{\geq 1}VVR^{\geq 3}$
V	VT	$\forall m,\ q = 1$	3	$VR^{\geq 1}VTR^{\geq 2}$
VV	V	$\forall\ m,\ q \in \{1,2\}$	3	$RVR^{\geq 1}VR^{\geq 3}$
VT	V	$\forall\ m,\ q \in \{1,2\}$	3	$VTR^{\geq 1}VR^{\geq 3}$
V	VVT	$\forall\ m,\ q = 1$	4	$VR^{\geq 1}VVR^{\geq 3}$
VV	VC	$\forall\ m,\ q = 1$	4	$RVR^{\geq 1}VR^{\geq 3}$
VT	VC	$\forall\ m,\ q = 1$	4	$VTR^{\geq 1}VR^{\geq 3}$
VTT	V	$m = 1, q \in \{1,2\}$ or $m = 2,\ q = 1$	4	$VTTR^{\geq 3}VR^{\geq 1}$ or $VTTR^{2}VR^{\geq 2}$ or $VTR^{\geq 1}VR^{\geq 3}$
VVT	V	$m = q = 1$	4	$RVTR^{\geq 2}VR^{\geq 1}$ or $RVTR^{\geq 1}VR^{\geq 2}$
VTT	VC	$m = q = 1$	5	$VTTR^{\geq 2}VR^{\geq 2}$ or $VTR^{\geq 1}VR^{\geq 3}$
VTTT	V	$m = q = 1$	5	$VTTTR^{\geq 4}$

The proof of Theorem 6.15 is as follows. Theorem 6.6 reduces Theorem 6.15 to the following lemmas.

LEMMA 6.16. *If* $(\alpha) \neq (\mathrm{RR...R})$ *is a regular* RVT *class which does not adjoin any of the fencing classes then* (α) *has one of the forms (6.3) - (6.6).*

LEMMA 6.17. *If an* RVT *class has one of the forms (6.3) - (6.6) and satisfies the condition of Theorem 6.15 then it does not adjoin any of the fencing classes.*

LEMMA 6.18. *If an* RVT *class has one of the forms (6.3) - (6.6) and does not adjoin any of the fencing classes then it satisfies the condition in Theorem 6.15.*

PROOF OF LEMMA 6.16. We will prove that if $(\alpha) \neq (\mathrm{RR...R})$ is a regular RVT class which does not have one of the forms (6.3) - (6.6) then (α) adjoins one of the fencing classes. To see this note that (α) can be expressed in one of the forms

$$(\alpha) = \mathrm{R}^{ge1}\omega^{(1)}\mathrm{R}^{\geq 1}\omega^{(2)} \cdots \mathrm{R}^{\geq 1}\omega^{(r)}\mathrm{R}^{q\geq 1}, \quad r \geq 3 \tag{6.7}$$

$$(\alpha) = \omega^{(1)}\mathrm{R}^{\geq 1}\omega^{(2)} \cdots \mathrm{R}^{\geq 1}\omega^{(r)}\mathrm{R}^{q\geq 1}, \quad r \geq 3 \tag{6.8}$$

where $\omega^{(i)}$ are entirely critical RVT codes. Since any critical (in particular entirely critical) code starts with V, we see that any class of the form (6.7) adjoins a fencing class of the form $\mathrm{R}^{\geq 1}\mathrm{VR}^{\geq 1}\mathrm{VR}^{\geq 3}$, and any class of the form (6.8) adjoins a fencing class of the form $\mathrm{VR}^{\geq 1}\mathrm{VR}^{\geq 1}\mathrm{VR}^{\geq 1}$.

The proof of Lemmas 6.17 and 6.18 is a straightforward RVT grammar analysis requiring patience but no ingenuity.

PROOF OF LEMMA 6.17. This lemma can be easily checked analyzing Tables 6.3 - 6.6. We leave the verification to the reader.

PROOF OF LEMMA 6.18. To prove this lemma we must check the following:

A. Let (α) be a regular RVT class of the form (6.3), respectively of the form (6.4), where ω is an entirely critical RVT code. The class (α) adjoins a fencing classes in each of the following cases:

A1. ω cannot be found in Table 6.3, respectively Table 6.4;

A2. ω is contained in Table 6.3, respectively Table 6.4, but q does not satisfy the constraint in the second column of this table, the row of ω.

B. Let (α) be a regular RVT class of the form (6.5), respectively of the form (6.6), where $\omega^{(1)}, \omega^{(2)}$ are entirely critical RVT codes. The class (α) adjoins one of the fencing classes in each of the following cases:

B1. the pair $(\omega^{(1)}, \omega^{(2)})$ cannot be found in the same row of Table 6.5, respectively Table 6.6;

B2. the pair $(\omega^{(1)}, \omega^{(2))}$ is contained in the same row of Table 6.5, respectively Table 6.6, but m, q do not satisfy the constraint in the third column of this table, the row of $\omega^{(1)}, \omega^{(2)}$.

PROOF OF STATEMENTS A2 AND B2. The proof is given in the last column of Tables 6.3 - 6.6.

In proving statements A1 and B1 we will use the following notations.

NOTATION. As in the Tables, C, C_i denote either critical letter V or T. By $|\omega|$ we denote the number of letters (= the length) of an entirely critical RVT code ω (equivalently, it is the codimension of the class (ω)).

PROOF OF STATEMENT A1 FOR CLASSES (6.3). Table 6.3 contains all entirely critical codes of length ≤ 4 and the entirely critical codes having the form $\mathrm{VVTC_1C_2}$ and of codimension 5. We must prove that a class (α) of the form $(\alpha) = \mathrm{R}^{\geq 1}\omega\mathrm{R}^{\geq 1}$ adjoins one of the fencing classes in each of cases pointed out in the first column of Table 6.7. The proof is given in the second column of Table 6.7.

TABLE 6.7. Proof of statement A1 for classes (6.3): $(\alpha) = \mathrm{R}^{\geq 1}\omega\mathrm{R}^{\geq 1}$

the case	(α) adjoins a fencing class of the form
$\lvert\omega\rvert \geq 5,\ \omega$ starts with VCV	$\mathrm{R}^{\geq 1}\mathrm{VRVR}^{\geq 3}$
$\lvert\omega\rvert \geq 5,\ \omega$ starts with VTT	$\mathrm{R}^{\geq 1}\mathrm{VTTR}^{\geq 3}$
$\lvert\omega\rvert \geq 6,\ \omega$ starts with VVT	$\mathrm{R}^{\geq 1}\mathrm{VVTR}^{\geq 4}$

PROOF OF STATEMENT A1 FOR CLASSES (6.4). Table 6.4 contains all entirely critical codes of length ≤ 4, all entirely critical codes of length 5 except the codes of the form $\mathrm{VC_1VC_2V}$, VCVTT, and all entirely critical codes of length 6 starting with VTTT. Therefore we must prove that an RVT class (α) of the form $(\alpha) = \omega\mathrm{R}^{\geq 1}$ adjoins one of the fencing classes in each of the following cases:

(a) ω starts with $\mathrm{VC_1VC_2V}$;

(b) ω starts with VCVTT;

(c) $|\omega| \geq 6$, ω starts *not* with VTTT;

(d) $|\omega| \geq 7$, ω starts with VTTT.

In case (a) the class (α) adjoins a fencing class of the form $\mathrm{VRVRVR}^{\geq 1}$. In case (b) the class (α) adjoins a fencing class of the form $\mathrm{VRVTR}^{\geq 2}$. In case (d) the class (α) adjoins a fencing class of the form $\mathrm{VTTTR}^{\geq 4}$. In case (c) the class (α) has one of the forms given in the first column of Table 6.8 and it adjoins a fencing class given in the second column of Table 6.8.

TABLE 6.8. Proof of statement A1 for classes (6.4),
the case $(\alpha) = \omega\mathrm{R}^{\geq 1}$, $|\omega| \geq 6$, ω starts *not* with VTTT

the case	adjoins a fencing class of the form
$\lvert\omega\rvert \geq 6,\ \omega$ starts with VCVV	$\mathrm{VRVVR}^{\geq 3}$
$\lvert\omega\rvert \geq 6,\ \omega$ starts with VCVT	$\mathrm{VRVTR}^{\geq 2}$
$\lvert\omega\rvert \geq 6,\ \omega$ starts with VVTV	$\mathrm{RVRVR}^{\geq 3}$
$\lvert\omega\rvert \geq 6,\ \omega$ starts with VTTV	$\mathrm{VTRVR}^{\geq 3}$
$\lvert\omega\rvert \geq 6,\ \omega$ starts with VVTT	$\mathrm{RVTTR}^{\geq 3}$

PROOF OF STATEMENT B1 FOR CLASSES (6.5). We must prove that a class (α) of the form $(\alpha) = \mathrm{R}^{\geq 1}\omega^{(1)}\mathrm{R}^{\geq 1}\omega^{(2)}\mathrm{R}^{\geq 1}$ adjoins one of the fencing classes in each of the cases pointed out in the first column of Table 6.9. The proof is given in the second column of Table 6.9.

TABLE 6.9. Proof of statement B1 for classes (6.5): $(\alpha) = \mathrm{R}^{\geq 1}\omega^{(1)}\mathrm{R}^{\geq 1}\omega^{(2)}\mathrm{R}^{\geq 1}$

the case	(α) adjoins a fencing class of the form
$\omega^{(1)} = \mathrm{V},\ \lvert\omega^{(2)}\rvert \geq 3$	$\mathrm{R}^{\geq 1}\mathrm{V}\mathrm{R}^{\geq 1}\mathrm{V}\mathrm{R}^{\geq 3}$
$\omega^{(1)} = \mathrm{VV},\ \lvert\omega^{(2)}\rvert \geq 3$	$\mathrm{R}^{\geq 1}\mathrm{V}\mathrm{R}^{\geq 1}\mathrm{V}\mathrm{R}^{\geq 3}$
$\omega^{(1)} = \mathrm{VT},\ \lvert\omega^{(2)}\rvert \geq 2$	$\mathrm{R}^{\geq 1}\mathrm{VTR}^{\geq 1}\mathrm{VR}^{\geq 2}$
$\omega^{(1)}$ starts with VVT, $\lvert\omega^{(2)}\rvert \geq 2$	$\mathrm{R}^{\geq 1}\mathrm{VVTR}^{\geq 4}$
$\omega^{(1)}$ starts with VVT, $\lvert\omega^{(1)}\rvert \geq 4$	$\mathrm{R}^{\geq 1}\mathrm{VVTR}^{\geq 4}$
$\omega^{(1)}$ starts with VCV	$\mathrm{R}^{\geq 1}\mathrm{VRVR}^{\geq 3}$
$\omega^{(1)}$ starts with VTT	$\mathrm{R}^{\geq 1}\mathrm{VTTR}^{\geq 3}$

PROOF OF STATEMENT B1 FOR CLASSES (6.4). We must prove that a class (α) of the form $(\alpha) = \omega^{(1)}\mathrm{R}^{\geq 1}\omega^{(2)}\mathrm{R}^{\geq 1}$ adjoins one of the fencing classes in each of cases pointed out in the first column of Table 6.10. The proof is given in the second column of of Table 6.10.

TABLE 6.10. Proof of statement B1 for classes (6.6) $(\alpha) = \omega^{(1)}\mathrm{R}^{\geq 1}\omega^{(2)}\mathrm{R}^{\geq 1}$

the case	(α) adjoins a fencing class of the form
$\omega^{(1)} = \mathrm{V},\ \omega^{(2)}$ starts with VCV	$\mathrm{VRVRVR}^{\geq 1}$
$\omega^{(1)} = \mathrm{V},\ \omega^{(2)}$ starts with VTT	$\mathrm{VR}^{\geq 1}\mathrm{VTR}^{\geq 2}$
$\omega^{(1)} = \mathrm{VC},\ \lvert\omega^{(2)}\rvert \geq 3$	$\mathrm{RVR}^{\geq 1}\mathrm{VR}^{\geq 3}$ if $\mathrm{C} = \mathrm{V}$ $\mathrm{VTR}^{\geq 1}\mathrm{VR}^{\geq 3}$ if $\mathrm{C} = \mathrm{T}$
$\omega^{(1)}$ starts with VCV	$\mathrm{VRVR}^{\geq 1}\mathrm{VR}^{\geq 1}$
$\omega^{(1)}$ starts with VVT, $\lvert\omega^{(1)}\rvert \geq 4$	$\mathrm{RVTR}^{\geq 2}\mathrm{VR}^{\geq 1}$
$\omega^{(1)}$ starts with VVT, $\lvert\omega^{(2)}\rvert \geq 2$	$\mathrm{RVTR}^{\geq 1}\mathrm{VR}^{\geq 2}$
$\omega^{(1)}$ starts with VTT, $\lvert\omega^{(2)}\rvert \geq 3$	$\mathrm{VTR}^{\geq 1}\mathrm{VR}^{\geq 3}$
$\omega^{(1)}$ starts with VTTV	$\mathrm{VTRVR}^{\geq 3}$
$\omega^{(1)}$ starts with VTTT, $\lvert\omega^{(2)}\rvert \geq 2$	$\mathrm{VTTTR}^{\geq 4}$

6.4. Local simplicity of RVT classes

We start the proof of Theorems 6.4 and 6.6. The proof is based on distinguishing certain RVT classes consisting of a finite number of orbits and distinguishing certain RVT classes which are not locally simple at any their points. The local simplicity of a set (in particular of a singularity class) is defined as follows.

DEFINITION 6.19. A set $Q \subset \mathbb{P}^k\mathbb{R}^2$ is locally simple at a point $p \in Q$ if there exists a neighbourhood U of p in $\mathbb{P}^k\mathbb{R}^2$ such that the set $U \cap Q$ is covered by a finite number of orbits. Similarly, a set $S \subset j^r\,\mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ is locally simple at a point $\xi \in S$ if there exists a neighbourhood U of ξ in $j^r\,\mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ such that the set $U \cap S$ is covered by a finite number of orbits.

In terms of local simplicity of an RVT class we can give the following necessary condition for the stage-simplicity of a point in the Monster.

PROPOSITION 6.20. *Let (α) and (β) be* RVT *classes in the same level. If (α) adjoins (β) and (β) is not locally simple at any of its point then (α) contains no simple points.*

PROOF. Take a point $p \in (\alpha)$. Any neighborhood U of p contains a point $\tilde{p} \in (\beta)$. Since (β) is not locally simple at $\tilde{p}$ the set $U \cap (\beta)$, and consequently the neighborhood U, is not covered by a finite number of orbits. □

REMARK 6.21. A singularity class can be locally simple at every one of its points, yet each of these points can fail to be simple. Indeed this property holds for the entirely critical classes (ω) having 8 or more letters. Such a class consists of a single orbit (Theorem 3.9), so that its points are trivially locally simple. But (ω), having 8 or more letters, has codimension greater than 7 and so none of its points are simple (Corollary 6.12). Proposition 6.20 is the reason underlying the failure of simplicity of (ω) (and underlying Corollary 6.12): (ω) adjoins a non-empty RVT class (one of our fencing classes) which is not locally simple at any its point.

REMARK 6.22. If a singularity class in the Monster consists of a finite number of orbits then obviously it is locally simple at any of its point. On the other hand, there exist singularity classes which can be divided into pieces, some of which are comprised of a finite number of orbits, and others of which are comprised of a continuum of orbits. As an example, consider the class of the entire Monster at any level $i \geq 8$. It consists of stage simple RVT classes (such as (RR...R)) which consist of only one orbit, as well as non-stage simple RVT classes (such as the fencing classes) comprised t of a continuum of orbits. See Example 6.14. Conjecturally, such “mixed structures” cannot occur within a single RVT class, i.e. an RVT class either is locally simple at any of its point or every relative neighbhorhood of every one of its point intersects a continuum of orbits.

THEOREM 6.23. *Let (α) be a regular* RVT *class with the jet-identification number r. If the class $j^r\,\mathrm{Leg}(\alpha)$ is not locally simple at any of its point then the class (α) is not locally simple at any of its point.*

Let us show that this theorem follows from Theorem 4.23 and the following proposition.

PROPOSITION 6.24. *Let (α) be a regular* RVT *class with jet-identification number r. Let $\gamma_j \in \mathrm{Leg}(\alpha)$ be a sequence of Legendrian curves such that $j^r\gamma_j \to j^r\gamma$*

where $\gamma \in \mathrm{Leg}(\alpha)$*. Then for any* $i \geq 1$ *the sequence of points* $\gamma_j^i(0)$ *tends to the point* $\gamma^i(0)$*.*

Theorem 6.23 from Proposition 6.24. Let $(1+k)$ be the level of (α). Proposition 6.24 with $i = k$ implies that for any open set $U \subset \mathbb{P}^{1+k}\mathbb{R}^2$ the set $j^r\,\mathrm{Leg}\big((\alpha)\cap U\big)$ is the intersection of the set $j^r\,\mathrm{Leg}(\alpha)$ with an open set in the space of r-jets of Legendrian curves on $\mathbb{P}^1\mathbb{R}^2$. Now Theorem 6.23 follows from Theorem 4.23.

Proof of Proposition 6.24. We need the following statement which follows from the way for calculating prolongations of integral curves in local coordinates in the Monster given in section 7.4.

Lemma 6.25. *Let* γ_j *be a sequence of polynomial Legendrian curves of the same degree which tends to a Legendrian curve* γ*. Assume that for any* $i \geq 1$ *the* RVT *type of the point* γ_j^i *does not depend on* j *and coincides with the type of the points* $\gamma^i(0)$*. Then for any fixed* $i \geq 1$ *the sequence of points* $\gamma_j^i(0)$ *tends to the point* $\gamma^i(0)$*.*

Proposition 6.24 follows from Lemma 6.25 and the definition of the jet-identification number. To see this it suffices to note that if two Legendrian curves $\gamma, \tilde{\gamma}$ belong to the Legendrization of the same RVT class $(\alpha_1, ..., \alpha_r)$ then for any $i \geq 1$ the points $\gamma^i(0)$ and $\tilde{\gamma}^i(0)$ have the same type (it is R if $i = 1$, it is α_{i-1} if $i \in \{2, ..., r+1\}$ and by Proposition 2.31, and the definition of $\mathrm{Leg}(p)$, it is R if $i \geq r+1$).

Remark 6.26. Proposition 6.24 means the continuity of the Monsterization *within a fixed* RVT *class*. In general the Monsterization is not a continuous operation: Lemma 6.25 does not hold without the assumption that for any fixed $i \geq 1$ the points $\gamma_j^i(0)$ have the same type. See section 9.1.

6.5. Proof of Theorem 6.4

The "if" part of Theorem 6.4 asserts that any point of any regular prolongation of any class from the list (6.1) is simple. It is easily checked that if (α) adjoins a regular prolongation of a class from the list then (α) itself is a regular prolongation of a class from the list. Combining this observation with the fact that the number of RVT classes in a fixed level is finite, we have reduced the "if" part of the Theorem to the assertion that any regular prolongation of any of the classes (6.1) consists of a finite number of orbits. The latter statement is a part of Theorem 5.2 corresponding to the case $q^* = \infty$ in Table 5.1. (Recall that for each class in the list, one has $q^* = \infty$ in the last column of Table 5.1).

Now we will prove the "only if" part of Theorem 6.4: a point not contained in a regular prolongation of a class from the list (6.1) is not tower simple. In view of Proposition 6.20 it suffices to prove:

Proposition 6.27. *If an* RVT *class* (α) *is not a regular prolongation of one of the classes* (6.1) *then some regular prolongation of* (α) *adjoins an* RVT *class which is not locally simple at any of its point.*

This proposition follows from Theorem 5.2 and the following lemma.

Lemma 6.28. *If an* RVT *class* (α) *is not a regular prolongation of one of the classes* (6.1) *then it adjoins a regular prolongation of one of the classes of Tables* 5.1, 5.2 *for which one has* $q^* < \infty$ *in the last column of these tables.*

Proposition 6.27 from Lemma 6.28. Let (α) be a class as in Proposition 6.27. By Lemma 6.28 for a sufficiently large l the class $(\alpha \mathrm{R}^l)$ adjoins a regular prolongation $(\beta \mathrm{R}^q)$ of a class (β) in one of the Tables 5.1, 5.2 for which one has $q^* < \infty$ in the last column. Note that $q \to \infty$ as $l \to \infty$. By the second statement of Theorem 5.2 the class $(\beta \mathrm{R}^q)$ is not locally simple at any of its point when $q > q^*$.

Proof of Lemma 6.28. Since Table 5.1 contains all critical classes of codimension ≤ 3, and since in this table one has $q^* = \infty$ only for the classes of the list (6.1), it suffices to prove Lemma 6.28 for classes (α) of codimension ≥ 4. The proof is by cases and is best done with the Table at hand. For the purposes of the proof, we will use the term "β-class" for any regular prolongation of one of the codimension 3 critical classes from Table 5.1 for which $q^* < \infty$.

(a) If (α) has codimension ≥ 4 and its code starts with R or VR or VV then (α) adjoins a β-class whose code also starts with R or VR or VV.

(b) If (α) has codimension ≥ 4 and its code starts with VT, but not with VTT then (α) adjoins a adjoins a β-class whose code starts with VR.

(c) If (α) has codimension ≥ 4 and its code starts with VTT, but not with VTTT then (α) adjoins a adjoins a β-class whose code starts with VTR.

One case remains:

(d) In the remaining case the code of (α) starts with VTTT. The class VTTT appears in Table 5.2 and has $q^* < \infty$.

6.6. Proof of Theorem 6.6

Recall that the RVT classes of Tables 6.1 or 6.2 are called fencing classes. We have to prove the following two statements:

A. If (α) does not adjoin any of the fencing classes then all its points are simple.

B. If (α) adjoins a fencing class then it contains no simple points.

Statement B is an immediate corollary of Proposition 6.20 and:

Theorem 6.29. *Each fencing class is not locally simple at any of its point.*

Proof. This theorem is a direct corollary of Theorem 5.2, and and the fact that every class (α) in Table 6.1 and 6.2 is a regular prolongation $(\alpha) = (\beta R^q)$ of a critical class (β) from Table 5.1 or Table 5.2 with q bigger than the number q^* in the last column of the table, the row of (β). To see the validity of this relation between the classes of the two sets of tables, first note that Table 5.1 is a complete list of critical classes of codimension ≤ 3 and that all but 3 of the classes listed in Tables 6.1 and 6.2 have codimension ≤ 3. The 3 classes not so covered occur at the bottom of Table 6.2, have codimension 4 and are regular prolongations (βR^q) of classes (β) from Table 5.2 with q bigger than the number q^* from the last column of that Table. □

Let us now show that statement A follows from:

Theorem 6.30. *An* RVT *class which does not adjoin any of the fencing classes consists of a finite number of orbits.*

This theorem is proved in sections 6.7.

LEMMA 6.31. *Let* (α) *be an* RVT *class. Assume that every* RVT *class which* (α) *adjoins consists of a finite number of orbits. Then every point of* (α) *is simple.*

PROOF OF LEMMA 6.31. If (α) does not adjoin (β) then $(\alpha) \cap (\bar{\beta}) = \emptyset$: no point of (α) adjoins (β). This property of RVT classes follows from Theorem 3.12. Each level of the Monster is the union of a *finite* number of RVT classes. Therefore a sufficiently small neighborhood of any point of the class (α) is covered by a finite number of orbits.

STATEMENT A FROM THEOREM 6.30 AND LEMMA 6.31. Suppose that (α) does not adjoin any of the fencing classes. Let $\mathcal{A}(\alpha)$ denote the collection of RVT classes which (α) *does* adjoin. Because "adjoin" defines a partial order on the set of RVT classes, it follows that every class $(\beta) \in \mathcal{A}(\alpha)$ satisfies the hypothesis of Theorem 6.30: (β) does not adjoin any class from either Table. By that Theorem every class in $\mathcal{A}(\alpha)$ consists of a finite number of orbits. Statement A now follows from Lemma 6.31.

6.7. Proof of Theorem 6.30

CLAIM 6.32. It suffices to prove Theorem 6.30 for the case that the class (α) is regular.

PROOF. Assume that the Theorem has been proved for regular classes. We must show it holds for critical classes. Take a critical RVT class (α). If it is entirely critical then it consists of a single orbit (Theorem 3.9). If, on the other hand, it contains at least one R then it has the form $(\alpha) = (\widehat{\alpha}\omega)$, where ω is entirely critical class and $(\widehat{\alpha})$ is a regular class. Assume that (α) does not adjoin any of the fencing classes. Then $\widehat{\alpha}$ does not adjoin any of the fencing classes, because any regular prolongation of a fencing class is also a fencing class (see Tables 6.1 and 6.2). Having assumed the validity of Theorem 6.30 for regular classes, we have that $(\widehat{\alpha})$ consists of a finite number of orbits. By Theorem 3.9 the class (α) consists of the same number of orbits. □

CLAIM 6.33. A regular RVT class of codimension ≤ 3 which does not adjoin any of the fencing classes consists of a finite number of orbits.

PROOF. If $(\alpha) \neq (\mathrm{RR...R})$ is a class of codimension ≤ 3 then $(\alpha) = (\beta \mathrm{R}^q)$, where (β) is a critical class from Table 5.1. It is easy to check that if (α) does not adjoin any of the fencing classes then $q \leq q^*$ where q^* is the number in the last column of Table 5.1, the row of β. Now the claim follows from Theorem 5.2. □

Claims 6.32 and 6.33 reduce Theorem 6.30 to the following statement.

PROPOSITION 6.34. *An* RVT *class of codimension* ≥ 4 *which does not adjoin any of the fencing classes consists of a finite number of orbits.*

To prove this proposition we use the explicit description of regular RVT classes which do not adjoin any of the fencing classes, see section 6.3.

PROPOSITION 6.35. *If* (α) *is a regular* RVT *class of codimension* ≥ 4 *which does not adjoin any of the fencing classes then* (α) *has one of the forms below, where* $\omega, \omega^{(1)}, \omega^{(2)}$ *denote arbitrary entirely critical classes:*

$$\mathrm{R}^{s\geq 0}\omega\mathrm{R}^{\geq 2} \tag{6.9}$$

$$\mathrm{R}^{s\geq 0}\omega^{(1)}\mathrm{R}\omega^{(2)}\mathrm{R} \tag{6.10}$$

$$\mathrm{VR}^{\geq 1}\omega\mathrm{R},\ \mathrm{VVR}^{\geq 1}\omega\mathrm{R},\ \mathrm{VTR}^{\geq 1}\omega\mathrm{R} \tag{6.11}$$

$$\mathrm{VTTTR}^{\leq 3},\ \mathrm{VTTRVR}^{\leq 2},\ \mathrm{VTTR}^2\mathrm{VR}. \tag{6.12}$$

Proof. By Lemmas 6.16 and 6.18 the class (α) in Proposition 6.35 has one of the forms (6.3) - (6.6) and satisfies the condition in Theorem 6.15. Analyzing codimension ≥ 4 classes in tables 6.3 - 6.6 we check that all codimension ≥ 4 classes in Table 6.3 have the form (6.9) with $s \geq 1$, all codimension ≥ 4 classes in Table 6.4 except the classes VTTRR$^{\leq 3}$ have the form (6.9) with $s = 0$, all codimension ≥ 4 classes in Table 6.5 have the form (6.10) with $s \geq 1$, and all codimension ≥ 4 classes in Table 6.6 except the classes VTTRVR$^{\leq 2}$, VTTR^{2}VR have either the form (6.10) with $s = 0$ or the form (6.11). □

In view of Proposition 6.35, to complete the proof of Proposition 6.34 (and consequently of Theorem 6.30 and Theorem 6.6) it suffices to prove:

Proposition 6.36. *Any* RVT *class appearing in (6.9) - (6.12) consists of a finite number of orbits.*

Proof. For classes of the form (6.12) this follows from Theorem 5.2 and Table 5.2. In fact, q^*, of the last column of Table 5.2, is 3 for VTTT, is 2 for VTTRV, and is 1 for VTTR^{2}V.

For classes (6.9) - (6.11) Proposition 6.36 follows from our results from section 5.6. Any class of the form (6.9) consists of ≤ 2 orbits by Proposition 5.9. The same holds for any class of the form (6.10) by Proposition 5.11. Finally the fact that any class of the form (6.11) consists of a finite number of orbits is a corollary of Proposition 5.10 and the assertion that each of the classes of the form VR$^{\geq 1}$, VVR$^{\geq 1}$, VTR$^{\geq 1}$ consists of a single orbit. This last assertion is a part of Theorem 5.8 on the classification of tower-simple points which was proved in section 5.5. □

CHAPTER 7

Local coordinate systems on the Monster

In [**KR**] Kumpera and Ruiz introduced special systems of coordinates designed to fit Goursat distributions. In section 7.1 we define these coordinates, henceforth called KR coordinates, and explain their projective meaning on the Monster. In sections 7.2 and 7.3 we relate the KR coordinates to critical curves, directions, points, and to RVT classes. In section 7.4 we show how to prolong a plane curves in KR coordinate terms .

7.1. The KR coordinate system

We will write a KR coordinate system for $\mathbb{P}^k\mathbb{R}^2$ as $(x, y, u_1, \ldots, u_k)$. The coordinates are such that

(1) the last coordinate u_k is an affine coordinate for the fiber of the circle bundle projection $\pi_{k,k-1} : \mathbb{P}^k\mathbb{R}^2 \to \mathbb{P}^{k-1}\mathbb{R}^2$.

(2) $\pi_{k,j}(x, y, u_1, \ldots, u_j, \ldots u_k) = (x, y, u_1, \ldots, u_j)$ is the coordinate representation of the projections $\pi_{k,j} : \mathbb{P}^k\mathbb{R}^2 \to \mathbb{P}^j\mathbb{R}^2$ (for $j \le k$).

Item (2) tells us that the u_i, $i \le j$, together with x, y also coordinatize $\mathbb{P}^j\mathbb{R}^2$.

In view of (2), we will not distinguish between u_j and $\pi^*_{k,j}u_j$, $k \ge j$.

We will be describing an inductive scheme for constructing the fiber coordinate u_{k+1}, given $(x, y, u_1, \ldots, u_k)$. The construction is based on homogeneous and affine coordinates for a projective line.

7.1.1. Coordinates for a projective line. Consider the projective line $\mathbb{P}(V)$, where V a two-dimensional vector space with a distinguished line $\ell_v \subset V$, called its "vertical line". Choose a basis θ^1, θ^2 for the dual space V^* of V such that $\ell_v = ker(\theta^1)$. This basis defines a set of linear coordinates such that the vertical line is the "y-axis". Then $[\theta^1, \theta^2]$ form homogeneous coordinates on $\mathbb{P}(V)$. In other words they define a map $\mathbb{P}(V) \to \mathbb{P}(\mathbb{R}^2)$ by sending each line $\ell = \text{span}(v) \in \mathbb{P}(V)$ to the line $[\theta^1(v), \theta^2(v)] = \text{span}\{(\theta^1(v), \theta^2(v)\} \in \mathbb{P}\mathbb{R}^2$. The vertical point $\ell_v \in \mathbb{P}(V)$ is sent to $[0, 1]$. Corresponding affine coordinates $u, \widetilde{u}$ are defined by dividing appropriately:

$$[\theta^1, \theta^2] = [1, \theta^2/\theta^1] = [1, u], \quad [\theta^1, \theta^2] = [\theta^1/\theta^2, 1] = [\widetilde{u}, 1].$$

So $u(\ell) = \theta^2(v)/\theta^1(v)$ is well-defined when the line ℓ is not vertical. To coordinatize a neighborhood of the vertical point, we use instead $\widetilde{u} = \theta^1/\theta^2$. These two affine coordinates are related by $\widetilde{u} = 1/u$.

If V is now a rank 2 vector bundle over a manifold M (such as $V = \Delta^j$ over $\mathbb{P}^j\mathbb{R}^2$) and if θ^1, θ^2 are local coframes for V, i.e smoothly varying bases for the dual vector bundle V^*, then the same formulae and relations hold, and allow us to define fiber-affine functions $u, \widetilde{u}$ on the $\mathbb{P}^1$ bundle $\mathbb{P}(V) \to M$. We call $u, \widetilde{u}$ the fiber-affine coordinates associated to the choice of local coframe.

7.1.2. Constructing the Kumpera-Ruiz coordinates. We apply these projective considerations to the Monster.

7.1.2.1. *The case* $k = 1$. This case was explained in section 1.2 which we recall now in slightly different notation. We are to construct coordinates in a neighborhood of a point $p^* = (m^*, \ell^*) \in \mathbb{P}^1\mathbb{R}^2$. Recall that $m^* \in \mathbb{R}^2$ and ℓ^* is a line in $\Delta^0(m^*) = T_{m^*}\mathbb{R}^2$. Take standard coordinates (x, y) on $\mathbb{R}^2$. Then $\{dx, dy\}$ is a coframe for $\Delta^0 = T\mathbb{R}^2$ so that $[dx, dy]$ forms fiber-homogeneous coordinates on $\Delta^0(x, y)$. The triple $(x, y, [dx, dy])$ defines a global diffeomorphism between $\mathbb{P}^1\mathbb{R}^2$ and $\mathbb{R}^2 \times \mathbb{P}^1$. There are two corresponding possibilities for the fiber-affine coordinates:

$$[dx, dy] = \left[1, \frac{dy}{dx}\right] \quad \text{or} \quad [dx, dy] = \left[\frac{dx}{dy}, 1\right].$$

We write either affine coordinates as u_1, deciding between the choices as follows, according to the "verticality" of ℓ^*:

$$u_1 = \begin{cases} \frac{dy}{dx} & \text{if} \ \ \ell^* \neq \operatorname{span}(\partial/\partial y) \\ \frac{dx}{dy} & \text{if} \ \ \ell^* = \operatorname{span}(\partial/\partial y). \end{cases}$$

In this way we cover $\mathbb{P}^1\mathbb{R}^2$ with two coordinate charts of the form (x, y, u_1). The coordinate transformation formula between the charts is $(x, y, u_1) \mapsto (x, y, 1/u_1)$.

If $u_1 = \frac{dy}{dx}$ then $dy - u_1 dx = 0$ and this relation defines the contact form Δ^1 on $\mathbb{P}^1\mathbb{R}^2$ within this chart. Similarly, when $u_1 = \frac{dx}{dy}$ the contact form is given by $dx - u_1 dy = 0$. In the first case, $\{dx, du_1\}$ coframes Δ^1. In the second case, $\{dy, du_1\}$ forms such a coframe.

7.1.2.2. *The case* $k = 2$. We proceed to the Engel case. Fix a point $p^* \in \mathbb{P}^2\mathbb{R}^2$ and write p_1^* for its projection to $\mathbb{P}^1\mathbb{R}^2$. Then $p^* = (p_1^*, \ell^*)$ where ℓ^* is a line in the 2-plane $\Delta^1(p_1^*)$. We have already defined the coordinates (x, y, u_1) in a neighborhood of p_1^*. Define the KR coordinate u_2 in a neighborhood of p^* as follows:

$$\text{if } u_1 = \frac{dy}{dx} \ \ \text{then} \ \ u_2 = \begin{cases} \frac{du_1}{dx} & \text{if } \ell^* \neq \operatorname{span}(\partial/\partial u_1) \\ \frac{dx}{du_1} & \text{if } \ell^* = \operatorname{span}(\partial/\partial u_1); \end{cases}$$

$$\text{if } u_1 = \frac{dx}{dy} \ \ \text{then} \ \ u_2 = \begin{cases} \frac{du_1}{dy} & \text{if } \ell^* \neq \operatorname{span}(\partial/\partial u_1) \\ \frac{dy}{du_1} & \text{if } \ell^* = \operatorname{span}(\partial/\partial u_1). \end{cases}$$

Remark 7.1. The constructed coordinates x, y, u_1, u_2 are not canonical. The coordinates x and y are chosen arbitrarily. Except this, we have arbitrarily chosen "distinguished" directions: the direction $\operatorname{span}(\partial/\partial y)$ in Δ^0 and the direction $\operatorname{span}(\partial/\partial u_1)$ in Δ^1. In fact, in view of the Darboux and Engel Theorem 1.3, there are no canonical coordinates u_1, u_2 in principle, even if x and y are fixed.

7.1.2.3. *From k to $k+1$ ($k \geq 2$).* Once the coordinates (x, y, u_1, u_2) are fixed the construction of the remaining coordinates $u_3, u_4, ...u_k$ near a point $p^* \in \mathbb{P}^k\mathbb{R}^2$ is canonical, and depends only on the verticality of p^* and its projections. At each step, we make a choice between one of two possible fiber affine coordinates, choosing depending on the verticality at that step.

We proceed inductively. Write p^* for a point in $\mathbb{P}^{1+k}\mathbb{R}^2$.

Inductive Hypothesis. Suppose that the KR coordinates $\{x, y, u_1, \ldots u_k\}$ near the projection $\pi_{k+1,k}(p^*)$ have been constructed and satisfy conditions (1) and (2) from the beginning of section 7.1. Use the notation:

$$u_0 = \begin{cases} x & \text{if } \ u_1 = \frac{dy}{dx} \\ y & \text{if } \ u_1 = \frac{dx}{dy}. \end{cases}$$

The inductive hypothesis for the k-th level of the Monster is that there exist a unique ordered pair of integers i, j between 0 and $k-1$ such that:

(a) the restriction of du_i, du_j to the 2-plane $\Delta^{k-1}(\pi_{k+1,k-1}(p^*))$ span the dual space for that 2-plane;

(b) one of i or j is $k-1$;

(c) $u_k = \frac{du_i}{du_j}$ is the corresponding fiber-affine coordinate at level k.

Observe that the coordinates on $\mathbb{P}^2\mathbb{R}^2$ above satisfy the inductive hypothesis.

Note that these hypothesis imply that

- the restrictions of du_k, du_j to the 2-plane $\Delta^k(\pi_{k+1,k}(p^*))$ span the dual space for that 2-plane;
- $\text{span}(\partial/\partial u_k)$ is the vertical line in $\Delta^k(\pi_{k+1,k}(p^*))$.

The inductive step. Let $p^* = (\pi_{k+1,k}(p^*), \ell^*) \in \mathbb{P}^{1+k}\mathbb{R}^2$, ℓ^* is a line in $\Delta^k(\pi_{k+1,k}(p^*))$. Let i, j be the numbers satisfying (a), (b), (c). Take

$$u_{k+1} = \begin{cases} \frac{du_k}{du_j} & \text{if } \ p^* \text{ is not vertical, that is, if } \ell^* \neq \text{span}(\partial/\partial u_k); \\ \frac{du_j}{du_k} & \text{if } \ p^* \text{ is vertical, that is, if } \ell^* = \text{span}(\partial/\partial u_k) \end{cases}$$

With u_{k+1} so defined, the inductive hypothesis holds for the $(k+1)$-st level of the Monster. This completes the construction of the Kumpera-Ruiz [KR]coordinate systems. There are 2^k KR coordinate charts at the kth level and these charts cover the Monster at that level.

7.1.3. The 2-distribution Δ^k. In KR local coordinate system for $\mathbb{P}^k\mathbb{R}^2$ the 2-distribution Δ^k is described by 1-forms $\theta_1, ..., \theta_k$ whose form corresponds to the structure of the coordinates u_i. One has

$$\theta_1 = \begin{cases} dy - u_1 dx & \text{if } \ u_1 = \frac{dy}{dx} \\ dx - u_1 dy & \text{if } \ u_1 = \frac{dx}{dy} \end{cases}.$$

And in general, at level $j \geq 2$ one has

$$u_j = \frac{du_{b_j}}{du_{a_j}}, \quad \theta_j = du_{b_j} - u_j du_{a_j}, \tag{7.1}$$
$$a_j, b_j \in \{0, ..., j-1\}, \quad u_0 \in \{x, y\}.$$

Notation. We will call equations (7.1) the "KR relations". They formally express the KR coordinate at level j in terms of derivatives of previous KR coordinates, and define a particular map $i \mapsto (a_i, b_i)$ from the finite set $\{1, \ldots, k\}$ to pairs (a, b) from the finite set $\{0, \ldots, k-1\}$.

Example 7.2. Let $p^* \in \mathbb{P}^4\mathbb{R}^2$ and assume that

(a) $\pi_{4,1}(p^*) = (m, \ell)$ where $\ell \neq \text{span}(\partial/\partial y)$;

(b) $\pi_{4,2}(p^*) = (\pi_{4,1}(p^*), \ell)$ where $\ell = \text{span}(\partial/\partial u_1)$;

(c) the point p^* belongs to the class VR, i.e. the point $\pi_{4,3}(p^*)$ is vertical and the point p^* itself is not vertical.

Then the KR local coordinate system at p^* is

$$u_1 = \frac{dy}{dx}, \quad u_2 = \frac{dx}{du_1}, \quad u_3 = \frac{du_1}{du_2}, \quad u_4 = \frac{du_3}{du_2},$$

and the 2-distribution Δ^4 is described by the vanishing of the 1-forms

$$dy - u_1 dx, \quad dx - u_2 du_1, \quad du_1 - u_3 du_2, \quad du_3 - u_4 du_2.$$

As we mentioned above, the first four coordinates, x, y, u_1, u_2 are not defined canonically because there are no "distinguished" directions in Δ^0 and Δ^1. Working with the KR coordinates it is convenient to fix the following point $O \in \mathbb{P}^2\mathbb{R}^2$.

Notation 7.3. Define the point $O \in \mathbb{P}^2\mathbb{R}^2$ by insisting that we are in the chart (x, y, u_1, u_2) with

$$u_1 = dy/dx, \quad u_2 = du_1/dx, \quad x(O) = y(O) = u_1(O) = u_2(O) = 0. \tag{7.2}$$

By the Engel Theorem 1.3 all points in $\mathbb{P}^2\mathbb{R}^2$ are equivalent. Therefore as far as equivalence is concerned we may work "above" O, i.e. to assume that the circle bundle projection of our point to the second level is the point O and consequently the KR relations at the first two levels are the relations (7.2).

Example 7.4. Suppose that $p^* \in \mathbb{P}^k\mathbb{R}^2$ represents a non-singular point, of the Monster, so that its RVT class is (RR...R), see Theorem 3.6 . Then none of the $\pi_{k,j}(p^*)$, $3 \leq j \leq k$ are vertical. Assume also that $\pi_{k,2}(p^*) = O$. Then

$$u_1 = \frac{dy}{dx}, \quad u_2 = \frac{du_1}{dx}, \quad \ldots, \quad u_k = \frac{du_{k-1}}{dx}$$

and the 2-distribution Δ^k is described by vanishing of the 1-forms

$$dy - u_1 dx, \quad du_1 - u_2 dx, \quad du_2 - u_3 dx, \quad \ldots, \quad du_k - u_{k-1} dx,$$

which is the classical Cartan normal form.

The KR local coordinates give a unique normal form for Δ^k at points $p^* \in \mathbb{P}^k\mathbb{R}^2$ above the point $O \in \mathbb{P}^2\mathbb{R}^2$ whose projection list $\pi_{k,3}(p^*), \pi_{k,4}(p^*), \ldots, \pi_{k,k}(p^*) = p^*$ have the same vertical-non-vertical types. This gives the impression that all such points p^* are equivalent, which, among other things, would imply a finite number of orbits at each level. This is not so. The reason is that some of the KR coordinates of a point $p^* \in \mathbb{P}^k\mathbb{R}^2$ need not be centered at 0. See Table 7.1 below. Indeed for $k \geq 8$ the values of the $u_j(p^*)$ for certain j can act as continuous moduli, see [**Mor3**] and Chapter 6.

7.2. Critical curves in the KR coordinates

The construction of the KR coordinates implies the following form of immersed vertical curves.

PROPOSITION 7.5. *In the* KR *coordinates* $x, y, u_1, ..., u_k$ *at* $p^* \in \mathbb{P}^k\mathbb{R}^2$, *the immersed vertical curve* γ_{vert} : $(x(t), y(t), u_1(t), \cdots, u_k(t))$ *through* p^* *has the form (unique up to reparameterization)*

$$\big(x(t), y(t), u_1(t), ..., u_{k-1}(t)\big) \equiv \big(x(0), y(0), u_1(0), ..., u_{k-1}(0)\big),$$
$$u_k(t) = u_k(0) + t.$$

Recall that an immersed critical curve germ $\Gamma : (\mathbb{R}, 0) \to (\mathbb{P}^k\mathbb{R}^2, p)$ is either vertical or the prolongation of a vertical curve V_j at the j-th level $2 \leq j < k$, see Theorem 2.20. The point $V_j^1(0)$ is vertical and the points $V_j^2(0), V_i^3(0), ...$ are not vertical (see Proposition 2.45). Therefore Proposition 7.5 and the mentioned results of Chapter 2 imply the following statement.

PROPOSITION 7.6. *Let* $\gamma_{\tan} : (\mathbb{R}, 0) \to (\mathbb{P}^k\mathbb{R}^2, p)$ *be the germ of an immersed critical non-vertical curve. Let* $x, y, u_1, ..., u_k$ *be* KR *coordinates at* p^*. *Then there exist* $j \in \{2, ..., k-1\}$ *and* $s \in \{0, 1, \ldots, j-1\}$ *such that*

$$u_{j+1} = \frac{du_s}{du_j},\ u_{j+2} = \frac{du_{j+1}}{du_j},\ u_{j+3} = \frac{du_{j+2}}{du_j}, \ldots, u_k = \frac{du_{k-1}}{du_j},$$

and the curve $\gamma_{\tan}(t) = \big(x(t), y(t), u_1(t), ..., u_k(t)\big)$ *has the form (unique up to reparameterization)*

$$\big(x(t), y(t), u_1(t), \cdots, u_{j-1}(t)\big) \equiv \big(x(0), y(0), u_1(0), ..., u_{j-1}(0)\big),$$
$$u_j(t) = u_j(0) + t, \quad u_{j+1}(t) = u_{j+2}(t) = \cdots = u_k(t) \equiv 0.$$

7.2.1. Regular, vertical and tangency points and directions. Propositions 7.5, 7.6 lets us distinguish regular, vertical, and tangency directions and points in terms of KR coordinates. Recall (Proposition 2.41) that if $p \in \mathbb{P}^k\mathbb{R}^2$ is a critical point then the plane $\Delta^k(p)$ contains exactly two critical lines - the vertical line and the tangency line.

PROPOSITION 7.7. *Let* $x, y, u_1, ..., u_k$ *be the local* KR *coordinates at a critical point* $p^* \in \mathbb{P}^k\mathbb{R}^2$. *Let* $u_k = \frac{du_i}{du_j}$. *Then:*

- $u_k(p^*) = 0$;
- *the vertical line in* $\Delta^k(p^*)$ *is spanned by* $\frac{\partial}{\partial u_k}$;
- *the tangency line in* $\Delta^k(p^*)$ *is spanned by* $\frac{\partial}{\partial u_j}$.

PROPOSITION 7.8. *Let* $x, y, u_1, ..., u_k$ *be the local* KR *coordinates at a point* $p^* \in \mathbb{P}^k\mathbb{R}^2$. *Assume that the projection* $\pi_{k,k-1}(p^*)$ *is a critical point in* $\mathbb{P}^{k-1}\mathbb{R}^2$. *If* $u_k(p^*) = 0$ *then* p^* *is a critical point in* $\mathbb{P}^k\mathbb{R}^2$.

REMARK 7.9. If p^* is a critical point then we can determine if it is a vertical or tangency point from the structure of the KR coordinates u_{k-1} and u_k. We have $u_{k-1} = \frac{du_i}{du_j}$, $i, j \in \{0, 1, ., k-2\}$. If $u_k = \frac{du_{k-1}}{du_j}$ then the critical point p^* is not vertical and consequently it is the unique tangency point in the fiber over $\pi_{k,k-1}(p^*)$. If $u_k = \frac{du_j}{du_{k-1}}$ then p^* is the vertical point in this fiber.

TABLE 7.1. The KR coordinates and the types of points in the Monster.

Type of p^* and $\pi_{k,k-1}(p^*)$	Coordinate u_k	$u_k(p_k^*)$
p^* is regular and $\pi_{k,k-1}(p^*)$ is regular	$u_k = \frac{du_{k-1}}{du_j}$	$*$
p^* is regular and $\pi_{k,k-1}(p^*)$ is critical	$u_k = \frac{du_{k-1}}{du_j}$	$\neq 0$
p^* is tangency and $\pi_{k,k-1}(p^*)$ is critical	$u_k = \frac{du_{k-1}}{du_j}$	0
p^* is vertical and $\pi_{k,k-1}(p^*)$ is of any type	$u_k = \frac{du_j}{du_{k-1}}$	0

Table 7.1 describes the relations between the KR coordinates u_{k-1}, u_k at a point $p^* \in \mathbb{P}^k\mathbb{R}^2$ and the RVT designation of p^* and its projection $\pi_{k,k-1}(p^*)$. The entry "*" in the first row, last column signifies that any value of that coordinate can be realized.

7.3. RVT classes and KR coordinates

The RVT class (α) of a point p^* does not contain information on its projections to the first two levels and so the choice of RVT class does not uniquely determine the relations among the KR coordinates. The KR relations at $p \in (\alpha)$ are uniquely defined by the RVT code if we fix the projection of p to the second level.

NOTATION 7.10. Given an RVT class $(\alpha) = (\alpha_1, ..., \alpha_k) \in \mathbb{P}^{k+2}\mathbb{R}^2$ we denote by $(\alpha)_O$ the set of points $(\alpha) \cap \pi_{k,2}^{-1}(O)$, where the point $O \in \mathbb{P}^2\mathbb{R}^2$ is defined in Notation 7.3.

With O fixed, we have $u_1 = dy/dx$, $u_2 = du_1/dx$ for the KR coordinates at any point $p \in (\alpha)_O$. Every subsequent KR coordinate u_j, $j > 2$ will have the form $u_j = du_{b_j}/du_{a_j}$ where $0 \le a_j, b_j < j$ and $u_0 = x$. The construction of the KR coordinates implies that the list of particular pairs a_j, b_j occurring the KR relations (7.1) for any point p^* of $(\alpha)_O$ will be the same. Note that by Propositions 7.7 and 7.8 one has the following:

1. if α_j is critical (either V or T) then $u_{j+2}(p^*) = 0$.
2. if α_{j-1} is critical and $\alpha_j = \text{R}$ then $u_{j+2}(p^*) \neq 0$.

EXAMPLE 7.11. Consider the RVT class $(\alpha_1, \dots, \alpha_6) = (\text{VTRRVT}) \subset \mathbb{P}^8\mathbb{R}^2$. The first four KR coordinates of any point $p^* \in (\alpha)_O$ correspond to the choice of the point O. They are x, y, $u_1 = dy/dx$, $u_2 = du_1/dx$ and their values at p^* are $x(p^*) = y(p^*) = u_1(p^*) = u_2(p^*) = 0$ since $\pi_{8,2}(p^*) = O$. The KR coordinate $u_j, j \ge 3$ corresponds to the letter α_{j-2}:

$$u_3 = \frac{dx}{du_2},\ u_4 = \frac{du_3}{du_2},\ u_5 = \frac{du_4}{du_2},\ u_6 = \frac{du_5}{du_2},\ u_7 = \frac{du_2}{du_6},\ u_8 = \frac{du_7}{du_6}.$$

According to statements 1. and 2. above one has

$$u_3(p^*) = u_4(p^*) = u_7(p^*) = u_8(p^*) = 0,\ u_5(p^*) \neq 0.$$

Whether or not $u_6(p^*) = 0$ depends on p^*. Both variants are possible. Therefore the KR coordinates allow us to associate to the equivalence class of a point $p^* \in (\alpha)$ two real numbers $C_1 = u_5(p^*) \neq 0$ and $C_2 = u_6(p^*)$. Nevertheless, C_1 and C_2 are not moduli. Using the symmetries one can reduce C_1 to 1 and C_2 to 0, as follows from the results of section 5.5. (See the third row of Table 5.4, the case $s = 0$).

7.4. Monsterization in KR coordinates

In this section we express the prolongations $c^i(t)$ of a plane curve germ

$$c : (\mathbb{R}, 0) \to \mathbb{R}^2,\quad c(t) = \big(x(t), y(t)\big) \tag{7.3}$$

in terms of KR coordinates. We will use the following notation.

NOTATION 7.12. Let $f = f(t)$ be an analytic function germ at $t = 0$. If f vanishes at 0 along with its first $(k-1)$ derivatives but its k-th derivative at 0 does not vanish we will use the notation $\mathrm{ord}(f) = k$. If $f(t) \equiv 0$ then $\mathrm{ord}(f) = \infty$.

Given a plane curve germ (7.3) let $u_1, ..., u_i$ be the KR coordinates at the point $c^i(0) \in \mathbb{P}^i\mathbb{R}^2$ and let

$$U_i(t) = u_i\big(c^i(t)\big),\ i \geq 1, \tag{7.4}$$

so that in these KR coordinates

$$c^i(t) = \big(x(t), y(t), U_1(t)..., U_i(t)\big). \tag{7.5}$$

Introduce also the coordinates u_{-1}, u_0 and the functions $U_{-1}(t), U_0(t)$ as follows:

$$\begin{aligned} &u_{-1} = y,\ u_0 = x,\ U_{-1}(t) = y(t),\ U_0(t) = x(t) \ \text{ if } \ \mathrm{ord}(y'(t)) \geq \mathrm{ord}(x'(t)) \\ &u_{-1} = x,\ u_0 = y,\ U_{-1}(t) = x(t),\ U_0(t) = y(t) \ \text{ if } \ \mathrm{ord}(y'(t)) < \mathrm{ord}(x'(t)). \end{aligned} \tag{7.6}$$

DEFINITION 7.13. We will say that $u_{-1}, u_0, u_1, u_2, ...$ are the KR coordinates and $U_{-1}(t), U_0(t), U_1(t), U_2(t), ...$ are the KR coordinate functions associated with the plane curve germ c.

The definition of the KR coordinates and the construction of the prolongation of integral curves imply the following construction of u_i and $U_i(t)$. The coordinates u_{-1}, u_0 and the functions $U_{-1}(t)$ and $U_0(t)$ are defined by (7.6). For $i \geq 1$ one has

$$u_i = \frac{du_{\beta_i}}{du_{\alpha_i}},\quad U_i(t) = \frac{U'_{\beta_i}(t)}{U'_{\alpha_i}(t)},\quad \alpha, \beta \in \{-1, 0, ..., i-1\}, \tag{7.7}$$

with $\beta_1 = -1,\ \ \alpha_1 = 0$ and $\alpha_{\geq 2}, \beta_{\geq 2}$ defined by the recursion formulae:

$$\alpha_i = \alpha_{i-1},\ \beta_i = i \ \text{ if } \ \mathrm{ord}(U'_i(t)) \geq \mathrm{ord}(U'_{\alpha_i}(t));$$

$$\alpha_i = i,\ \beta_i = \alpha_{i-1} \ \text{ if } \ \mathrm{ord}(U'_i(t)) < \mathrm{ord}(U'_{\alpha_i}(t)).$$

This construction and Table 7.1 gives an efficient way to compute the RVT types of points $c^i(0)$ and consequently the RVT-code of c. All points in the first and the second levels are regular. Table 7.2 shows how the RVT type of $c^{i+1}(0)$ is determined from that of $c^i(0)$ and the orders of $U_i(t)$ and $U_\alpha(t)$.

TABLE 7.2. The RVT type of the point $c^k(0)$. Here $u_i = \frac{du_{\beta_i}}{du_{\alpha_i}}$

Type of $c^i(0)$	Relation between $U'_i(t))$ and $U'_{\alpha_i}(t))$	Type of $c^{i+1}(0)$
R	$\mathrm{ord}(U'_i(t)) \geq \mathrm{ord}(U'_{\alpha_i}(t))$	R
R	$\mathrm{ord}(U'_i(t)) < \mathrm{ord}(U'_{\alpha_i}(t))$	V
V or T	$\mathrm{ord}(U'_i(t)) > \mathrm{ord}(U'_{\alpha_i}(t))$	T
V or T	$\mathrm{ord}(U'_i(t)) = \mathrm{ord}(U'_{\alpha_i}(t))$	R
V or T	$\mathrm{ord}(U'_i(t)) < \mathrm{ord}(U'_{\alpha_i}(t))$	V

The regularization level of c (see defintion 3.16) can now be determined as follows.

PROPOSITION 7.14. *Let $c = (x(t), y(t))$ be a plane curve germ at $t = 0$ such that the prolongation c^1 is not immersed. Let u_i and $U_i(t)$ be the* KR *coordinates and the* KR *coordinate functions associated with c. Consider the* KR *relations (7.7). Let k be the smallest positive integer such that*

$$U'_{\alpha_k}(0) \neq 0, \quad U'_k(0) \neq 0$$

Then k is the regularization level of c.

PROOF. Let $s \geq 2$ be the first integer such that the curve c^s is immersed. Then

$$U'_1(0) = \cdots = U'_{s-1}(0) = 0, \; U'_s(0) \neq 0.$$

Since $U'_{\alpha_s}(0) = 0$ and $U'_s(0) \neq 0$ it follows that the k of the proposition is greater than s, and that, upon using the recursion relations 7.7, that for indices between s and k we have KR relations determined by $\alpha_{s+1} = \alpha_{s+2} = \cdots = s$, and

$$u_s = \frac{du_{\beta_s}}{du_{\alpha_s}}, \; u_{s+1} = \frac{du_{\alpha_s}}{du_s}, \; u_{s+2} = \frac{du_{s+1}}{du_s}, \; u_{s+3} = \frac{du_{s+2}}{du_s}, \; \cdots \; u_k = \frac{du_{k-1}}{du_s}. \tag{7.8}$$

It follows that the curve c^s has vertical direction at $t = 0$ and the point $c^{s+1}(0)$ is vertical. According to the definition of k and s one has

$$U'_s(0) \neq 0, \; U'_{\alpha_s}(0) = 0, \; U'_{s+1}(0) = \cdots = U'_{k-1}(0) = 0, \; U'_k(0) \neq 0.$$

Consider the case $k = s + 1$. Then $U'_s(0) \neq 0, U'_{s+1}(0) \neq 0$ and by Proposition 7.7 the curve $c^k = c^{s+1}$ is regular. The curves $c^1, ..., c^{s-1}$ are not regular because they are not immersed. The immersed curve c^s is not regular because it has vertical direction. Thus $k = s + 1$ is the regularization level of c.

Consider now the case $k \geq s + 2$. In this case $U'_{s+1}(0) = 0$ and by Proposition 7.7 the curve c^{s+1} has tangency direction at $t = 0$ hence $c^{s+2}(0)$ is a tangency point. If $k \geq s + 3$ then again by Proposition 7.7 the curve c^{s+2} has tangency direction at $t = 0$ hence $c^{s+3}(0)$ is a tangency point. Continuing using Proposition 7.7 we obtain that the curves $c^{s+1}, ..., c^{k-1}$ have tangency directions at $t = 0$ and $c^{s+2}(0), ..., c^k(0)$ are tangency points. Therefore the curve $c^i, i < k$ are not regular. Using Proposition 7.7 one more time and the conditions $U'_s(0) \neq 0, U'_k(0) \neq 0$ we obtain that c^k is a regular curve. Thus k is the regularization level of c. $\square$

EXAMPLE 7.15. Consider the plane curve germ $c : x(t) = t^6, y(t) = t^{14} + t^{15}$. It is easy to calculate that the associated KR coordinates and KR functions u_i have the form given in the second and the third column of Table 7.3, where $k_1, k_2, \ldots$ are certain *non-zero* numerical coefficients. The last column is the type of the point $c^i(0)$ calculated according to Table 7.2. By Proposition 7.14 the curve c regularizes at level 6. It follows that the RVT code of c is VTRV.

TABLE 7.3. The prolongations of the curve
$c : x(t) = t^6, y(t) = t^{14} + t^{15}$.
Here k_i are certain non-zero numerical coefficients.

Prol.	KR coord.	KR function	Type of $c^i(0)$
c^1	$u_1 = \frac{dy}{dx}$	$U_1(t) = y'(t)/x'(t)$, $U_1(t) = k_1 t^8 + k_2 t^9$	R
c^2	$u_2 = \frac{du_1}{dx}$	$U_2(t) = U_1'(t)/x'(t)$, $U_2(t) = k_3 t^2 + k_4 t^3$	R
c^3	$u_3 = \frac{dx}{du_2}$	$U_3(t) = x'(t)/U_2'(t)$, $U_3(t) = k_5 t^4 + k_6 t^5 + o(t^5)$	V
c^4	$u_4 = \frac{du_3}{du_2}$	$U_4(t) = U_3'(t)/U_2'(t)$, $U_4(t) = k_7 t^2 + k_8 t^3 + o(t^3)$	T
c^5	$u_5 = \frac{du_4}{du_2}$	$U_5(t) = U_4'(t)/U_2'(t)$, $U_5(t) = k_9 + k_{10} t + o(t)$	R
c^6	$u_6 = \frac{du_2}{du_5}$	$U_6(t) = U_2'(t)/U_5'(t)$, $U_6(t) = k_{11} t + o(t)$	V
c^7	$u_7 = \frac{du_6}{du_5}$	$U_7(t) = U_6'(t)/U_5'(t)$, $U_7(t) = k_{12} + o(1)$	R
$c^i,\ i \geq 7$	$u_i = \frac{du_{i-1}}{du_5}$	$U_i(t) = U_{i-1}'(t)/U_5'(t)$	R

The RVT code of the curve c from example 7.15 could have been obtained directly from Theorem **A** (section 3.8). But that theorem has not yet been proved. Indeed, the proof of Theorem **A** is an application of the construction we have just given of the KR coordinates and coordinate functions associated to a plane curve and of its RVT code. This proof appears in the next chapter, in section 8.5.

CHAPTER 8

Prolongations and directional blow-up. Proof of Theorems A and B

KR coordinates allow us to express the prolongations of plane curves in terms of iterations of an operator on the space of plane curve germs.

Recall that for an analytic function germ $f(t)$ at $t = 0$ we use the notation $\mathrm{ord}(f) = k$ if f vanishes at 0 along with its first $(k-1)$ derivatives but its k-th derivative at 0 does not vanish. If $f(0) \neq 0$ then $\mathrm{ord}(f) = 0$. If $f(t) \equiv 0$ then $\mathrm{ord}(f) = \infty$.

Definition 8.1. Denote by $\mathcal{P}$ the set of analytic well-parameterized plane curve germs at $t = 0$ having the form

$$\mathcal{P}: \ (x(t), y(t)): \quad \mathrm{ord}(x'(t)) \leq \mathrm{ord}(y'(t)) < \infty.$$

For $(x(t), y(t)) \in \mathcal{P}$ write

$$u(t) = \frac{y'(t)}{x'(t)}$$

and define the operator $\mathcal{B} : \mathcal{P} \to \mathcal{P}$ by:

$$\mathcal{B}(x(t), y(t)) = \begin{cases} \big(x(t),\ u(t)\big) \text{ if } \mathrm{ord}\, u'(t) \geq \mathrm{ord}(x'(t)) \\ \big(u(t),\ x(t)\big) \text{ if } \mathrm{ord}\, u'(t) < \mathrm{ord}(x'(t)). \end{cases}$$

The operator $\mathcal{B}$ will be called "directional blow-up".

Directional blow-up is a well-defined map from $\mathcal{P}$ to itself. In section 8.1. we use its iterates to describe the prolongation and the regularization level of a plane curve germ. In section 8.2 we relate directional blow-up to the maps $\mathbb{E}_{\mathrm{T}}, \mathbb{E}_{\mathrm{V}}$ and L used in the construction of the map RVT$\,(\Lambda)$. These relations lead to the proof of Theorem **A** of section 3.8 for the simplest Puiseux characteristics $[\lambda_0; \lambda_1]$ (section 8.3). In section 8.4 we present certain properties of the operator $\mathcal{B}$. These properties are proved in section 8.7 and are used to prove Theorem **A** for arbitrary Puiseux characteristics in section 8.5 and Theorem **B** (from section 4.8) in section 8.6.

Throughout this section we will denote by u_i and $U_i(t)$ the KR coordinates and the KR coordinate functions associated with a plane curve $c = (x(t), y(t))$, and we will denote by α_i, β_i the indices such that the KR relations have the form (7.7):

$$u_i = \frac{du_{\beta_i}}{du_{\alpha_i}}, \quad U_i(t) = \frac{U'_{\beta_i}(t)}{U'_{\alpha_i}(t)}, \quad \alpha_i, \beta_i \in \{-1, 0, ..., i-1\}, \quad \alpha_1 = 0,\ \beta_1 = -1.$$

Recall that $\{u_{-1}, u_0\} = \{x, y\}$ and $\{U_{-1}(t), U_0(t)\} = \{x(t), y(t)\}$ up to the order in these equalities, see (7.6).

8.1. Directional blow-up and KR coordinates

Write $\mathcal{B}^i = \mathcal{B} \circ \mathcal{B} \circ \ldots \mathcal{B}$ (i times) for the ith iterate of $\mathcal{B}$.

PROPOSITION 8.2. *For any curve $c = (x(t), y(t)) \in \mathcal{P}$ one has*

$$\big(U_{\alpha_i}(t),\ U_{\beta_i}(t)\big) = \mathcal{B}^{i-1}(c). \tag{8.1}$$

PROOF. Since $\operatorname{ord}(x(t)) \le \operatorname{ord}(y(t))$ for any curve $c \in \mathcal{P}$ and $(\alpha_1, \beta_1) = (0, -1)$, then $\big(U_{\alpha_1}(t),\ U_{\beta_1}(t)\big) = (x(t), y(t))$ hence (8.1) holds for $i = 1$. Let us prove (8.1) for $i = r + 1$ assuming that it holds for $i = r$. If $\operatorname{ord}(U_r'(t)) \ge \operatorname{ord} U_{\alpha_r}'(t))$ then according to section 7.4 and the definition of the operator $\mathcal{B}$ one has

$$(U_{\alpha_{r+1}}(t), U_{\beta r+1}(t)) = (U_{\alpha_r}(t), U_r(t)),$$
$$\mathcal{B}^r(c) = \mathcal{B}(\mathcal{B}^{r-1}(c)) = \mathcal{B}((U_{\alpha_r}(t), U_{\beta_r}(t)) = (U_{\alpha_r}(t), U_r(t)).$$

If $\operatorname{ord}(U_r'(t)) < \operatorname{ord} U_{\alpha_r}'(t))$ then

$$(U_{\alpha_{r+1}}(t), U_{\beta r+1}(t)) = (U_r(t), U_{\alpha_r}(t)),$$
$$\mathcal{B}^r(c) = \mathcal{B}(\mathcal{B}^{r-1}(c)) = \mathcal{B}((U_{\alpha_r}(t), U_{\beta_r}(t)) = (U_r(t), U_{\alpha_r}(t)).$$

In either case we have (8.1) with $i = r + 1$. $\square$

EXAMPLE 8.3. The first six directional blow-ups of the curve

$$c : x(t) = t^6, \quad y(t) = t^{14} + t^{15}$$

are as follows, where k_i are certain non-zero numerical coefficients and $U_i(t)$ are the KR functions associated with c, see Example 7.15.

$$\mathcal{B}^1(c) = \big(t^6,\ k_1 t^8 + k_2 t^9\big) = \big(x(t), U_1(t)\big);$$
$$\mathcal{B}^2(c) = \big(k_3 t^2 + k_4 t^3,\ t^6\big) = \big(U_2(t), x(t)\big);$$
$$\mathcal{B}^3(c) = \big(k_3 t^2 + k_4 t^3,\ k_5 t^4 + k_6 t^5 + o(t^5)\big) = \big(U_2(t), U_3(t)\big);$$
$$\mathcal{B}^4(c) = \big(k_3 t^2 + k_4 t^3,\ k_7 t^2 + k_8 t^3\big) = \big(x(t), U_1(t)\big);$$
$$\mathcal{B}^5(c) = \big(k_9 + k_{10} t + o(t),\ k_3 t^2 + k_4 t^3\big) = \big(U_5(t), U_2(t)\big);$$
$$\mathcal{B}^6(c) = \big(k_9 + k_{10} t + o(t),\ k_{11} t + o(t)\big) = \big(U_5(t), U_6(t)\big).$$

Now we characterize the regularization level of a plane curve in terms of the operator $\mathcal{B}$.

DEFINITION 8.4. Let $\mathcal{P}(1) \subset \mathcal{P}$ denote the subset of plane curve germs

$$\mathcal{P}(1) = \Big\{\big(x(t), y(t)\big) \in \mathcal{P} : \quad x'(0) \neq 0, \quad y'(0) \neq 0\Big\}.$$

PROPOSITION 8.5. *Let $c \in \mathcal{P}$ be a plane curve whose first prolongation c^1 is not immersed. Let k be the minimal integer such that $\mathcal{B}^k(c) \in \mathcal{P}(1)$. Then k equals the regularization level r of c.*

PROOF. Fix $i \le k$. Then $\mathcal{B}^{i-1}(c) \notin \mathcal{P}(1)$. By equation (8.1) of Proposition 8.2 either $U_{\alpha_i}'(0) = 0$ or $U_{\beta_i}'(0) = 0$. Now the set of indices $\{\alpha_i, \beta_i\}$ equals the set $\{i-1, \alpha_{i-1}\}$. It follows that one of $U_{i-1}'(0)$, $U_{\alpha_{i-1}}'(0)$ is 0. Consequently, by Proposition 7.14 $i - 1 < r$, where r is the regularization level, so that $k \le r$. Since $\mathcal{B}^k(c) \in \mathcal{P}(1)$ one has by equation (8.1) of Proposition 8.2 in the case $i = k+1$ that $U_{\alpha_{k+1}}'(0) \neq 0$, $U_{\beta_{k+1}}'(0) \neq 0$. Now the sets $\{\alpha_{k+1}, \beta_{k+1}\}$ and $\{k, \alpha_k\}$ are equal. It follows that $U_k'(0) \neq 0$ and $U_{\alpha_k}'(0) \neq 0$. By Proposition 7.14 the curve c regularizes at level k. $\square$

The regularization level can also be characterized as follows.

PROPOSITION 8.6. *Let $c \in \mathcal{P}$ be a plane curve such that the prolongation c^1 is not immersed. If $\mathcal{B}^s(c) \in \mathcal{P}(1)$ and $c^s(0)$ is a critical point then $s = r$ is the regularization level of c.*

PROOF. Let r be the regularization level of c. By Proposition 8.5 one has $r \leq s$. By definition of "regularization level" the curve c^r is regular. Since $r \leq s$ it follows from Proposition 2.31 the curve c^s is regular. But $c^s(0)$ is a critical point, so by Proposition 3.18 one has $r = s$. □

In what follows we also will need the following property of the operator $\mathcal{B}$. Recall that the sign = (repar.) = between two curves or jets of curves means that these curves or jets are the same up to reparameterization.

LEMMA 8.7. *Let $c, c^* \in \mathcal{P}$ and $s \geq 0$ an integer. Suppose that $c^{s+1}(0) = (c^*)^{s+1}(0)$. Then*

1. *If $\mathcal{B}^s(c) = (\text{repar.}) = \mathcal{B}^s(c^*)$ then $c = (\text{repar.}) = c^*$.*
2. *If $j^r\mathcal{B}^s(c) = (\text{repar.}) = j^r\mathcal{B}^s(c^*)$ then $j^r c^s = (\text{repar.}) = j^r(c^*)^s$.*

PROOF. By induction on s. The case $s = 0$ is trivial, since $\mathcal{B}^0(c) = c$. Suppose now that both implications 1 and 2 of the lemma hold for some $s \geq 0$. We must prove that they hold for $s + 1$. Thus, let us suppose that $c^{s+2}(0) = (c^*)^{s+2}(0)$.

VALIDITY OF ASSERTION 1. To prove the validity of assertion 1 for $s+1$, assume that $\mathcal{B}^{s+1}(c) = (\text{repar.}) = \mathcal{B}^{s+1}(c^*)$. We must prove that that $c = (\text{repar.}) = c^*$. By the inductive hypothesis, it is enough to prove that $\mathcal{B}^s(c) = (\text{repar.}) = \mathcal{B}^s(c^*)$, the condition $c^{s+1}(0) = (c^*)^{s+1}(0)$ automatically following from $c^{s+2}(0) = (c^*)^{s+2}(0)$.

The KR coordinates associated to the curves c and c^* agree up to level $s + 2$ because $c^{s+2}(0) = c^{*s+2}(0)$. Write these coordinates as $u_i, i = 1, \ldots, s + 2$ and the corresponding coordinate functions of the curves as $U_i(t)$ and $U_i^*(t)$. Thus $U_i(t) = U'_{\beta_i}(t)/U'_{\alpha_i}(t)$, $U_i^*(t) = U^{*\prime}_{\beta_i^*}(t)/U^{*\prime}_{\alpha_i^*}(t)$ and $\alpha_i = \alpha_i^*, \beta_i = \beta_i^*$ for $i \leq s + 1$ According to equation 8.1 of proposition 8.2 we have

$$\mathcal{B}^{s+1}(c) = (U_{\alpha_s}(t), U_{\beta_s}(t)), \mathcal{B}^{s+1}(c^*) = (U^*_{\alpha_s}(t), U^*_{\beta_s}(t))$$
$$\mathcal{B}^{s}(c) = (U_{\alpha_{s-1}}(t), U_{\beta_{s-1}}(t)), \mathcal{B}^{s}(c^*) = (U^*_{\alpha_{s-1}}(t), U^*_{\beta_{s-1}}(t)$$

Warning. We need $c^{s+2}(0) = c^{*s+2}(0)$. The equality $c^{s+1}(0) = c^{*s+1}(0)$ is not enough. For if equality occurs for $s + 1$ but not for $s + 2$ then it may happen that $U_{s+2} = dU_{s+1}/dU_\kappa$ while $dU^*_{s+1} = dU^*_\kappa/dU^*_{s+1}$, which would mean that $(\alpha_{s+1}, \beta_{s+1}) = (\beta^*_{s+1}, \alpha^*_{s+1})$. Such a circumstance can force c and c^* to be inequivalent, despite the fact that $\mathcal{B}^{s+1}(c) = \mathcal{B}^{*s+1}(c^*)$. See Example 8.8 immediately following the proof below.

The proof proceeds by observing that one of the two component functions of $\mathcal{B}^s(c)$ already appears in $\mathcal{B}^{s+1}(c)$ while the other one can be reconstructed from the component functions of $\mathcal{B}^{s+1}(c)$ by an integration. The constant of integration arising for the two curves is the same since $c^{s+1}(0) = c^{*s+1}(0)$.

We proceed with the details by case. First, observe that the pairs (α_i, β_i) associated to c and to c^* are identical for $i \leq s + 1$, since the KR coordinates agree up to level $s+2$. Next note that that the pairs $\{\alpha_s, \beta_s\}, \{\alpha_{s-1}, \beta_{s-1}\}$ have precisely one element in common, this being either $s-1$ or some index $\kappa < s-1$, depending on whether or not $c^s(0)$ is vertical. If $c^s(0)$ is not vertical, then $u_s = du_{s-1}/du_\kappa$ for some index $\kappa < s-1$. In this case we have either $u_{s+1} = du_s/du_\kappa$ or $u_{s+1} = du_\kappa/du_s$ (depending on the verticality of $c^{s+1}(0)$). The shared index is κ. On the other hand, if $c^s(0)$ is vertical, then $u_s = du_\kappa/du_{s-1}$ for some index $\kappa < s-1$. In this case we have either $u_{s+1} = du_s/du_{s-1}$ or $u_{s+1} = du_{s-1}/du_s$ and the shared index is $s-1$.

If $c^s(0)$ is not vertical, so that the shared index is κ we have that $\mathcal{B}^s(c) = (U_\kappa(t), U_{s-1}(t))$ while $\mathcal{B}^{s+1}(c)$ is either $(U_\kappa(t), U_s(t))$ or $(U_s(t), U_\kappa(t))$. Given $\mathcal{B}^{s+1}(c)$ we reconstruct $\mathcal{B}^s(c)$ as follows. The component $U_\kappa(t)$ of $\mathcal{B}^s(c)$ already appears as a component function in $\mathcal{B}^{s+1}(c)$. The other component function U_{s-1} satisifies the relation $U'_{s-1} = U_s U'_\kappa$. Integrating, starting from the value $U_{s-1}(0)$ which we know from $c^s(0)$ we see that

$$U_{s-1}(t) = U_{s-1}(0) + \int_0^t U_s(v)U'_\kappa(v)dv. \tag{8.2}$$

Now suppose that $\mathcal{B}^{s+1}(c)$ and $\mathcal{B}^{s+1}(c^*)$ are equal after a reparameterization ϕ. Then $U^*_\kappa = U_\kappa \circ \phi(t)$ and $U^*_s(t) = U_s \circ \phi(t)$. We must show that $U^*_{s-1} = U_{s-1} \circ \phi$. Plugging in to the integration formula we see that $U^*_{s-1}(t) = U_{s-1}(0) + \int_0^t U_s(\phi(v))U'_\kappa(\phi(v))\phi'(v)dv$. Upon making the subsitution $t = \phi(u)$ the change of variables formula for integration now yields the desired result.

If, on the other hand, $c^s(0)$ is vertical so that the shared index is $s-1$, we proceed as in the preceding paragraph but with the roles of κ and $s-1$ reversed. We have $\mathcal{B}^s(c) = (U_{s-1}(t), U_\kappa(t))$ while $\mathcal{B}^{s+1}(c)$ is either $(U_s(t), U_{s-1}(t))$ or $(U_{s-1}(t), U_s(t))$. Given $\mathcal{B}^{s+1}(c)$ we reconstruct $\mathcal{B}^s(c)$ by noting that we already know the component function U_{s-1}, while U_κ is obtained by integrating the relation $U'_\kappa = U_s U'_{s-1}$. As in preceding paragraph, we see that if $\mathcal{B}^{s+1}(c^*) = \mathcal{B}^{s+1}(c) \circ \phi$ then $\mathcal{B}^s(c^*) = \mathcal{B}^s(c) \circ \phi$. We have proved the validity of assertion 1 by induction.

Validity of Assertion 2. The proof is essentially the same as that of Assertion 1, except that we observe that the integration formula (8.2) is well-defined on the jet level. That is to say, if we add arbitrary terms of the form ϵt^{r+1} to the component functions of $\mathcal{B}^{s+1}(c)$, then the integrand of that formula changes by a term of the form $\epsilon t^r U(0)$ and so upon integrating, the resulting component function changes by a term of the form ϵt^{r+1}. Thus the integration formula holds on the level of r-jets, and the r-jet of $\mathcal{B}^{s+1}(c)$ (together with the knowledge of $c^{s+1}(0)$) determines the r-jet of $\mathcal{B}^s(c)$. Assertion 2 is proved. $\square$

Example 8.8. Set

$$c(t) = (t^2, t^5), \quad c^*(t) = ((5/2)t^3, (3/2)t^5).$$

Then for both c and c^* we have $u_1 = dy/dx$ so that

$$c^1(t) = (t^2, t^5, (5/2)t^3), \quad (c^*)^1(t) = ((5/2)t^3, (3/2)t^5, t^2),$$

so that $c^1(0) = c^{*1}(0)$. Looking at the orders, we see that

$$\mathcal{B}(c)(t) = \mathcal{B}(c^*)(t) = (t^2, (5/2)t^3).$$

But it is false that $c(t) = c^*(t)$. Consequently we cannot replace the hypothesis $c^{s+1}(0) = c^{*s+1}(0)$ by the weaker hypothesis $c^s(0) = c^{*s}(0)$ in Lemma 8.7. What has happened is the "meaning" of the components of $\mathcal{B}^1(c)$ and $\mathcal{B}^1(c^*)$ in terms of (x, y)-coordinates has been reversed: one is $(dy/dx, x)$ the other $(x, dy/dx)$. Note that $c^2(0) \neq c^{*2}(0)$. Indeed $c^2(0)$ is not vertical, and has associated KR coordinate du_1/dx, while $c^{*2}(0)$ is vertical with KR coordinate $u_2 = dx/du_1$.

8.2. Directional blow-up and the maps $\mathbb{E}_{\mathrm{T}}, \mathbb{E}_{\mathrm{V}}, \mathrm{L}$

In this section we relate the directional blow-up with the maps $(a,b) \to \mathrm{L}(a,b)$, $(a,b) \to \mathbb{E}_{\mathrm{T}}^{-1}(a,b), (a,b) \to \mathbb{E}_{\mathrm{V}}^{-1}(a,b)$ from section 3.8.5. Recall these maps:

$$\mathrm{L}(a,b) = \begin{cases} \mathrm{T} \text{ if } b > 2a \\ \mathrm{V} \text{ if } b < 2a \end{cases}, \quad \mathbb{E}_{\mathrm{T}}^{-1}(a,b) = (a, b-a), \ \mathbb{E}_{\mathrm{V}}^{-1}(a,b) = (b-a, a).$$

Here a and b are the integers and $a < b$. If $b = 2a$ then $\mathrm{L}(a,b)$ is not defined.

Given a plane curve germ $c \in \mathcal{P}$ define the integers a_i, b_i as follows:

$$\begin{gathered} \mathcal{B}^i(c) = \big(f_i(t), g_i(t)\big), \\ a_i = \mathrm{ord}(f_i(t)), \quad b_i = \mathrm{ord}\, g_i(t)). \end{gathered} \tag{8.3}$$

Proposition 8.9. *Let $c \in \mathcal{P}$ and let a_i, b_i be integers defined by* (8.3)*. Assume that $p \geq 1$, $a_p \geq 1$, $b_p \neq a_p$ and $b_p \neq 2a_p$. Let $l = \mathrm{L}(a_p, b_p)$.*

(i) $(a_{p+1}, b_{p+1}) = \mathbb{E}_l^{-1}(a_p, b_p)$;

(ii) *If $l = \mathrm{V}$ then $c^{p+2}(0)$ is a vertical point;*

(iii) *If $l = \mathrm{T}$ and the point $c^{p+1}(0)$ is critical then $c^{p+2}(0)$ is a tangency point.*

Proof. Let u_i and $U_i(t)$ be the KR coordinates and fuctions associated with the curve c. Consider the KR relations at level $(p+1)$:

$$u_{p+1} = \frac{du_{\alpha_{p+1}}}{du_{\beta_{p+1}}}, \quad U_{p+1}(t) = \frac{U'_{\beta_{p+1}}(t)}{U'_{\alpha_{p+1}}(t)}.$$

Let $f_i(t), g_i(t), a_i, b_i$ be as in Proposition 8.9. By Proposition 8.2 one has $U_{\alpha_{p+1}}(t) = f_p(t)$ and $U_{\beta_{p+1}}(t) = g_p(t)$. Therefore

$$\mathrm{ord}\big(U_{\alpha_{p+1}}(t)\big) = a_p, \quad \mathrm{ord}\big(U_{\beta_{p+1}}(t)\big) = b_p.$$

Since $a_p \geq 1$ and $b_p \neq a_p$ then

$$\mathrm{ord}\big(U_{p+1}(t)\big) = b_p - a_p \geq 1.$$

Consider the case $\mathrm{L}(a_p, b_p) = \mathrm{V}$ which holds if $b_p < 2a_p$. In this case

$$\mathrm{ord}\big(U'_{p+1}(t)\big) = b_p - a_p - 1 < a_p - 1 = \mathrm{ord}\big(U'_{\alpha_{p+1}}(t)\big).$$

According to Table 7.2 the point $c^{p+2}(0)$ is vertical. The curve $\mathcal{B}^{p+1}(c)$ has the form

$$\mathcal{B}^{p+1}(c) = \mathcal{B}\big(f_p(t), g_p(t)\big) = \mathcal{B}\big(U_{\alpha_{p+1}}(t), U_{\beta_{p+1}}(t)\big) = \big(U_{p+1}(t), U_{\alpha_{p+1}}(t)\big)$$

and it follows that $(a_{p+1}, b_{p+1}) = (b_p - a_p, a_p) = \mathbb{E}_{\mathrm{V}}^{-1}(a_p, b_p)$.

Consider now the case $\mathrm{L}(a_p, b_p) = T$ which holds if $b_p > 2a_p$. In this case $\mathrm{ord}\big(U'_{p+1}(t)\big) > \mathrm{ord}\big(U'_{\alpha_{p+1}}(t)\big)$. According to Table 7.2 the point $c^{p+2}(0)$ is a tangency point provided that the point $c^{p+1}(0)$ is critical. Independently of the type of the point $c^{p+1}(0)$ the curve $\mathcal{B}^{p+1}(c)$ has the form

$$\mathcal{B}^{p+1}(c) = \mathcal{B}\big(f_p(t), g_p(t)\big) = \mathcal{B}\big(U_{\alpha_{p+1}}(t), U_{\beta_{p+1}}(t)\big) = \big(U_{\alpha_{p+1}}(t), U_{p+1}(t)\big)$$

and it follows that $(a_{p+1}, b_{p+1}) = (a_p, b_p - a_p) = \mathbb{E}_{\mathrm{T}}^{-1}(a_p, b_p)$. □

8.3. Proof of Theorem A for Puiseux characteristics $[\lambda_0; \lambda_1]$

Let c be a plane curve with such a Puiseux characteristic , with $\lambda_1 > 2\lambda_0$. Let

$$\lambda_1 = q\lambda_0 + r, \quad q \geq 2, \quad r < \lambda_0, \quad \omega(\lambda_0, \lambda_0 + r) = (\omega_1, ..., \omega_l).$$

We have to prove that c regularizes at level $(q + l)$ and that the RVT code of c is $(\mathrm{R}^{q-2}\omega_1, ..., \omega_l)$. By Theorem 3.21 we may replace c be any curve RL-equivalent to c, so we may assume that c has the form

$$c: \quad x(t) = t^{\lambda_0}, \quad y(t) = t^{\lambda_1} + \text{h.o.t.}$$

Let u_i and $U_i(t)$ be the KR coordinates and functions associated with c. The KR relations at the first q levels have the form

$$u_1 = \frac{dy}{dx}, \quad u_2 = \frac{du_1}{dx}, \quad \cdots \quad , u_{q-1} = \frac{du_{q-2}}{dx}, \; u_q = \frac{du_{q-1}}{dx}.$$

and it follows that the points $c^i(0), i \leq q$ are regular (see section 7.4).

Define a_i and b_i by (8.3). The curve $\mathcal{B}^{q-1}(c)$ has the form $(t^{\lambda_0}, \; \kappa t^{\lambda_0 + r} + \text{h.o.t.})$ where $\kappa \neq 0$ hence

$$a_{q-1} = \lambda_0, \quad b_{q-1} = \lambda_0 + r.$$

One has $\mathrm{L}(\lambda_0, \lambda_0 + r) = \mathrm{V}$ since $r < \lambda_0$. By Proposition 8.9, (ii) with $p = q - 1$ the point $c^{q+1}(0)$ is vertical. Recall that any critical (in particular entirely critical) RVT code starts with V. Therefore the type of $c^{q+1}(0)$ is $\omega_1 = \mathrm{V}$. By Proposition 8.9, (ii) with $p = q - 1$ one has

$$(a_q, b_q) = E^{-1}_{\omega_1}(a_{q-1}, b_{q-1}) = E^{-1}_{\omega_1}(\lambda_0, \lambda_0 + r).$$

The equation $\omega(\lambda_0, \lambda_0 + r) = (\omega_1, ..., \omega_l)$ implies

$$(\lambda_0, \lambda_0 + r) = E_{\omega_1} \circ E_{\omega_2} \circ \cdots \circ E_{\omega_l}(1, 2)$$

and we obtain

$$(a_q, b_q) = E_{\omega_2} \circ E_{\omega_3} \circ \cdots \circ E_{\omega_l}(1, 2). \tag{8.4}$$

It follows $L(a_q, b_q) = \omega_2$. The point $c^{q+1}(0)$ is critical. By Proposition 8.9, (ii) or (iii) with $p = q$ the point $c^{q+2}(0)$ has type ω_2. Proposition 8.9, (i) with $p = q$ and (8.4) imply

$$(a_{q+1}, b_{q+1}) = E_{\omega_3} \circ E_{\omega_4} \circ \cdots \circ E_{\omega_l}(1, 2).$$

Continuing using Proposition 8.9 in the same way we obtain

$$\begin{aligned} &(a_{q+i}, b_{q+i}) = E_{\omega_{i+2}} \circ \cdots E_{\omega_l}(1, 2), \quad i = 0, 1, ..., l - 2, \\ &(a_{q+l-1}, b_{q+l-1}) = (1, 2) \end{aligned} \tag{8.5}$$

and we obtain that the point $c^{q+i}(0)$ has type ω_i, $i = 1, ..., l$. It remains to show that $(q + l)$ is the regularization level of c. The equation $(a_{q+l-1}, b_{q+l-1}) = (1, 2)$ implies $(a_{q+l}, b_{q+l}) = (1, 1)$ which means that the curve $\mathcal{B}^{q+l}(c)$ belongs to the set $\mathcal{P}(1)$. By Proposition 8.6 the regularization level of c is $q + l$.

8.4. Further properties of the directional blow-up

In this section we formulate and prove two propositions on the operator $\mathcal{B}$. Proposition 8.11 gives an alternative proof of Theorem 2.36 on the regularization of a well-parameterized plane curve (independent of Theorem **A**). Propositions 8.11 and 8.12 will be used in section 8.5 for the proof of Theorem **A** for Puiseux characteristics $[\lambda_0; \lambda_1, ..., \lambda_m]$ with $m \geq 2$. These propositions also will be used in section 8.6 for the proof of Theorem **B**.

To formulate Propositions 8.11 and 8.12 we construct a stratification $\mathcal{P}(i), i = 1, 2, \ldots$ of $\mathcal{P}$ which includes $\mathcal{P}(1) \subset \mathcal{P}$.

DEFINITION 8.10. For $d \geq 1$, let $\mathcal{P}(d)$ denote the union of the following two sets of $\mathcal{P}$. The first subset consists of non-immersed curves whose order of good parameterization is d. The second subset consists of immersed curves $(x(t), y(t))$ having $\mathrm{ord}(x'(t)) = 0$ and $\mathrm{ord}(y'(t)) = d - 1$.

EXAMPLE. A non-immersed plane curve germ of the form $(t^2, at^4 + bt^5 + o(t^5))$ where $b \neq 0$ belongs to $\mathcal{P}(5)$ for any a. An immersed plane curve germ of the form $(t, at^4 + bt^5 + o(t^5))$ belong to $\mathcal{P}(4)$ if $a \neq 0$ and belongs to $\mathcal{P}(5)$ if $a = 0$ and $b \neq 0$.

REMARKS. Definition 8.10 for $d = 1$ coincides with Definition 8.4. $\mathcal{P}(1)$ and $\mathcal{P}(2)$ consist entirely of immersed curve germs. Any curve in $\mathcal{P}$ belongs to some $\mathcal{P}(d)$ and $\mathcal{P}$ is the disjoint union of the $\mathcal{P}(i)$.

PROPOSITION 8.11. *Let $c \in \mathcal{P}(d)$, $d \geq 2$. Then $\mathcal{B}(c) \in \mathcal{P}(d_1)$ for some integer $d_1 < d$.*

Proposition 8.11 is proved in section 8.7.

REGULARIZATION THEOREM 2.36 FROM PROPOSITION 8.11. In section 2.10 we reduced the Regularization Theorem 2.36 to the case of plane curves. Any analytic non-immersed well-parameterized plane curve germ c belongs to the set $\mathcal{P}(d)$ where d is the order of good parameterization of c, up to a change of coordinated $(x, y) \to (y, x)$. By repeated application of Proposition 8.11 there exists a positive integer $k < d$ such that $\mathcal{B}^k(c) \in \mathcal{P}(1)$ and $\mathcal{B}^i(c) \notin \mathcal{P}(1)$ for $i < k$. By Proposition 8.5 the curve c regularizes at level k.

REMARK. As a by-product we have obtained that *the regularization level of a well-parameterized curve is smaller than its order of good parameterization.*

PROPOSITION 8.12. *Let*

$$d \geq 2, \quad c^* \in \mathcal{P}(d), \quad \mathcal{B}(c^*) \in \mathcal{P}(d_1)$$

as per Proposition 8.11*, so that $d_1 < d$. Let r, $\mu \geq 1$ and let c be a plane curve germ of the form*

$$c(t) = (\text{repar.}) = c^*(t^\mu) + \big(0,\ bt^{d\cdot\mu+r} + \text{h.o.t.}\big), \quad b \in \mathbb{R}. \tag{8.6}$$

Then

$$\mathcal{B}(c)(t) = (\text{repar.}) = \mathcal{B}(c^*)(t^\mu) + \big(0,\ \kappa bt^{d_1\cdot\mu+r} + \text{h.o.t.}\big), \quad \kappa \neq 0. \tag{8.7}$$

The components of $\mathcal{B}(c^*)$ are in the same relation to those of $c^* = (x^*(t), y^*(t))$ as the components of $\mathcal{B}(c))$ are in relation to those of $c = (x(t), y(t))$:

$$\mathcal{B}(c^*) = \Big(x^*(t),\ (y^*)'(t)/(x^*)'(t)\Big) \implies \mathcal{B}(c) = \Big(x(t),\ y'(t)/x'(t)\Big),$$
$$B(c^*) = \Big((y^*)'(t)/(x^*)'(t),\ \ x^*(t)\Big) \implies \mathcal{B}(c) = \Big(y'(t)/x'(t),\ x(t)\Big).$$

Proposition 8.12 is proved in section 8.7.

8.4.1. Examples. The following examples illustrate how Propositions 8.11 and 8.12 work together.

Example 8.13. Take $c^* = (t^3, t^7)$ so that $c^*(t) \in \mathcal{P}(7)$ and $d = 7$ in Proposition 8.11. Then $\mathcal{B}(c^*) = (t^3, (7/3)t^4) \in \mathcal{P}(4)$ so that $d_1 = 4$ in that Proposition. Now, with Proposition 8.12 in mind, set

$$c(t) = c^*(t^2) + (0,\ bt^{17} + \text{h.o.t.}),$$

corresponding to the values $\mu = 2, r = 3$ of the Proposition. Compute

$$\mathcal{B}(c) = (t^6, (7/3)t^8 + (17/6)bt^{11} + \text{h.o.t.})$$

illustrating that the Proposition holds, with $\kappa = 17/6$.

In the previous example, the right hand side of equation (8.7) held with no reparameterization necessary. The next example illustrates that sometimes a reparameterization is necessary for that equation to be valid. We proceed with a more telegraphic notation than that of the previous example.

Example 8.14. Proposition 8.11:

$$c^* = (\text{repar.}) = (t^3, t^5) \in \mathcal{P}(5) \text{ so } d = 5.$$
$$\mathcal{B}(c^*) = (\text{repar.}) = ((5/3)t^2, t^3) \in \mathcal{P}(3) \text{ so } d_1 = 3.$$

Proposition 8.12: Taking this c^*, and $\mu = 2, r = 3$ we have

$$c(t) = c^*(t^2) + (0,\ bt^{13} + \text{h.o.t.}) = (\text{repar.}) = (t^6, t^{10} + bt^{13} + \text{h.o.t.})$$

and compute

$$\mathcal{B}(c) = ((5/3)t^4 + (13/6)bt^7 + \text{h.o.t.},\ t^6). \tag{8.8}$$

But, according to equation (8.7) of Proposition 8.12 we have

$$\mathcal{B}(c) = (\text{repar.}) = \big((5/3)t^4,\ t^6 + \kappa bt^9 + \text{h.o.t.}\big), \quad \kappa \neq 0. \tag{8.9}$$

To see that curve (8.8) can indeed be reparameterized to have the form (8.9) use the reparameterization $t \to t(1 - \delta bt^3 + \text{h.o.t.})$ where $\delta = (13/6) : (20/3)$.

Example 8.15. In Proposition 8.11 take

$$c^* = (\text{repar.}) = (t^3, t^4) \in \mathcal{P}(4) \text{ so } d = 4. \tag{8.10}$$

Compute that

$$\mathcal{B}(c^*) = (\text{repar.}) = ((4/3)t, t^3) \in \mathcal{P}(3) \text{ so } d_1 = 3; \tag{8.11}$$

Now, in Proposition 8.12 take

$$\mu = 2, r = 3, \quad c = c(t^2) + (0, bt^{11} + \text{h.o.t.}) = (t^6,\ t^8 + bt^{11} + \text{h.o.t.}) \tag{8.12}$$

And compute

$$\mathcal{B}(c) = (\text{repar.}) = \big((4/3)t^2 + (11/6)bt^5,\ t^6\big). \tag{8.13}$$

On the other hand by Proposition 8.12 with $\mu = 2, r = 3$

$$\mathcal{B}(c) = (\text{repar.}) = \big((4/3)t^4,\ t^6 + \kappa b t^9 + \text{h.o.t.}\big),\quad \kappa \neq 0. \tag{8.14}$$

This means that curve (8.13) can be reparameterized into the form (8.14). In fact, (8.8) can be brought into the form (8.9) by a reparameterization of the form $t \to t(1 - \delta b t^3 + \text{h.o.t.})$ where $\delta = (11/6) : (4/3)$.

We proved Theorem **A** for Puiseux characteristics of minimal length in section 8.3. Its proof for Puiseux characteristics of any length, and the proof of Theorem **B** are based on the following corollaries of the Propositions 8.11 and 8.12.

8.4.2. Corollaries. Let c^* and c be plane curves as in Proposition 8.12. Let $k \geq 3$ be the regularization level of c^*. Let d_i be the positive integers such that $\mathcal{B}^i(c) \in \mathcal{P}(d_i)$. By Proposition 8.5 d_i form a decreasing list, so there is a first k such that $d_k = 1$. Iterating Proposition 8.12 k times we obtain the following corollary.

PROPOSITION 8.16. *Let $c^* \in \mathcal{P}(d)$ and let $k \geq 3$ be the regularization level of c^*. Let $r, \mu \geq 1$ and let c be a plane curve germ of the form*

$$c = (\text{repar.}) = c^*(t^\mu) + \big(0,\ b t^{d\cdot\mu + r} + \text{h.o.t.}\big),\quad b \in \mathbb{R}. \tag{8.15}$$

Then the first k directional blow-ups of c have the form

$$\begin{aligned}\mathcal{B}^i(c) = (\text{repar.}) = \mathcal{B}^i(c^*)(t^\mu) + \big(0,\ \kappa_i b t^{d_i\mu + r} + \text{h.o.t.}\big),\\ i \leq k-1,\ \kappa_i \neq 0,\ d_i \geq 2;\end{aligned} \tag{8.16}$$

$$\mathcal{B}^k(c) = (\text{repar.}) = \mathcal{B}^k(c^*)(t^\mu) + \big(0,\ \kappa b t^{\mu + r} + \text{h.o.t.}\big),\ \kappa \neq 0. \tag{8.17}$$

Moreover, the pairs of indices (α, β) appearing in $\mathcal{B}^{i+1}(c) = (U_{\alpha_i}, U_{\beta_i})$ and $\mathcal{B}^{i+1}(c^*) = (U^*_{\alpha^*_i}, U^*_{\beta^*_i})$ are identical: $\alpha_i = \alpha^*_i$ and $\beta_i = \beta^*_i$ for $i \leq k$.

Let c, c^*, d, μ, r be as in Proposition 8.16. By the final parts of this Proposition and of Proposition 8.12 and by the definition of KR coordinates, we see that the KR coordinates associated to c and to c^* are equal up to level k. Since $r \geq 1$, equations (8.16) - (8.17) and Proposition 8.2 imply

$$j^{d_i\mu}\Big(U_{\alpha_{i+1}},\ U_{\beta_{i+1}}\Big) = (\text{repar.}) = j^{d_i\mu}\Big(U^*_{\alpha^*_{i+1}}(t^\mu),\ U^*_{\beta^*_{i+1}}(t^\mu)\Big),\quad i \leq k-1. \tag{8.18}$$

$$j^{\mu}\Big(U_{\alpha_{k+1}},\ U_{\beta_{k+1}}\Big) = (\text{repar.}) = j^{\mu}\Big(U^*_{\alpha^*_{k+1}}(t^\mu),\ U^*_{\beta^*_{k+1}}(t^\mu)\Big). \tag{8.19}$$

Since either $\alpha_i = i-1$ or $\beta_i = i-1$ (for any i), equations (8.18) and (8.19) also imply that for $i \leq k$ one has $U_i(0) = U^*_i(0)$. The fact that k is the regularization level of c^* gives more. It implies $\alpha^*_{k+1} = \alpha^*_k$ and by Proposition 7.14 $U'_{\alpha_k}(0) \neq 0$. Now equation (8.19) implies $u_{k+1} = u^*_{k+1}$ and $U_{k+1}(0) = U^*_{k+1}(0)$. These conclusions hold for any $r \geq 1$, in particular for $r = 1$. We obtain the following statement.

PROPOSITION 8.17. *Let $k \geq 3$ be the regularization level of a plane curve germ $c^* \in \mathcal{P}(d)$, and μ a positive integer. If c is a plane curve germ whose $(d\mu)$-jet satisfies: $j^{d\mu}(c(t)) = (\text{repar.}) = j^{d\mu}(c^*(t^\mu))$ then $c^{k+1}(0) = (c^*)^{k+1}(0)$.*

Consider now the case $\mu = 1$. The following statement is a direct corollary of Propositions 8.16 and 8.17.

Proposition 8.18. *Let $c^* \in \mathcal{P}(d)$ and let $k \geq 3$ be the regularization level of c^*. Let $r \geq 1$ and let c be a plane curve germ of the form*

$$c(t) = (\text{repar.}) = c^*(t) + \big(0,\ bt^{d+r} + \text{h.o.t.}\big), \quad b \in \mathbb{R}. \tag{8.20}$$

Then $c^{k+1}(0) = (c^*)^{k+1}(0)$ and the curve $\mathcal{B}^k(c)$ has the form

$$\mathcal{B}^k(c) = (\text{repar.}) = \mathcal{B}^k(c^*) + \big(0,\ \kappa bt^{1+r} + \text{h.o.t.}\big),\ \kappa \neq 0.$$

Finally, we will need the following corollary of Proposition 8.18.

Proposition 8.19. *Let $c^* \in \mathcal{P}(d)$ and let $k \geq 3$ be the regularization level of c^*. Let $q \geq 1$ and let C be a plane curve germ such that $j^qC = j^q\mathcal{B}^k(c)$. There exists a plane curve germ c such that $j^{d+q-1}c = j^{d+q-1}c^*$ and $\mathcal{B}^k(c) = C$.*

Proof. The existence of c on the level of formal power series follows from Proposition 8.18 applied with $r = 1, 2, \ldots$. The convergence of this power series can be easily obtained from the condition that $C = \mathcal{B}^k(c)$ is an analytic curve (see the proof of Lemma 8.7). □

8.5. Proof of Theorem A for arbitrary Puiseux characteristics

In this section we prove Theorem **A** for an arbitrary Puiseux characteristic $\Lambda = [\lambda_0; \lambda_1, ..., \lambda_m]$ by induction on its length $m+1$. The case $m = 1$ was proved in section 8.3 and we take this as the base step of the induction. We now prove the inductive step, from $(m-1)$ to m.

A plane curve germ with the Puiseux characteristic Λ is RL-equivalent to a curve of the form

$$c = c^*(t^\mu) + (0, bt^{\lambda_m} + \text{h.o.t.}), \quad b \neq 0,$$
$$\mu = \text{g.c.d.}(\lambda_0, ..., \lambda_{m-1}),$$

where c^* is a well-parameterized plane curve germ with the Puiseux characteristic

$$\text{Puiseux characteristic of } c^* : \quad [\frac{\lambda_0}{\mu}; \quad \frac{\lambda_1}{\mu}, \ldots, \frac{\lambda_{m-1}}{\mu}],$$

see section 3.8.5.2. Assume that c^* regularizes at level k and has RVT code

$$\text{RVT code of } c^* = (\alpha_1, ..., \alpha_{k-2}). \tag{8.21}$$

To prove the inductive step we must show that c has RVT code

$$\text{RVT code of } c = \big(\alpha_1, ..., \alpha_{k-2}\mathrm{R}^{s+1}\omega(\mu, \mu + \mu_1)\big), \tag{8.22}$$

where s and μ_1 are defined by the equation

$$\lambda_m = \lambda_{m-1} + s\mu + \mu_1, \quad s \geq 0, \quad \mu_1 < \mu.$$

Lemma 8.20. *If $3 \leq i \leq k$ then the point $c^i(0)$ has the type α_{i-2}. The point $c^{k+1}(0)$ is regular.*

Proof. The order of good parameterization of a plane curve is the last number in its Puiseux characteristic . Hence $c^* \in \mathcal{P}(d)$ with $d = \lambda_{m-1}/\mu$. One has $\lambda_m = \mu d + s\mu + \mu_1$. Since $\mu_1 > 0$, by Proposition 8.17 one has $c^{k+1}(0) = (c^*)^{k+1}(0)$ and consequently $c^i(0) = (c^*)^i(0)$ for $i \leq k+1$. The assumption that c^* has RVT code (8.21) implies that the point $(c^*)^i(0)$ has the type α_{i-2} as $3 \leq i \leq k$ and that the curve $(c^*)^k$ is regular. Therefore the point $(c^*)^{k+1}(0)$ is regular. □

Thus the RVT code of c starts with $\alpha_1, ..., \alpha_{k-2}\mathrm{R}$. To prove (8.22) we have to prove that its remaining part is $\mathrm{R}^s\omega(\mu, \mu+\mu_1)$.

Let u_i and $U_i(t)$ be the KR coordinates and the KR coordinate functions associated with c. Fix the indexes α_i, β_i such that $u_i = du_{\beta_i}/du_{\alpha_i}$, $i \le k+1$. Consider the curve

$$C = \mathcal{B}^{i-1}(c) = \big(U_{\alpha_k}(t), U_{\beta_k}(t)\big).$$

Let $\widehat{U}_i(t)$ be the KR functions associated with C. Then, according to the construction in section 7.4 and Proposition 8.2 one has $U_{k+i}(t) = \widehat{U}_{1+i}(t)$, $i \ge 0$. It follows that the curves c and C are related as follows:

1. For $i \ge 2$ the point $(c)^{k+i}(0)$ has the same type as the point $C^{i+1}(0)$.
2. For $i \ge 1$ the curve $(c)^{k+i}$ is regular if and only if the curve C^{i+2} is regular.

It follows that to prove that the remaining part of the RVT code of c is $\mathrm{R}^s\omega(\mu, \mu+\mu_1)$ one has to prove that

$$\text{RVT code of } C = \mathrm{R}^s\omega(\mu, \mu+\mu_1). \tag{8.23}$$

Theorem **A** for Puiseux characteristics $[\lambda_0; \lambda_1]$ of length two (the base step of the induction, proved in section 8.3), reduces (8.23) to the claim:

$$\text{Puiseux characteristic of } C = [\mu;\ (s+2)\mu + \mu_1]. \tag{8.24}$$

To prove (8.24) we will use Proposition 8.16. Observe the following:

1. Since $c^{k+1}(0) = (c^*)^{k+1}(0)$ the KR coordinates $u_i, i \le k+1$ associated with c^* are the same as the KR coordinates associated with c.
2. The integers d_i in (8.16) are defined by the condition $\mathcal{B}^i(c^*) \in \mathcal{P}(d_i)$. Since $\mathcal{B}^k(c^*) \in \mathcal{P}(1)$ and $\mathcal{B}^{k-1}(c^*) \notin \mathcal{P}(1)$ (by Proposition 8.5), it follows that $\mathcal{B}^{k-1}(c^*) \in \mathcal{P}(2)$, i.e. $d_{k-1} = 2$.

These observations, equation (8.16) with $i = k-1$, and Proposition 8.2 imply the equation

$$\begin{aligned} C &= \big(U_{\alpha_k}(t),\ U_{\beta_k}(t)\big) = (\text{repar.}) = \\ &\big(U^*_{\alpha_k}(t^\mu),\ U^*_{\beta_k}(t^\mu) + \kappa t^{(s+2)\mu+\mu_1} + \text{h.o.t.}\big), \quad \kappa \ne 0, \end{aligned} \tag{8.25}$$

where $U^*_i(t)$ are the KR coordinate functions associated with the curve c^*. By Proposition 7.14 $(U^*_{\alpha_k})'(0) \ne 0$. Therefore the curve (8.25) is RL-equivalent to a curve of the form $\big(t^\mu,\ \pm t^{(s+2)\mu+\mu_1} + \text{h.o.t.}\big)$ and (8.24) follows.

8.6. Proof of Theorem B of section 4.8

Let c^* be a plane curve germ whose order of good parameterization is m and whose regularization level is $k \ge 3$. We have to prove the following statements for any $k \ge 1$ and any plane curve germ c and any positive integer q:

$$j^{m+q-1}c = (\text{repar.}) = j^{m+q-1}c^* \implies j^q c^k = (\text{repar.}) = j^q(c^*)^k; \tag{8.26}$$

$$j^q c^k = (\text{repar.}) = j^q(c^*)^k \implies j^{m+q-1}c = (\text{repar.}) = j^{m+q-1}c^*. \tag{8.27}$$

Proof of (8.26). There is no loss of generality in assuming that $c^* \in \mathcal{P}(m)$. The assumption $j^{m+q-1}c = (\text{repar.}) = j^{m+q-1}c^*$ implies

$$c = (\text{repar.}) = c^* + (0,\ f(t)), \quad j^{m+q-1}f(t) = 0.$$

By Proposition 8.18 $c^{k+1}(0) = (c^*)^{k+1}(0)$ and $j^q\mathcal{B}^k(c) = (\text{repar.}) = j^q\mathcal{B}^k(c^*)$. Now by Lemma 8.7 with $r = q, s = k$ one has $j^q c^k = (\text{repar.}) = j^q(c^*)^k$.

Proof of (8.27). The assumption

$$j^q c^k = (\text{repar.}) = j^q(c^*)^k \tag{8.28}$$

and Proposition 8.19 imply that there exists a plane curve $\tilde{c}$ such that

$$j^{m+q-1}\tilde{c} = j^{m+q-1}c^*, \tag{8.29}$$

$$\mathcal{B}^k(\tilde{c}) = (\text{repar.}) = \mathcal{B}^k(c). \tag{8.30}$$

Let us prove that these equations imply

$$\tilde{c} = (\text{repar.}) = c \tag{8.31}$$

and consequently $j^{m+q-1}c = (\text{repar.}) = j^{m+q-1}c^*$ as required. Lemma 8.7 states that (8.30) implies (8.31) provided that

$$\tilde{c}^{k+1}(0) = c^{k+1}(0). \tag{8.32}$$

Thus we have to prove (8.32). To prove it, we will prove that

$$c^{k+1}(0) = (c^*)^{k+1}(0), \quad \tilde{c}^{k+1}(0) = (c^*)^{k+1}(0). \tag{8.33}$$

By equation (8.29) and implication (8.26) (just proved), but with c replaced by $\tilde{c}$, we have

$$j^q\tilde{c}^k = (\text{repar.}) = j^q(c^*)^k. \tag{8.34}$$

Since $q \geq 1$, equations (8.28) and (8.34) imply

$$j^1 c^k = (\text{repar.}) = j^1(c^*)^k, \quad j^1\tilde{c}^k = (\text{repar.}) = j^1(c^*)^k. \tag{8.35}$$

Since k is the regularization level of c^*, the curve $(c^*)^k$ is immersed and (8.35) implies the required equations (8.33).

8.7. Proof of Propositions 8.10 and 8.11

Note that if $c \in \mathcal{P}$ then $\tilde{c} = c \circ \phi \in \mathcal{P}$ and $\mathcal{B}(\tilde{c}) = \mathcal{B}(c) \circ \phi$ for ϕ a reparameterization. This observation allows us to replace c^* in Proposition 8.11 and 8.12 by any curve c related to it by a reparameterization.

Consider first the case that $c^* \in \mathcal{P}(d)$ is immersed. In this case we may assume that it has the form

$$c^* = (x^*(t), y^*(t)) = \big(x_0 + t,\ y_0 + at^d + \text{h.o.t.}\big), \quad a \neq 0.$$

The curve $\mathcal{B}(c^*)$ has the form

$$\mathcal{B}(c^*) = \big(x_0 + t,\ dat^{d-1} + \text{h.o.t.}\big).$$

Since $d \geq 2$ then Proposition 8.11 holds with $d_1 = d - 1$. Let now

$$c = c^*(t^\mu) + \big(0,\ bt^{d\cdot\mu+r} + \text{h.o.t.}\big) = \big(x(t), y(t)\big) = \big(x_0 + t^\mu,\ y_0 + at^{d\cdot\mu} + bt^{d\cdot\mu+r} + \text{h.o.t.}\big).$$

This curve is immersed if $\mu = 1$ and non-immersed if $\mu \geq 2$. Since $d \geq 2$ then in either of these cases

$$\mathcal{B}(c) = \big(x(t),\ y'(t)/x'(t)\big) = c(t^\mu) + \big(0,\ \kappa_1 bt^{(d-1)\cdot\mu+r} + \text{h.o.t.}\big), \quad \kappa_1 = (d\mu + r)/\mu.$$

Since $(d-1) = d_1$, Proposition 8.12 holds with $\kappa = \kappa_1$.

Now consider the case that $c^* \in \mathcal{P}(d)$ is singular. After a reparameterization, we may put c^* in to the form

$$
\begin{aligned}
x^*(t) &= x_0 + a_0 t^{\lambda_0}, \quad a_0 \neq 0 \\
y^*(t) &= y_0 + a_1 t^{\lambda_1} + \cdots + a_m t^{\lambda_m} + \text{h.o.t.}, \quad a_i \neq 0
\end{aligned} \tag{8.36}
$$

where $[\lambda_0; \lambda_1, ..., \lambda_m]$ is a Puiseux characteristic . By Example 4.28 the order of good parameterization of c^* is λ_m. Thus $d = \lambda_m$.

The new function U^* appearing as one of the components of $\mathcal{B}(c^*)$ has the form

$$
U^*(t) = \frac{(y^*)'(t)}{(x^*)'(t)} = b_1 t^{\lambda_1 - \lambda_0} + \cdots + b_m t^{\lambda_m - \lambda_0} + \text{h.o.t.}, \quad b_i \neq 0. \tag{8.37}
$$

Set

$$
\lambda^* = \begin{cases} \lambda_1 - \lambda_0 \text{ if } \lambda_1 > \lambda_0 \\ \lambda_2 - \lambda_0 \text{ if } \lambda_1 = \lambda_0. \end{cases}
$$

for the smallest nonzero exponent occuring in the expansion of U^*. We proceed by cases, according to the size of λ^*.

Case 1: $\lambda^* \geq \lambda_0$. Then $\mathcal{B}(c^*) = (x^*(t), U^*(t))$, is not immersed, and has Puiseux characteristic $[\lambda_0, \lambda^*, \ldots, \lambda_m - \lambda_0]$. According to Example 4.28 the order of good parameterization of $\mathcal{B}(c^*)$ is $\lambda_m - \lambda_0$ which is less than $d = \lambda_m$ and Proposition 8.11 holds with

$$
d_1 = \lambda_m - \lambda_0 = d - \lambda_0.
$$

Moving on to the proof of Proposition 8.12 in this case, consider a curve of the form

$$
c = (x(t), y(t)), \quad x(t) = x^*(t^\mu), \quad y(t) = y^*(t^\mu) + b t^{\mu \lambda_m + r} + \text{h.o.t.} \tag{8.38}
$$

The function $U(t) = y'(t)/x'(t)$ is of the form

$$
\begin{aligned}
U(t) = y'(t)/x'(t) = U^*(t^\mu) + \kappa_2 t^{\mu(\lambda_m - \lambda_0) + r} + \text{h.o.t.}, \\
\kappa_2 = (\mu \lambda_m + r)/a_0 \mu \lambda_0.
\end{aligned} \tag{8.39}
$$

We have $\mathcal{B}(c) = (x(t), U(t))$ and since $\lambda_m - \lambda_0 = d_1$, Proposition 8.12 holds with $\kappa = \kappa_2$.

Case 2: $\lambda^* = 1$. In this case $x^*(t) = x_0 + a_0 t^{\lambda_0}$, $U^*(t) = u_0 + a^* t + \text{h.o.t.}$ with $a^* \neq 0$ and $\mathcal{B}(c^*) = (U^*(t), x^*(t))$ is immersed and belongs to $\mathcal{P}(\lambda_0)$. Proposition 8.11 holds with

$$
d_1 = \lambda_0 < d = \lambda_m.
$$

Moving on to Proposition 8.12 in this case, consider a curve c of the form (8.38) with corresponding function U of the form (8.39). Note that

$$
\text{ord}(U'(t)) = \mu - 1 < \text{ord}(x'(t)) = \mu \lambda_0 - 1.
$$

Therefore the curve $\mathcal{B}(c)$ has the form (8.7) and can be reparameterized into the form:

$$
\big(U^*(t^\mu), \quad x^*(t^\mu) + \kappa_4 b t^{\mu(\lambda_m - \lambda^*) + r} + \text{h.o.t.}\big), \quad \kappa_4 \neq 0. \tag{8.40}
$$

To prove Proposition 8.12 we have to show that when $\lambda^* = 1$ one has $\lambda_m - 1 = \lambda_0$. To show this, it suffices to note that g.c.d. $(\lambda_0, \lambda_0 + 1) = 1$ and consequently within the case $\lambda^* = 1$ there are only two possibilities: either $m = 1$ and $\lambda_1 = \lambda_0 + 1$, or $m = 2$, $\lambda_1 = \lambda_0$, $\lambda_2 = \lambda_0 + 1$.

CASE 3: The remaining case is that of $2 \leq \lambda^* < \lambda_0$. In this case $\mathcal{B}(c^*) = (U^*(t), x^*(t))$ which is a singular curve with $U^*(t)$ of the form of equation 8.37. At this point we need a lemma

LEMMA 8.21. *Let $(u(t), v(t))$ be an analytic plane curve germ with power series expansion*

$$u(t) = a_1 t^{n_1} + a_2 t^{n_2} + \dots a_m t^{n_m} + \dots, \qquad v(t) = bt^{\lambda_0}$$

with all the a_i and b non-vanishing. Suppose that $n_1 < \lambda_0$ and that n_m is the smallest exponent occurring in the expansion of u that is relatively prime to λ_0. Then the order of good parameterization of the curve is $n_m + \lambda_0 - n_1$.

PROOF. Introduce a new parameter τ such that $u(t) = a_1\tau^{n_1}$. The reparameterization has the form

$$t = \tau\big(1 + f(\tau^{n_1}) + c\tau^{n_m - n_1} + o(t\tau n_m - n_1)\big), \quad c \neq 0$$

where $f(\cdot)$ is some function of one variable. This reparameterization brings the curve $(u(t), v(t))$ to the form

$$u = a_1\tau^{n_1}, \quad v = b\tau^{\lambda_0} \cdot \big(1 + f(\tau^{n_1}) + c\tau^{n_m - n_1} + o(t^{n_m - n_1})\big)^{\lambda_0}, \quad c \neq 0.$$

Note that the smallest exponent occurring in this expansion of v which is relatively prime to the exponent n_1 is $n_m + (\lambda_0 - n_1)$. By Example 4.28 and the RL invariance of the order of good parameterization number, we have that the order of good parameterization is $n_m + \lambda_0 - n_1$ as claimed. $\square$

According to this lemma, the order of good parameterization of $\mathcal{B}(c^*)$ in Case 3 is then $(\lambda_m - \lambda_0) + \lambda_0 - \lambda_* = \lambda_m - \lambda_*$, which is less than $d = \lambda_m$. Proposition 8.12 holds with

$$d_1 = \lambda_m - \lambda_*.$$

Moving on to the proof of Proposition 8.12 in this case, consider the curve (8.38) and the function (8.39). The curve $\mathcal{B}(c)$ has the form

$$\mathcal{B}(c) = (U(t), x(t)) = \big(U^*(t^\mu) + \kappa_2 b t^{\mu(\lambda_m - \lambda_0) + r} + \text{h.o.t.},\ x^*(t^\mu)\big). \tag{8.41}$$

Note that $U^*(t^\mu) - U(0) = b^* t^{\mu\lambda^*} + \text{h.o.t.}$ where $b^* \neq 0$. Therefore the first component in $\mathcal{B}(c)$ can be brought to $U^*(t^\mu)$ by a reparameterization of the form

$$t \to t\big(1 + \kappa_3 b t^{\mu(\lambda_m - \lambda_0 - \lambda^*) + r} + \text{h.o.t.}\big), \quad \kappa_3 \neq 0.$$

The function $x^*(t^\mu) - x(0)$ has the form $a_0 t^{\mu\lambda_0} + \text{h.o.t.}$ so this reparameterization brings the $\mathcal{B}(c)$ to the form

$$\big(U^*(t^\mu), \quad x^*(t^\mu) + \kappa_4 b t^{\mu(\lambda_m - \lambda_*) + r} + \text{h.o.t.}\big), \quad \kappa_4 \neq 0.$$

Since $\lambda_m - \lambda_* = d_1$, Proposition 8.12 holds with $\kappa = \kappa_4$.

CHAPTER 9

Open questions

9.1. Unfolding versus prolongation.

The A_2-singularity is represented by the plane curve $c(t) = (t^2, t^3)$. Its unfolding is

$$c_\epsilon(t): \; x(t) = t^2, \; y(t) = t^3 - \epsilon t.$$

As $\epsilon \to 0$ the curve c_ϵ tends to the curve c in the C^∞ Whitney topology. However its prolongation fail to converge to the prolongation of c.

We can remedy this failure of convergence by replacing the prolongation c^1 of c by the multi-curve $\hat{c}^1 = im(c^1) \cup \mathbb{P}^1_0 \subset \mathbb{P}^1\mathbb{R}^2$, viewed as subset of $\mathbb{P}^1\mathbb{R}^2$. Here $im(c^1)$ is the image of the old prolongationc^1 and $\mathbb{P}^1_0$ is the fiber over the singular point 0. Then $c^1_\epsilon \to \hat{c}^1$ in the sense of Hausdorff convergence of unparameterized subsets of $\mathbb{P}^1\mathbb{R}^2$. The multicurve $\hat{c}^1$ is the algebraic geometric blow-up of c. The original prolongation c^1 parameterizes the "proper transform" of c . The fiber $\mathbb{P}^1_0$ is the "exceptional fiber" in blow-up terminology, and can be thought of as the prolongation of the singular point 0.

We can do a little bit better than Hausdorff (setwise) convergence by observing a few details of the shape of c_ϵ. The point $t = \pm\sqrt{\epsilon}$ is a double point ($\epsilon > 0$) and the arc of c_ϵ between these values of t is a kink whose tangent lines rotate through an angle of a little bit over 180 degrees. In other words, in the short time $2\sqrt{\epsilon}$ the prolongation of c^ϵ has gone once around the $\mathbb{RP}^1$-fiber of $\mathbb{P}^1\mathbb{R}^2 \to \mathbb{R}^2$. Rescale ("blow up") time according to $\epsilon\tau = t$. Then the interval $[-\sqrt{\epsilon}, \sqrt{\epsilon}]$ becomes $[-1/\sqrt{\epsilon}, 1/\sqrt{\epsilon}]$ and in the rescaled time the fiber coordinate $u = dy/dx$ of c^1_ϵ behaves like $u = -1/\tau + O(\epsilon)$. Consequently the prolongation of $\phi_\epsilon(\tau) = c_\epsilon(\tau/\epsilon)$ tends to $\mathbb{P}^1_0$ as a parameterized curve. Outside of the kink, we use the original time t, and $c^{(1)}_\epsilon(t)$ converges away from $t = 0$ to $c^1(t)$.

This example suggests several questions concerning the relations between unfolding and prolongation.

Question 9.1. Does an analogue of the picture for the A_2 unfolding just described exist for the unfolding of any plane curve singularity? Can one construct a general framework under which prolongation of an unfolding becomes continuous in the unfolding parameter?

9.2. Prolongation = Blow-up?

The preceding example suggests a close relation between prolongation and the blow-up of algebraic geometers. Both procedures are methods for resolving singularities. By resolving a plane curve singularity we will mean some process for replacing the singular curve by a non-singular curve in a new space, together with a projection from that new space to the plane containing the original curve. We showed (Theorem 2.36, see also Chapter 8) that prolongation is a method of resolution: after a finite number of prolongations a non-constant singular analytic plane curve becomes a regular curve. Blow-up (reviewed below) is the algebraic geometer's method of resolving singularities.

QUESTION 9.2. What are the relations between resolution of singularities by blow-up and by prolongation? Can a precise correspondence be set up between blown-up curves and prolonged curves? How does one attach and keep track of the analogues of exceptional fibers in prolongation?

The two methods of resolution have different starting points as to what is meant by "a curve". In blow-up, one takes the algebraic geometer's point of view: a "curve" is the zero level set of a function $f(x, y)$ on the plane, typically a polynomial function. In prolongation as we defined it, coming from a differential topology perspective, curves are taken to as parameterized sets $t \mapsto (x(t), y(t))$. To match up the methods, we must match up their starting points. We could, with little difficulty, generalize the definition of prolongation so that it applies to curves defined via level sets of analytic functions. Alternatively we could use the Newton-Puiseux method to parameterize a curve as a level set. In the Newton-Puiseux expansion we typically have to choose a number of different parameterizations to cover a neighborhood of a singular point, since typical singular points have several different branches of curves coming in to them, each branch with its own separate parameterization (Puiseux expansion). Think of the figure eight. If we take the Newton-Puiseux route, then we deal with prolonging "multi-curves" the finite unions of parameterized curves. Each branch of the multi-curve is then prolonged separately and the results put together to form the resolved multi-curve.

In algebraic geometry the variables t, x, y of the paragraph above are typically viewed as complex variables, or even coming from an arbitrary field. The function $f(x, y)$ defining the curve is then a polynomial over that field. Prolongation can still be defined. Derivatives of polynomials still make sense, but strange problems can occur with derivatives over fields of characteristic p. If we switch to $\mathbb{C}$ from $\mathbb{R}$, the Monster tower can be defined in a manner identical to what was done in this book. All the variables, lines, curves, germs, etc. are simply complex. The same thing appears to work over an arbitrary field but we have not checked. But in the discussion below we are thinking of the case where the field is either $\mathbb{R}$ or $\mathbb{C}$.

REVIEW OF BLOW-UP. Suppose we have a curve C lying on a surface $S = S_0$ and having a singular point a point $p_0 \in C \subset S$. We construct a new surface $S_1 = \mathrm{Bl}(S_0; p_0)$, called the blow-up of S_0 at p_0, together with a map $\beta_1 : S_1 \to S_0$, called the blow-down map, which is a diffeomorphism everywhere except over p_0. The fiber $\beta^{-1}(p_0)$ over p_0 is a $\mathbb{P}^1$ and is called the exceptional fiber. The blow-up of C is the inverse image $\beta^{-1}(C)$ and consists of two parts, the exceptional fiber, and the rest, called "the proper transform" of C which is the closure of $\beta^{-1}(C \setminus \{p_0\})$.

We describe blow-up in the case $S_0 = \mathbb{R}^2$ is the plane and p_0 is the origin 0. Then we define $S_1 = \mathrm{Bl}(\mathbb{R}^2, 0) \subset \mathbb{P}^1\mathbb{R}^2$ to consist of those pairs $(point, line)$ such that the line *passes through* the origin. Let (x, y, u) be our standard coordinates on $\mathbb{P}^1\mathbb{R}^2$ so that (x, y) coordinatize the plane and u is the slope of a line through the point (x, y). If $(point, line)$ are in S_1 then the line must pass through the origin so the line is defined by

$$y = ux \tag{9.1}$$

The pair (x, u) are coordinates on $\mathrm{Bl}(\mathbb{R}^2, 0)$. Their coordinate neighborhood is all of $\mathrm{Bl}(\mathbb{R}^2, 0)$ with the exception of those pairs (point, line) of the form $((0, y), y-\text{axis})$. The relation (9.1) simultaneously defines $\mathrm{Bl}(\mathbb{R}^2, 0)$ as a subvariety of $\mathbb{P}^1\mathbb{R}^2$ and tells us what the blowdown map is:

$$\beta_1(x, u) = (x, xu) = (x, y). \tag{9.2}$$

The exceptional fiber is is the set of all pairs (point, line) where the point is 0. It is the fiber of $\mathbb{P}^1\mathbb{R}^2 \to \mathbb{R}^2$ over the origin.

If $C \subset \mathbb{R}^2$ is a parameterized curve $(x(t), y(t))$ in the plane with singular point at the origin, and with $ord(x) \leq ord(y)$ so that its limiting tangent line is not the y-axis, then in the blown up coordinates C is represented by $(x(t), u(t))$ with $u(t) = y(t)/x(t)$. $p = 0$. These, with the limit $t \to 0$ then parameterize the proper transform of C.

Difference between methods. We can now see one of the main analytic difference between blow-up and prolongation. In blow-up, the "new" coordinate function of the new curve is related to the coordinates $(x(t), y(t))$ of the old curve by quotient: $u(t) = y(t)/x(t)$, while in prolongation, this new coordinate function is related by quotients *of derivatives* $u_1(t) = y'(t)/x'(t)$. Geometrically speaking, in blow-up we replace the curve by its pointed secant lines (through the origin) while in prolongation we replace it by its pointed tangent lines. (By a "pointed line" we mean a line, with a point on that line.)

Continuing with the review of Blow-up. To blow-up a surface S_0 at a point p_0 we choose local coordinates (x, y) centered at that point, and proceed as if the coordinate patch for x, y were the whole plane. The fact that the blow-down map is a diffeomorphism away from the singular fiber shows us how to attach the blown-up coordinate patch to the "old" surface.

We return to the general setting of a curve $C \subset S_0$ on a surface with singular point $p_0 \in C$. After one blow-up the proper transform of C is typically still singular. We may have to perform blow up a number of times: $S_k \to S_{k-1} \to \ldots S_i \to \ldots S_0 = (S, p_0)$ before the proper transform of C becomes non-singular. Here $S_{i+1} = \mathrm{Bl}(S_i, p_i)$ and $p_i \in C_i$ is a singular point on the ith blow-up C_i of C. One stops the procedure when the proper transform is smooth, and all intersections of additional fibers along the way are "normal".

Definition 9.3. A multi-curve consisting of some finite number of embedded curves is called "normal" if all intersections among its components are transverse and there are no triple intersections.

Observe that the definition of "normal" also makes sense for integral multi-curves in rank 2 distributions, such as our Δ^i.

The method of blow-up requires selecting singular points p_i along the way. It can happen that there may be several choices of such points available at certain steps. In this case the final resolution can depend on these choices. Hence the resolution by blow-up is not always unique.

Comparing the Results. As we have defined it, the result of resolution by prolongation of a planar curve germ is an immersed curve germ in some high-dimensional manifold. In resolution by blow-up, the proper transform of the curve is an immersed integral curve germ in a surface. There is not much to compare: a piece of immersed curve germ one place is basically the same as a piece of integral curve any other place. The only real difference between the two resolutions is then embedding dimension: $i+2$ versus 2, where i is the regularization level. There seems to be nothing of interest to say.

To obtain results of interest, **we must keep track of the exceptional curves** created along the way, and their combinatorics. The prolongation analogue of an exceptional curve will be a critical curve: either a vertical curve, or one of its prolongations, a tangency curve. Let us agree that the prolongation of a point is the vertical curve over it. Then, in prolonging a singular curve, we agree to take, along with the prolongation as we have been defining it, the prolongation of every singular point along the curve. The result of this new prolongation is a multi-curve, and it agrees with the $\hat{c}^1$ as discussed at the beginning of the present chapter. Now, we have interesting objects to compare: curves with many branches, or "multi-curves" and the associated combinatorics of their intersections.

The simplest class of singularities with which to compare the two methods are the "unibranched" singularites: those consisting of a single branch. The simplest examples of these are $x^p - y^q = 0$, with p, q relatively prime. For unibranched singularities no choices appear in blow-up: at each step there is but one singular point. At each step exactly one new exceptional curve is created. The crucial features of a unibranched singularitiy are encoded by its Puiseux characteristic .

In both methods the resolution defines a graph. The edges represent the exceptional curves (or critical curves in the case of prolongation) created along the way, with one extra edge for the proper transform (the old prolongation). Vertices represent intersections.

Theorem 9.4 (Swaminathan). *For a unibranched plane singular curve germ the resolution graphs for prolongation and blow-up are isomorphic.*

This theorem comprises the thesis of Vidya Swaminathan, UCSC. Her thesis was started while we were finishing the first draft of the current book.

In formulating this theorem, care had to be taken in defining incidence of edges for the prolongation graph. Transverse (within the distribution!) intersections of integral curves always vanish upon one prolongation, but in blow-up such intersections persist, unless the intersection point agrees with the point being blown up. It follows that care must be taken in defining and keeping track of intersections for the prolongation graph. We do this by by recording the level and way in which the intersection occurs relative to the "proper prolongation".

In algebraic geometry, the edges of the graph can be labelled by two integers: the "degree" and "self-intersection number". We have found prolongation analogues of these integers. The theorem goes on to assert that the two graphs are isomorphic as labelled graphs.

Does this theorem hold for general algebraic singularities? Almost certainly not! In resolution by blow-up of multi-branched singularities one typically has to make choices along the way of which point to blow up at next, with the end results being different. There are various resolution graphs in blow-up. There is only one resolution graph in prolongation.

There seem to be at least two ways for salvaging uniqueness out of resolution by blow-up. One is through the concept of "minimal resolution". The other is what is known as the "stable reduction". Both seem to be achievable by "shrinking" a resolution by blow-up as described above. See [**Ha**] and references therein. Does resolution by prolongation correspond to one of these two resolutions?

Perhaps it is worth going deeper in to the subject than proving that the two labelled graphs isomorphic. With either method, the end result is a "curve" in a manifold, where "curve" can be interpreted either in the differential topological sense of a multi-curve: a union of embedded curves, or in an algebraic geometric sense. Are these two "curves" actually "isomorphic", rather than simply their graphs being isomorphic?

If we take the differential topological viewpoint then it seems that the answer is "yes": the graph of the curve determines its isomorphism class. Isomorphism takes some care to define, because the curves lie in ambient spaces of different dimensions, 2 for blow-up, and $2+i$ for prolongation. Such a definition of isomorphism can be made by making use of submersions of neighborhoods of the curves onto a third surface.

If we want to take the modern algebraic geometric viewpoint , which is to say, using schemes, then we are out of our depth. Neither of us trained in algebraic geometry. "Curve" would mean one-dimensional scheme. One would have to go back to the drawing board and define the prolongation of one-dimensional schemes. Is there sufficient motivation, beyond generality for generality's sake? The book of Kollar may be useful here [**Ko**], Do we even want to bother defining prolongation for curves in characteristic p? We would like to thank Ravi Vakil for suggesting these thoughts.

A Warning. We saw that the first blow-up $Bl(\mathbb{R}^2, 0)$ sat inside the first level of the Monster. In our initial attempts to relate the prolongation and blow-up methods we tried to embed the higher blow-ups of the plane into the higher levels of the Monster, in such a way that the prolongations of the curve could be isotoped to its proper transforms. This approach appears to be impossible. There are plane curve singularities whose first prolongations are "non-planar" – meaning they do not lie on *any* embedded surface. The prolongation of such a curve cannot be isotoped to its blow-up, which is planar. An example of such a curve is provided by $(t^3, t^{10} + t^{11})$.

9.3. Puiseux characteristic of Legendrian curves

The results from the beginning of section 3.8 allow us to define the Puiseux characteristic of a singular Legendrian curve germ.

Definition 9.5. Let γ be a well-parameterized non-immersed Legendrian curve germ in a contact manifold (M^3, ω), where ω is a contact form. Let c be a plane curve germ satisfying the following conditions:

a). the one-step-prolongation c^1 is a Legendrian curve germ in $\mathbb{P}^1\mathbb{R}^2$ which is RL-contact equivalent to γ, i.e. there exists a local diffeomorphism $\Phi : \mathbb{P}^1\mathbb{R}^2 \to M^3$

and a local reparameterization $\phi : \mathbb{R} \to \mathbb{R}$ such that Φ brings the contact structure described by the 1-form ω to the canonical contact structure Δ^1 and $\gamma = \Phi \circ c^1 \circ \phi$.

b). c is a well-parameterized plane curve germ of the form

$$(at^q + \text{h.o.t.},\ bt^p + \text{h.o.t.}),\quad q \geq 2,\ p > 2q,\quad a, b \neq 0.$$

The Puiseux characteristic of the plane curve germ c will be called the Puiseux characteristic of the Legendrian curve germ γ.

Theorem 9.6. *Definition 9.5 is correct: given a Legendrian curve germ γ there is a plane curve germ c satisfying a) and b) of the definition above. All such plane curve germs have the same Puiseux characteristic . Thus the Puiseux characteristic of a Legendrian curve germ is an invariant with respect to the RL-contact equivalence.*

Proof. The existence of c satisfying a) and b) follows from Lemma 3.22. The fact that the Puiseux characteristic does not depend on the choice of c and that it is an invariant with respect to the RL-contact equivalence follows from Theorem 3.21 and Theorem **A** of section 3.8 . In fact, any plane curve germ satisfying b) has Puiseux characteristic of the form $[\lambda_0; \lambda_1, ..., \lambda_m]$ where $\lambda_1 > 2\lambda_0$. Let c and $\tilde{c}$ be plane curve germs with Puiseux characteristic Λ and $\widetilde{\Lambda}$ of this form and such that c^1 and $\tilde{c}^1$ are RL-contact equivalent curves. Then by Theorem 3.21 the curves c and $\tilde{c}$ have the same RVT-code and then by Theorem **A** one has $\Lambda = \widetilde{\Lambda}$. □

Thus we have a discrete invariant of a Legendrian curve germ with respect to RL-contact equivalence. The proof of Theorem 9.6 involves the Monster tower whereas Definition 9.5 involves not more than the first level of the Monster.

Question 9.7. Is there a direct proof, not involving the Monster, that the Puiseux characteristic of a Legendrian curve according to Definition 9.5 is an invariant with respect to RL-contact equivalence?

The classical Puiseux characteristic is an invariant not only with respect to the RL-equivalence of analytic plane curves, but also with respect to a weaker equivalence: topological equivalence of their complexifications. See [**W**], where Wall calls two curve germs sharing the same Puiseux characteristic "equisingular" and in Proposition 4.3.8 there gives eight different combinatorial invariants whose constancy is equivalent to equisingularity. In Proposition 5.5.8 he proves that two curve germs are equisingular if and only if the pairs $(D_\epsilon, C), (D_\epsilon, \tilde{C})$ are homeomorphic for ϵ sufficiently small. Here $C, \tilde{C}$ are the images of the two curve germs, and B_ϵ is a small ball about the singular point in the ambient *complex* plane $\mathbb{C}^2$.

In view of this result and Theorem 9.6 it is natural to ask the following question.

Question 9.8. Is there a natural complex contact-topological equivalence for Legendrian curves in $\mathbb{C}^3$, endowed with its standard complex contact structure, with the property that the Puiseux characteristic of a (complex) Legendrian curve (according to Definition 9.5) is a complete invariant with respect to this equivalence?

One candidate for such an equivalence relation is the one generated by the group of homeomorphisms which take Legendrian curves to Legendrian curves.

9.4. The infinite Monster

The infinite Monster $\mathbb{P}^\infty\mathbb{R}^2$ is the projective limit of the finite Monsters $\mathbb{P}^i\mathbb{R}^2$ by way of the sequence of projections $\ldots \to \mathbb{P}^{i+1}\mathbb{R}^2 \to \mathbb{P}^i\mathbb{R}^2 \to \ldots \mathbb{P}^1\mathbb{R}^2 \to \mathbb{R}^2$. A point of the infinite Monster is thus a sequence of points $(p_1, p_2, p_3, ...)$, where $p_i \in \mathbb{P}^i\mathbb{R}^2$ such that $p_i = \pi_{i,j}(p_j)$ whenever $i < j$. Two points $P = (p_1, p_2, ...)$ and $\widetilde{P} = (\tilde{p}_1, \tilde{p}_2, ...)$ of $\mathbb{P}^\infty\mathbb{R}^2$ are called equivalent if for any finite k there exists a local symmetry Φ_k of $\mathbb{P}^k\mathbb{R}^2$ (a local diffeomorphism preserving the canonical 2-distribution Δ^i) sending p_k to $\tilde{p}_k$.

The prolongations of a non-constant analytic Legendrian curve germ γ define a unique point $P = \mathrm{Monster}(\gamma) \in \mathbb{P}^\infty\mathbb{R}^2$ by setting $p_i = \gamma^i(0)$.

Definition 9.9. A point $P = (p_1, p_2, ...) \in \mathbb{P}^\infty$ is regular if there is some i_0 such that for all $j > i_0$ the points p_j are regular points of $\mathbb{P}^j\mathbb{R}^2$.

Theorem 9.10. *Let γ be a non-constant analytic Legendrian curve germ. Then the point* $\mathrm{Monster}(\gamma) \in \mathbb{P}^\infty\mathbb{R}^2$ *is regular.*

Remark. In this theorem the curve γ may be badly parameterized.

Proof. Let $\gamma = \gamma(t)$ be the germ at $t = 0$. Since γ is analytic and not constant, there exists a non-constant map germ $\phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ and a well-parameterized Legendrian curve germ γ such that $\gamma(t) = \tilde{\gamma}(\phi(t))$. By Proposition 2.6 one has $\mathrm{Monster}(\gamma) = \mathrm{Monster}(\tilde{\gamma})$. Since $\tilde{\gamma}$ is analytic and well-parameterized, by Theorem 2.36 there exists k such that the curve γ^k is regular and consequently the point $\gamma^{k+i}(0)$ is a regular point in $\mathbb{P}^{k+i}\mathbb{R}^2$, for any $i \geq 1$ (see Proposition 2.31). Therefore the point $\mathrm{Monster}(\gamma) = \mathrm{Monster}(\tilde{\gamma})$ is a regular point in the infinite Monster. □

Not every point P in the infinite Monster corresponds to an analytic curve. To see this, consider a regular point $P = (p_1, p_2, \dots)$ such that all points p_i are regular. By translating and rotating if necessary, we may suppose that the first three coordinates of P are $x = y = 0$ and $u_1 = dy/dx$. Then the remaining KR coordinates u_j, $j > 1$ satisfy $u_{j+1} = du_j/dx$. An immersed integral curve γ through P (or through any one of its finite truncations p_j) can be parameterized by x, so that $y = y(x)$, and $u_j = d^jy/dx^j$, $j > 1$. It follows that the Taylor expansion of $y = y(x)$ at $x = 0$ is $\Sigma\frac{1}{j!}u_j(0)x^j$ with $u_0 = y$ and the KR coordinates of P can be interpreted as the Taylor coefficients of the real-valued function $y(x)$. If that function is analytic, then $y = \Sigma a_j x^j$ with the $|a_j|$ bounded. It follows that the point P is touched by an analytic curve if and only if its KR coordinates satisfy $j!|u_j| \leq C$ for some constant C.

Question 9.11. It is easy to prove that on the level of formal power series any regular point P of the infinite Monster is the Monsterization of some C^∞ Legendrian curve γ: $P = \mathrm{Monster}(\gamma)$. Is there a nice canonical terms the analytic part $\mathbb{P}^\omega\mathbb{R}^2 \subset \mathbb{P}^\infty\mathbb{R}^2$, i.e. the set of points of $\mathbb{P}^\infty\mathbb{R}^2$ which is the Monsterization of analytic Legendrian curve germs (and so touched by some analytic curve)?

Question 9.12. A non-regular point P of $\mathbb{P}^\infty R^2$ is one whose RVT code does not stabilize on R. Do such points play any geometric or analytic role? The set of these non-regular curves has infinite codimension in $\mathbb{P}^\infty$ and by Theorem 9.10 a non-regular point does not correspond to any analytic curves. Is there some generalized curves whose appropriate Monsterization represents such a non-regular point P?

Theorem 4.40 on the evaluation of the jet-identification number implies that if $P = (p_1, p_2, ...)$ is a regular point in $\mathbb{P}^\infty\mathbb{R}^2$ then for sufficiently large i the jet-identification number of the point p_i is $\mathrm{const} + i$. This suggests

Conjecture 9.13. *Two well-parameterized analytic Legendrian curve germs γ and $\tilde{\gamma}$ are* RL-*contact equivalent if and only if the corresponding regular points* $\mathrm{Monster}(\gamma), \mathrm{Monster}(\tilde{\gamma}) \in \mathbb{P}^\infty\mathbb{R}^2$ *are equivalent.*

9.5. Moduli and projective geometry

As discussed in the introduction, moduli (continuous parameters) occur in the problem of classifying points in $\mathbb{P}^i\mathbb{R}^2$ as soon as $i \geq 8$.

Question 9.14. What is the geometry underlying these moduli? Can they be explained in projective geometric terms?

Here is a plausible beginning to such a projective geometric explanation which has so far resisted all our attempts at completion. The group of symmetries of $\mathbb{P}^i\mathbb{R}^2$ fixing the point $p \in \mathbb{P}^i\mathbb{R}^2$ acts by projective transformations on the projective line $\mathbb{P}\big(\Delta_i(p)\big)$, which is the fiber over p for $\mathbb{P}^{i+1}\mathbb{R}^2 \to \mathbb{P}^i\mathbb{R}^2$. If p is a critical point we know that there are two distinguished points on this line, the vertical point v_p and the tangency point t_p. If we were able to distinguish another such point, say d_p, then we could use the cross-ratio $[a, b, c, d]$ to uniquely parameterize *all* the points of the fiber, and in this way obtain moduli. This parameterization would be done using the projective transformation $\phi : \mathbb{RP}^1 = \mathbb{R} \cup \{\infty\} \to \mathbb{P}\big(\Delta_i(p)\big)$ which takes $\{0, 1, \infty\}$ to $\{v_p, t_p, d_p\}$. This transformation is unique by the fundamental theorem of projective geometry. The real parameter $t \in \mathbb{R} \subset \mathbb{RP}^1$ would form a modulus for all the points $q = \phi(t) \in \mathbb{P}\big(\Delta_i(p)\big)$ of the fiber, with $[0, 1, \infty, t] = [v_p, t_p, d_p, q]$ in cross-ratio terms. All our attempts to distinguish such a third point d_p have so far failed.

9.6. RVT and the growth vector

There is another discrete invariant of points of the Monster besides their RVT class. This invariant is the (small) growth vector of the distribution at that point.

The growth vector at a point p of a distribution D is the sequence $g_1, g_2, ...$, where g_i is the dimension of the space spanned by all vectors of the form

$$[X_1, [X_2, [X_3, ..., X_j]]...](p), \quad X_1, ..., X_j \in D, \quad j \leq i,$$

where $X_1, ..., X_j$ are germs at p of vector fields. For Goursat 2-distributions on $\mathbb{R}^n$ (as well as for all non-holonomic distributions on n-manifolds) $g_l = n$ for some finite l and so the growth vector is an l-tuple of integers starting with 2 and ending with n. Some of the integers in the tuple might be repeated. The number l as well as the growth vector g may depend on the point p.

Definition 9.15. The growth vector of a point p in the kth level of the Monster is the growth vector of the Goursat 2-distribution Δ^k. (See the first statement of Theorem 1.2.)

Translating to the Monster known results on the growth vector of Goursat 2-distributions, we obtain the following statements.

THEOREM 9.16 (corollary of Murray's theorem, see [**Mu**]). *Any point in the first level of the Monster has growth vector* $(2, 3)$. *Any point in the second level of the Monster has growth vector* $(2, 3, 4)$. *Let* $k \geq 1$. *A point* p *in level* $(k+2)$, $k \geq 1$, *of the Monster has growth vector* $(2, 3, 4, 5, ..., k + 4)$ *if and only if it belongs to the open* RVT *class* R^k.

The growth vector $(2, 3, 4, 5, ...,)$ is the fastest growth vector for Goursat distributions. The growth vector of a singular point of a Goursat 2-distribution (and consequently of a singular point of the Monster) can be quite complicated. One of examples is $(2, 3, 4, 4, 5, 5, 5, 6, 6, 6, 6, 6, 7, ..., 7, 8, ..., 8, 9)$ where 7 is repeated 8 times and 8 is repeated 13 times. See [**Mor2**].

In [**J**] Jean defined certain classes in the kinematic model of a car pulling i trailers and proved that all configurations of the trailers within a single class are described by Goursat 2-distributions with the same growth vector. The kinematic model of a car pulling i trailers is isomorphic to $\mathbb{P}^{i+1}\mathbb{R}^2$. (See [**MZ**], Appendix D) and the classes defined by Jean coincide, up to this isomorphism, with the RVT classes. Therefore one has the following statement.

THEOREM 9.17 (Corollary of Jean's results in [**J**]). *All points of the Monster in a fixed* RVT *class have the same growth vector.*

It follows that there is a map

$$\mathrm{GW} : (\alpha) \rightarrow \text{ growth vector of some (and hence any) point of } (\alpha)$$

defined on the set of all RVT codes. Jean's results give an explicit recursion formulae for the map GW. In [**Mor2**] Mormul proved that the map GW is injective. He described its image (a rather involved set) and constructed the inverse map. The main result of the work [**Mor2**], translated from Goursat distributions to the Monster, can be formulated as follows.

THEOREM 9.18 (Corollary of Mormul's results in [**Mor2**]). *Two points in the Monster have the same* RVT *code if and only if they have the same growth vector.*

QUESTION 9.19.

1. Is it possible to prove Theorem 9.17 and possibly a stronger Theorem 9.18 in pure Monster terms using the developed geometry of points and curves in the Monster? Such a proof would probably lead to a series of new applications of the growth vector.

2. How does one distinguish growth vectors corresponding to regular RVT classes from those corresponding to critical RVT classes? From those corresponding to entirely critical classes? What is the relation between the growth vector of an RVT class and the growth vector of its regular prolongations?

3. In section 3.8 we constructed the map $(\alpha) \rightarrow \mathrm{Pc}\,(\alpha)$ and its inverse $\Lambda \rightarrow \mathrm{RVT}\,(\Lambda)$ which give canonical 1-1 correspondences between the set of all critical RVT classes and the set of all Puiseux characteristics of the form $[\lambda_0; \lambda_1, ..., \lambda_m]$ with $\lambda_1 > 2\lambda_0$. How do these maps relate to the map $(\alpha) \rightarrow \mathrm{GW}(\alpha)$? Perhaps the composition $\mathrm{GW} \circ \mathrm{RVT}$ is a simpler 1-1 correspondence than any of the maps Pc, RVT, or GW?

APPENDIX A

Classification of integral Engel curves

In this appendix we determine and classify the simple singularities of immersed integral curves in an Engel manifold. We include this appendix both for its intrinsic interest, and as an illustration of our methods. The key to applying our method is Theorem 2.2 which reduces the classification of integral curve germs in a Goursat manifold to the classification of Legendrian curve germs in $\mathbb{P}^1\mathbb{R}^2$.

An Engel manifold is a 4-dimensional Goursat manifold, i.e. a 4-manifold M endowed with a rank 2 Goursat distribution D. The Engel normal form theorem asserts that (M, D) is locally diffeomorphic to $(\mathbb{P}^2\mathbb{R}^2, \Delta^2)$. Equivalently, (M, D) is locally diffeomorphic to the *standard Engel space*

$$(\mathbb{R}^4, D), \quad D : dy - u_1 dx = 0, du_1 - u_2 dx = 0.$$

Let (M, D) and $(\widetilde{M}, \widetilde{D})$ be Engel manifolds. Two curve germs $\Gamma : (\mathbb{R}, 0) \to (M, p)$, $\widetilde{\Gamma} : (\mathbb{R}, 0) \to (\widetilde{M}, \tilde{p})$ are called RL-Engel-equivalent if there exists a local diffeomorphism $E : (M, p) \to (\widetilde{M}, \tilde{p})$ sending D to $\widetilde{D}$ and a local reparameterization $\phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $E\big(\Gamma(t)\big) = \widetilde{\Gamma}\big(\phi(t)\big)$. An integral curve (or an Engel curve) is a curve everywhere tangent to D.

We will classify immersed integral curves with respect to RL-Engel-equivalence. The most basic invariant of such a curve germ is its order of contact with the characteristic foliation $\mathcal{F}$. We recall that an Engel manifold is canonically foliated by a family of non-singular curves. The curves of the characteristic foliation $\mathcal{F}$ for the model example $\mathbb{P}^2\mathbb{R}^2$ are the fibers of $\mathbb{P}^2\mathbb{R}^2 \to \mathbb{P}^1\mathbb{R}^2$. And in the standard Engel space the curves of $\mathcal{F}$ are the u_2 curves. (See [**Mon1, Mon2**] for other descriptions of the characteristic foliation.)

We also recall the order of contact between two immersed curves passing through the same point P. The order of contact is zero if their tangent lines at P do not agree. If their tangent lines are equal then we may choose coordinates $(x^1, \ldots, x^n)$ centered at P so that the x^n-axis is this tangent line and one of the curves is the x^n-curve $(x^1 = 0 = \ldots x^{n-1})$. The other curve can then be parameterized as a graph over the x^n-axis: $x^i = c_i(t), i < n$, $x^n = t$. Then the order of contact between the two curves is N if the vector function $(c_1, \ldots c_{n-1})$ has vanishing N-jet but non-vanishing $N+1$-jet.

The order of contact between an immersed curve germ Γ and the foliation $\mathcal{F}$ is the order of contact at $P = \Gamma(0)$ between Γ and the leaf of $\mathcal{F}$ through P.

Theorem A.1 determines and classifies all *simple germs* of immersed integral curves in an Engel manifold. An integral curve germ Γ is called *simple* if there exists a $k < \infty$ such that some integral k-jet neighborhood of γ is covered by a finite number of RL-Engel-equivalence classes. By an integral k-jet neighborhood we mean a set consisting of all integral curve germs whose k-jet at $t = 0$ is sufficiently close to that of γ.

THEOREM A.1. *An immersed integral curve germ in an Engel manifold is simple if and only if its order of contact with the characteristic foliation does not exceed* 3. *Any such simple curve germ is* RL-*Engel-equivalent to one and only one of the curve germs in the standard Engel space presented in Table* A.1.

TABLE A.1. Classification of simple immersed integral curve germs. ord denotes the order of contact between an integral curve and the characteristic foliation

Normal form	ord
$\Gamma_0: \quad x = t,\ y = u_1 = u_2 = 0$	0
$\Gamma_1: \quad x = t^2,\ y = t^5,\ u_1 = y'(t)/x'(t),\ u_2 = u_1'(t)/x'(t)$	1
$\Gamma_2: \quad x = t^3,\ y = t^7,\ u_1 = y'(t)/x'(t),\ u_2 = u_1'(t)/x'(t)$	2
$\Gamma_3^{\pm}: \quad x = t^4,\ y = t^9 \pm t^{11},\ u_1 = y'(t)/x'(t),\ u_2 = u_1'(t)/x'(t)$	3
$\Gamma_3^0: \quad x = t^4,\ y = t^9,\ u_1 = y'(t)/x'(t),\ u_2 = u_1'(t)/x'(t)$	3

The proof is based on the following lemma. Consider the projection

$$\pi : \mathbb{R}^4(x, y, u_1, u_2) \to \mathbb{R}^3(x, y, u_1), \quad \pi(x, y, u_1, u_2) = (x, y, u_1)$$

from the standard Engel space to the contact 3-manifold $(\mathbb{R}^3(x, y, u_1), dy - u_1 dx)$.

LEMMA A.2. *Let $\Gamma, \widetilde{\Gamma} : (\mathbb{R}, 0) \to (\mathbb{R}^4, 0)$ be integral curve germs in the standard Engel space such that $\pi(\Gamma(t)) \not\equiv 0$ and $\pi(\widetilde{\Gamma}(t)) \not\equiv 0$. The germs Γ and $\widetilde{\Gamma}$ are* RL-*Engel-equivalent if and only if the germs $\pi(\Gamma(t))$ and $\pi(\widetilde{\Gamma}(t))$ are* RL-*contact equivalent with respect to the contact structure $dy - u_1 dx$.*

PROOF. Lemma A.2 is the case $k = 1$ of Theorem 2.2. □

Continuing with the proof of the theorem, consider the germ

$$\Gamma : (\mathbb{R}, 0) \to (\mathbb{R}^4, 0), \quad \Gamma(t) = \big(x(t), y(t), u_1(t), u_2(t)\big)$$

of an immersed integral curve in the standard Engel space. The leaves of the characteristic foliation $\mathcal{F}$ in the standard Engel space are the fibers of the projection π. If the order of contact between Γ and $\mathcal{F}$ is N then $\pi(\Gamma) = \big(x(t), y(t), u_1(t)\big)$ has zero N-jet and non-zero $(N + 1)$-jet.

The case $N = 0$. If $N = 0$ then $\pi(\Gamma)$ is an immersed Legendrian curve. All such Legendrian curves are locally RL-contact equivalent. By Lemma A.2 Γ is RL-Engel-equivalent to the curve germ Γ_0 of Table A.1.

The case $N > 0$. Suppose that $N > 0$. Write $O(t^k)$ for the set of function germs with zero $(k - 1)$-jet and non-zero k-jet. Suppose $x(t) \in O(t^q), y(t) \in O(t^p)$. The condition for γ to be integral is that $y'(t) = u_1(t)x'(t)$ and $u_1'(t) = u_2(t)x'(t)$. These equations imply that $p \geq 2q+1$, $u_1(t) \in O(t^{p-q})$ and $u_2(t) \in O(t^{p-2q})$. Since $\mathrm{ord}(\Gamma, \mathcal{F}) = N > 0$, the function germs $x(t), y(t), u_1(t)$ have zero N-jet and at least one of them has non-zero $(N+1)$-jet. It follows that $q = N+1$. Since $N > 0$ and Γ is immersed we must have $p - 2q = 1$. It follows that Γ has the form

$$x(t) = O(t^{N+1}),\ y(t) = O(t^{2N+3}),\ u_1(t) = y'(t)/x'(t),\ u_2(t) = u_1'(t)/x'(t).$$

Consider the set of germs of Legendrian curves

$$Q_N: \quad x(t) \in O(t^{N+1}),\ y(t) \in O(t^{2N+3}),\ u_1(t) = y'(t)/x'(t)$$

in the contact space $\left(\mathbb{R}^3(x, y, u_1), dy - u_1 dx\right)$. Theorem A.1 and the remaining $(N > 0)$ entries of Table A.1 now follow from Lemma A.2 and the following results on the contact classification of Legendrian curves in $(\mathbb{R}^3, dy - u_1 dx)$:

(1) Any Legendrian curve germ in Q_1 is RL-contact equivalent to the germ $\pi(\gamma_1)$ where γ_1 is the curve in Table A.1;

(2) Any Legendrian curve germ in Q_2 is RL-contact equivalent to the germ $\pi(\gamma_2)$, where γ_2 is the curve in Table A.1;

(3) Any Legendrian curve germ in the set Q_3 is RL-contact equivalent to one and only one of the germs $\pi(\gamma_3^{\pm})$, $\pi(\gamma_3^0)$, where $\gamma_3^{\pm}$ and γ_3^0) are curves in Table A.1;

(4) There is a modulus in the RL-contact classification of Legendrian curve germs in the set Q_4.

These statements are proved in Appendix B. See Examples B.4 and B.5 for the proof of statement (1). Statement (2) is a part of Proposition 5.6 which is proved in sections B.3 and B.4. Statement (3) is a part of Proposition B.13 (see the fourth row of Table B.2).

APPENDIX B

Contact classification of Legendrian curves

In this appendix we reduce the local classification problem for Legendrian curve germs or their jets to the corresponding problems for plane and space curves. We then use known solutions to these latter problems to prove Proposition 5.5 and Proposition 5.6.

Recall from Chapters 1 and 2 that the prolongation of a plane curve is a Legendrian curve in the contact manifold $(\mathbb{P}^1\mathbb{R}^2, \Delta^1)$. It is easy to prove that all Legendrian curve germs (in any contact manifold) can be obtained by such a prolongation, up to a contactomorphism.

Proposition B.1. *Let M^3 be a contact manifold. Any non-constant analytic Legendrian curve germ $\gamma : (\mathbb{R}, 0) \to M^3$ is contactomorphic to the one-step-prolongation c^1 of some plane curve germ c.*

To say that γ is contactomorphic to c^1 means that there exists a local diffeomorphism $\Phi : \mathbb{P}^1\mathbb{R}^2 \to M^3$ which brings the contact structure on M^3 to the contact structure Δ^1 and which brings the curve c^1 to the curve γ: $\gamma = \Phi \circ c^1$.

Proof. By the classical Darboux theorem stating that all contact structures are locally diffeomorphic the curve γ is contactomorphic to some Legendrian curve germ $\tilde{\gamma} : (\mathbb{R}, 0) \to \mathbb{P}^1\mathbb{R}^2$. Let c be the circle bundle projection of γ to $\mathbb{R}^2$. If c is not a constant curve then by Proposition 2.6 $\tilde{\gamma} = c^1$ and we are done. Consider the case that c is a constant curve. Take local coordinates x, y centered at the point $c(0)$ and such that the fiber coordinate $u = dy/dx$ is well-defined near the point $\gamma(0)$. (See section 7.1.2). In the coordinates x, y, u the contact structure Δ^1 is described by the 1-form $dy - udx$ and $\tilde{\gamma} = (0, 0, u(t))$. The map $(x, y, u) \to (u, -x, y - xu)$ is a local contactomorphism which brings γ to the form $\hat{\gamma} = (u(t), 0, 0)$. The projection of $\hat{\gamma}$ to $\mathbb{R}^2$ is a non-constant curve $\hat{c} = (u(t), 0)$, therefore $\hat{\gamma} = \hat{c}^1$. □

B.1. Reduction theorems for curves

Theorem B.2. *If two non-constant analytic plane curve germs are* RL-*equivalent then their first prolongations are* RL-*contact equivalent curve germs.*

Proof. See Proposition 2.8, the case $k = 1$. □

Theorem B.3. *Two analytic Legendrian curve germs are* RL-*contact equivalent if and only if they are* RL-*equivalent.*

Proof. See [**Z1, Z2**]. See [**I**] for the proof in the holomorphic category. □

We illustrate the use of these theorems by a few examples.

EXAMPLE B.4. Any plane curve germ of the form $\big(t^2, t^{2s+1}f(t)\big)$, $f(0) \neq 0$, is RL-equivalent to $A_{2s} = (t^2, t^{2s+1})$. By Theorem B.2, any Legendrian curve germ of the form $\big(t^2, t^{2s+1}f(t)\big)^1$, $f(0) \neq 0$, is RL-contact equivalent to the curve $(A_{2s})^1$.

EXAMPLE B.5. Fix $m \geq 0$. According to [**BG**] the set of plane curve germs of the form $(t^3, t^{3m+7}f(t))$, $f(0) \neq 0$, intersects a finite number of orbits with respect to the RL-equivalence, and the same is true for the set of plane curve germs of the form $(t^3, t^{3m+8}f(t))$, $f(0) \neq 0$. By Theorem B.2, the set of all Legendrian curve germs of the form $(t^3, t^{3m+7}f(t))^1$, $f(0) \neq 0$ and of the form $(t^3, t^{3m+8}f(t))^1$, $f(00 \neq 0$ intersect a finite number of orbits with respect to RL-contact equivalence.

Consider the case $m = 0$. According to [**BG**] any plane curve germ of the form $\big(t^3, t^7 f(t)\big)$, $f(0) \neq 0$ is RL-equivalent to either (t^3, t^7) or to $(t^3, t^7 + t^8)$ while any plane curve germ of the form $\big(t^3, t^8 f(t)$, $f(0) \neq 0$ is RL-equivalent to either (t^3, t^8) or to $(t^3, t^8 \pm t^{10})$. By Theorem B.2, it follows that the set of all Legendrian curve germs of the form $\big(t^3, t^7 f(t)\big)^1$, $f(0) \neq 0$ decomposes into *at most* two orbits, while those of the form $(t^3, t^8 f(t))^1$, $f(0) \neq 0$ decomposes into at most three orbits. We now use Theorem B.3 and some known theorems about space curve singularities to show that in each case there is in fact only one orbit. Upon prolongation

$$\big(t^3, t^7 f(t)\big)^1 = \big(t^3, t^7 f(t), (7/3)f(0) \cdot t^4 + o(t^4)\big); \tag{B.1}$$

$$\big(t^3, t^8 f(t)\big)^1 = \big(t^3, t^8 f(t), (8/3)f(0) \cdot t^5 + o(t^5)\big) \tag{B.2}$$

in our standard contact coordinates, $x, y, u = dy/dx$. According to classification results in [**GH, A2**] any space curve germ of the form (B.1) is RL-equivalent to $(t^3, t^4, 0)$ and any space curve germ of the form (B.2) is RL-equivalent to $(t^3, t^5, 0)$. The RL-contact equivalence of all Legendrian curve germs of the form $(t^3, t^7 + \text{h.o.t.})^1$, respectively of the form $(t^3, t^8 + \text{h.o.t.})^1$ now follows from Theorem B.3.

EXAMPLE B.6. Fix $m \geq 1$. Legendrian curve germs of the form

$$\big(t^4,\ t^{10}f(t^2) + t^{9+2m}g(t)\big)^1, \quad f(0), g(0) \neq 0$$

where m is a fixed integer are all RL-contact equivalent to each other. This follows from Theorem B.3 and the classification results in [**GH, A2**] which imply that any space curve germ of this form is RL-equivalent to $(t^4, t^6 + t^{5+2m}, t^{9+2m})$.

B.2. Reduction theorems for jets

THEOREM B.7. *Fix integers $q < p < r$ and real numbers $b_{p+1}, \ldots, b_{r-1}$. Consider the family of Legendrian curve germs of the form*

$$\gamma_a = \Big(t^q,\ t^p + b_{p+1}t^{p+1} + \cdots + b_{r-1}t^{r-1} + at^r + o(t^r)\Big)^1,\ a \in \mathbb{R}. \tag{B.3}$$

Assume that one or more of the following conditions holds:

(a) $r = k_1 q + k_2 p$ *for some non-negative integers* k_1, k_2*;*

(b) $r = 2p - q$*;*

(c) $r = 2p - 2q$.

Then we can kill the at^r term: γ_a is RL-contact equivalent to a Legendrian curve germ of the form

$$\gamma_0 = \Big(t^q,\ t^p + b_{p+1}t^{p+1} + \cdots + b_{r-1}t^{r-1} + o(t^r)\Big)^1. \tag{B.4}$$

This theorem can be reformulated as follows: under any of the assumptions (a), (b), (c) the $(r-q)$-jets of γ_a is RL-contact equivalent to the $(r-q)$-jet of γ_0.

Example B.8. Consider the Legendrian curve germ $\gamma = (t^5, t^{12} + \cdots)^1$. Since $14 = 2(12-5)$ and $15 = 3 \cdot 5$ then by Theorem B.7 γ is RL-contact equivalent to a Legendrian curve of the form $\big(t^5, t^{12} + bt^{13} + o(t^{15})\big)^1$ for some b. Equivalently the 10-jet of γ is RL-contact equivalent to the 10-jet of the curve $(t^5, t^{12} + bt^{13})^1$.

The proof of Theorem B.7 requires the following lemma from the classification of plane curves.

Lemma B.9. *Fix integers $q < p < r$ and real numbers $b_{p+1}, ..., b_{r-1}$. Consider the family of plane curve germs*

$$c_a: \quad x = t^q,\ y = t^p + b_{p+1}t^{p+1} + \cdots + b_{r-1}t^{r-1} + at^r + o(t^r).$$

In case (a) *or* (b) *of Theorem* B.7 *any plane curve germ of the form c_a is* RL-*equivalent to a plane curve germ of the form c_0.*

Proof. In case (a) a diffeomorphism of the form $(x, y) \to (x, y + \alpha x^{k_1} y^{k_2})$ with a suitable α converts c_a to c_0. In case (b) a reparameterization $t \to t + \alpha t^{p-q+1}$ followed by a linear coordinate change $(x, y) \to (x + ry, y)$ with $\alpha = -a/p$ and $r = -q\alpha$ will convert c_a, $a \neq 0$, to a curve of the form

$$x = t^q + o(t^p),\ y = t^p + b_{p+1}t^{p+1} + \cdots + b_{r-1}t^{r-1} + o(t^{2p-q}).$$

A final reparameterization of the $t \to t + o(t^{p-q+1})$ converts $x(t)$ back to t^q, keeping $y(t)$ the same, up to $o(t^{2p-q})$. □

Proof of Theorem B.7. In cases (a) and (b) Theorem B.7 follows immediately from Theorem B.2 and Lemma B.9. To prove the Theorem in case (c) fix standard contact coordinates, $x, y, u = dy/dx$, and a curve of the form (B.3). Project the curve to the (x, u_1) plane to get

$$x = t^q,$$
$$u_1 = (p/q)t^{p-q} + d_{p-q+1}t^{p-q+1} + \cdots + d_{r-q-1}t^{r-q-1} + \alpha t^{2p-3q} + o(t^{2p-3q}),$$
$$d_i = b_{i+q} \cdot (i+q)/q, \quad \alpha = (2p-2q)a/q.$$

Apply Lemma B.9, and its proof, to this plane curve to see that the parameter α can be reduced to 0 by a reparameterization $t \to \phi(t)$ and a linear shear diffeomorphism $(x, u_1) \to (x + \kappa u_1, u_1)$ with a certain κ. Note that $2p - 3q = 2(p-q) - q$ so the exponents fit the pattern of case (b) in the Lemma. This diffeomorphism is a symplectomorphism of the (x, u_1)-plane with respect to the symplectic structure $dx \wedge du_1$ and consequently induces a contactomorphism Ψ. Explicitly $\Psi(x, y, u_1) = \big(x + \kappa u_1, y + (\kappa/2)u_1^2, u_1)\big)$. Modulo y translations, a Legendrian curve is uniquely determined by its projection to the (x, u_1)-plane. The constant y is fixed by the requirement that the germ pass through $(0, 0, 0)$. It follows that the contactomorphism Ψ and reparameterization ϕ take (B.3) to a Legendrian curve of the form (B.4).

Remark. A more general theorem than Theorem B.7 is proved in [**Z1**].

THEOREM B.10. *Let $q < p < s < r$ and let $s > r - q$. Consider the family of Legendrian curve germs*

$$(t^q, t^p + \delta t^s + \mathbf{b}t^r)^1, \quad \delta \in \{\pm 1, 0\} \tag{B.5}$$

and the family of plane curve germs

$$(t^q, t^{p-q} + \delta t^{s-q} + \mathbf{c}t^{r-q}). \tag{B.6}$$

If $\mathbf{c}$ *is a modulus in* (B.6) *with respect to* RL-*equivalence of* $(r-q)$-*jets of plane curves then* $\mathbf{b}$ *is a modulus in* (B.5) *with respect to* RL-*equivalence* (*and consequently* RL-*contact equivalence*) *of* $(r-q)$-*jets of Legendrian curves.*

PROOF. Since $s > r - q$ then in standard contact coordinates $x, y, u = dy/dx$ the family (B.5) has the form

$$x = t^q, \ y = t^p + o(t^{r-q}), \ u = k_1 t^{p-q} + k_2 \delta t^{s-q} + k_3 \mathbf{b} t^{r-q},$$

where $k_1 = p/q, \ k_2 = s/q, \ k_3 = r/q$. Any curve of this form can be reduced to a curve of the form

$$x = t^q, \ y = o(t^{r-q}), \ u = k_1 t^{p-q} + k_2 \delta t^{s-q} + k_3 \mathbf{b} t^{r-q} \tag{B.7}$$

by the diffeomorphism $(x, y, u_1) \to (x, y - xu/k_1, u)$. It is clear from the bounds on the exponents that the $(r-q)$-jets of two space curves of the form (B.7) are RL-equivalent if and only if the $(r-q)$-jets of two plane curves of the form

$$x = t^q, \ u = k_1 t^{p-q} + k_2 \delta t^{s-q} + k_3 \mathbf{b} t^{r-q} \tag{B.8}$$

are equivalent. Finally, a scaling change of coordinates

$$(t, x, u) \to (\kappa t, \ \kappa^q x, \ \kappa^{p-q} k_1 u)$$

with $\kappa = (k_1/k_2)^{p-s}$ reduces (B.8) to (B.6) with $\mathbf{c} = r \cdot \mathbf{b}$ for certain fixed $r \neq 0$. □

We also need the following

LEMMA B.11. *Let $q < p < r$ and $a \in \mathbb{R}$. The Legendrian curve $(t^q, t^p + at^r)^1$ is* RL-*contact equivalent to one of the curves $(t^q, t^p + \delta t^r)^1$, where $\delta \in \{1, 0\}$ if $r - p$ is odd and $\delta \in \{-1, 1, 0\}$ if $r - q$ is even.*

PROOF. If $a \neq 0$ and $r - p$ is an odd (respectively even) number then a can be reduced to 1 (respectively ± 1) by a reparameterization $t \to kt$ and a contactomorphism $(x, y, u_1) \to (k^{-q}x, k^{-p}y, k^{q-p}u_1)$ for suitable k. □

B.3. Proof of Proposition 5.6, part (i)

For the Legendrian curves $(t^2, t^{2s+5}f(t))^1$, $(t^3, t^7 f(t))^1$, $(t^3, t^8 f(t))^1$, and $(t^4, t^{10} f(t^2) + t^{9+2m} g(t))^1$ (corresponding to the classes $\mathrm{R}^s\mathrm{V}, \mathrm{VT}, \mathrm{VV}, \mathrm{VR}^m\mathrm{V}$) the proof is found in Examples B.4 - B.6 above.

To prove Proposition 5.6, (i) for the Legendrian curves $(t^3, t^{3s+7} f(t))^1$ and $(t^3, t^{3s+8} f(t))^1$, $s \geq 1$ (corresponding to the classes $\mathrm{R}^s\mathrm{VT}$ and $\mathrm{R}^s\mathrm{VV}$) we combine Theorem B.2 with the classification of plane curve germs found in [**BG**] to see that $(t^3, t^{3s+7} + \cdots)^1$ (respectively $(t^3, t^{3s+8} + \cdots)^1$ is RL-contact equivalent to one of the germs $(E_{s,j})^1, (t^3, t^7)^1$ (respectively $(E'_{s,j})^1, (t^3, t^8)^1$), where j varies from 0 to s. Classification results for space curves found in [**GH**] imply that the space curve germs $(E_{s,s})^1$ and $(t^3, t^7)^1$ (respectively $(E'_{s,s})^1$ and $(t^3, t^8)^1$) are RL-equivalent. Now apply Theorem B.3.

It remains to prove Proposition 5.6, (i) for the Legendrian curves $(t^4, t^9 + \cdots)^1$ and $(t^4, t^{11} + \cdots)^1$ (corresponding to the classes VTT and VVT). By Theorem B.7 and Lemma B.11 any Legendrian curve germ of the form $(t^4, t^9 + \cdots)^1$ is RL-contact equivalent to a curve of the form

$$\left(t^4, t^9 + \delta t^{11} + o(t^{11})\right)^1, \ \delta \in \{\pm 1, 0\} \tag{B.9}$$

and any Legendrian germ of the form $(t^4, t^{11} + \cdots)^1$ is RL-contact equivalent to a curve of one of the forms

$$\left(t^4, t^{11} + \delta t^{13} + o(t^{13})\right)^1, \quad \delta \in \{\pm 1\}, \tag{B.10}$$

$$\left(t^4, t^{11} + \delta t^{17} + o(t^{17})\right)^1, \quad \delta \in \{\pm 1, 0\}. \tag{B.11}$$

Fix a δ appropriate to the case, (B.9), (B.10), or (B.11). Applying the classification results of [**GH**] we obtain:

1. A space curve germ of the form (B.9) is RL-equivalent to $(t^4, t^5 + \delta t^7, t^{11})$;
2. A space curve germ of the form (B.10) is RL-equivalent to $(t^4, t^7 + \delta t^9, t^{13})$;
3. A space curve germ of the form (B.11) is RL-equivalent to $(t^4, t^7 + \delta t^{13}, t^{17})$.

Proposition 5.6,(i) follows from these statements and Theorem B.3.

B.4. Proof of Proposition 5.6, part (ii)

We have to prove that the i-jets of two space curve germs $\gamma, \tilde{\gamma}$ of the set $\widehat{\mathrm{Leg}}(\alpha)$ are not RL-equivalent unless $j^i\gamma = j^i\tilde{\gamma}$. This statement, involving no contact structure, can be checked using basic techniques in the classification of plane and space curves.

EXAMPLE B.12. Consider two Legendrian curves of the set $\widehat{\mathrm{Leg}}(\mathrm{VVT})$:

$$\gamma = (t^4, t^{11} + t^{17})^1, \quad \tilde{\gamma} = (t^4, t^{11})^1$$

In standard contact coordinates these curves have the form

$$\gamma: \ x = t^4, \ y = t^{11} + t^{17}, \ u = (11/4)t^7 + (17/4)t^{13};$$
$$\tilde{\gamma}: x = t^4, \ y = t^{11}, \ u = (11/4)t^7.$$

Their i-jets are different if and only if $i \geq 13$. It is easy to show that the 13-jets of γ and $\tilde{\gamma}$ are RL-equivalent to the 13-jets of the curves $(t^4, t^7 + t^{13}, 0)$ and $(t^4, t^7, 0)$ respectively. (To get rid of the t^{11} term use a transformation of the form $(x, y, u) \mapsto (x, y - cxu, u)$.) It is easy to prove that the 13-jets of the plane curves $(t^4, t^7 + t^{13})$ and (t^4, t^7) are not RL-equivalent. See [**BG**]. Therefore the 13-gets of γ and $\tilde{\gamma}$ are not RL-equivalent.

B.5. Proof of Proposition 5.5

The case $q^* = \infty$ is covered by Proposition 5.6, (i), so assume $q^* < \infty$. The sets of Legendrian curves given in Tables 5.1 and 5.2 are repeated in the first column of Tables B.1 and B.2. They are parameterized by one, two, or three function germs $f_i(t)$, $f_i(0) \neq 0$. The number r in the second column of these tables is equal to $r = d + q^* - 1$, where d is the parameterization number from Tables 5.1 and 5.2. Proposition 5.5 is a direct corollary of the following statement.

Proposition B.13. *Consider a row from Table B.1 or Table B.2, and the corresponding set of Legendrian curves Q (first column) and integer r (second column) of that row. In describing Q certain arbitrary functions $f_i(t)$ are used, with $f_i(0) \neq 0$. Then:*

(i) *The r-jet of any Legendrian curve germ in Q is RL-contact equivalent to the r-jet of a Legendrian curve with $f_i(t)$ as given by the third column.*

(ii) *The $(r+1)$-jet of a generic Legendrian curve germ in Q is RL-contact equivalent to the $(r+1)$-jet of a Legendrian curve with $f_i(t)$ as given in the last column. Here "generic" means that in the normal form for the r-jet (the third column) the parameter $\delta \in \{1, 0\}$ takes value 1. If the normal form in the third column does not contain δ then "generic" means "any".*

(iii) *The last column expresses a 1-parameter family of Legendrian curves in Q for which the parameter $\mathbf{b} \in \mathbb{R}$ is a modulus with respect to* RL*-contact equivalence of $(r+1)$-jets of Legendrian curves.*

Proposition B.13, (i) and (ii) is a direct corollary of Theorem B.7 and Lemma B.11. We illustrate with an example.

Example B.14. Proposition B.13, (i) and (ii) the case of the set Q consisting of Legendrian curve germs of the form $(t^5, t^{13}f(t))^1$, $f(0) \neq 0$, which is the 6th row of Table B.1. We have to prove that the following statements for any curve $\gamma \in Q$:

(a) the 11-jet of γ is RL-contact equivalent to the 11-jet of the Legendrian curve $(t^5, t^{13} + \delta t^{14})^1$, $\delta \in \{1, 0\}$.

(b) if in this normal form $\delta \neq 0$ then the 12-jet of γ is RL-contact equivalent to the 12-jet of a Legendrian curve of the form $(t^5, t^{13} + t^{14} + \mathbf{b}t^{17})^1$, $\mathbf{b} \in \mathbb{R}$.

The i-jets of two curves $(t^5, t^{13}f(t))^1$ and $(t^5, t^{13}\tilde{f}(t))^1$ are the same if and only if the $(i+5)$-jets of the functions $t^{13}f(t)$ and $t^{13}\tilde{f}(t)$ are the same. Therefore to prove (a) and (b) we have to prove that any Legendrian curve germ of the form

$$\left(t^5, a_{13}t^{13} + a_{14}t^{14} + a_{15}t^{15} + a_{16}t^{16} + \cdots\right)^1, \quad a_{13} \neq 0$$

is RL-contact equivalent to a Legendrian curve germ of the form

$$\left(t^5, t^{13} + \delta t^{14} + o(t^{16})\right)^1, \quad \delta \in \{1, 0\}$$

i.e. the parameter a_{13} can be reduced to 1, a_{14} can be reduced to δ, and a_{15} and a_{16} can be reduced to 0. The fact that the couple (a_{13}, a_{14}) reduces to $(1, \delta)$ follows from Lemma B.11. The fact that a_{15} can be reduced to 0 follows from Theorem B.7, part (a) because $15 = 5 \cdot 3$. The fact that a_{16} can be reduced to 0 follows from Theorem B.7, part (c) because $16 = 2 \cdot (13 - 5)$.

Proposition B.13, (ii) is a direct corollary of Theorem B.10 and the techniques in [**BG**] for the classification of plane curves. We illustrate with an example.

Example B.15. Proposition B.13, (ii), the case of the set Q consisting of Legendrian curve germs of the form $(t^5, t^{12}f(t))^1$, $f(0) \neq 0$, which is the 4th row of Table B.1. We have to prove that in the family $(t^5, t^{12} + t^{13} + \mathbf{b}t^{16})^1$ the parameter $\mathbf{b}$ is a modulus with respect to RL-contact equivalence of 11-jets of Legendrian curves. Theorem B.10 reduces this statement to the following claim: in the family $(t^5, t^7 + t^8 + \mathbf{c}t^{11})$ the parameter $\mathbf{c}$ is a modulus with respect to RL-equivalence of 11-jets of plane curves. This claim can be easily proved using the techniques in [**BG**].

TABLE B.1. Proof of Proposition 5.5 . The indices: s and m are arbitrary positive integers unless otherwise constrained as per the first column

A set Q of Legen. curves	r	Normal form for j^rQ $\delta \in \{1, 0\}$	Normal form for a generic jet in $j^{r+1}Q$ **b** is a modulus
$\big(t^4,\ \ t^{4s+10}f_1(t^2)+ t^{4s+9+2m}f_2(t)\big)^1$	$4s+2m+6$	$f_1(t)=1$, $f_2(t)=1$	$f_1(t)=1$, $f_2(t)=1+\mathbf{b}t^2$
$\big(t^4, t^{9+4s}f_1(t)\big)^1$	$4s+6$	$f_1(t)=1+\delta t$	$f_1(t)=1+t+\mathbf{b}t^2$
$\big(t^4, t^{11+4s}f_1(t)\big)^1$	$4s+9$	$f_1(t)=1\pm\delta t^2$	$f_1(t)=1\pm t^2+\mathbf{b}t^3$
$\big(t^5, t^{12}f_1(t)\big)^1$	10	$f_1(t)=1+\delta t$	$f_1(t)=1+t+\mathbf{b}t^4$
$\big(t^5, t^{12+5s}f_1(t)\big)^1$	$5s+8$	$f_1(t)=1+\delta t$	$f_1(t)=1+t+\mathbf{b}t^2$
$\big(t^5, t^{13}f_1(t)\big)^1$	11	$f_1(t)=1+\delta t$	$f_1(t)=1+t+\mathbf{b}t^4$
$\big(t^5, t^{13+5s}f_1(t)\big)^1$	$5s+10$	$f_1(t)=1+\delta t$	$f_1(t)=1+t+\mathbf{b}t^3$
$\big(t^6, t^{14}f_1(t^2)+ t^{13+2m}f_2(t)\big)^1$	$2m+8$	$f_1(t)=1$, $f_2(t)=1$	$f_1(t)=1$, $f_2(t)=1+\mathbf{b}t^2$
$\big(t^6, t^{6s+14}f_1(t^2)+ t^{6s+15}f_2(t)\big)^1$	$6s+9$	$f_1(t)=1$ $f_2(t)=1$	$f_1(t)=1$ $f_2(t)=1+\mathbf{b}t$
$\big(t^6, t^{6s+14}f_1(t^2)+ t^{6s+13+2m}f_2(t)\big)^1$, $m\geq 2$	$6s+2m$ $+6$	$f_1(t)=1\pm\delta t$ $f_2(t)=0$	$f_1(t)=1\pm t$ $f_2(t)=\mathbf{b}$
$\big(t^6, t^{6s+15}f_1(t^3)+ t^{6s+13+3m}f_2(t)\big)^1$, $s\geq 0$	$6s+3m$ $+7$	$f_1(t)=1$ $f_2(t)=\pm 1$	$f_1(t)=1$ $f_2(t)=\pm 1+\mathbf{b}t$
$\big(t^6, t^{6s+15}f_1(t^3)+ t^{6s+14+3m}f_2(t)\big)^1$, $s\geq 0$	$6s+3m$ $+9$	$f_1(t)=1$ $f_2(t)=\pm 1$	$f_1(t)=1$ $f_2(t)=\pm 1+\mathbf{b}t^2$
$\big(t^6, t^{16}f_1(t^2)+ t^{15+2m}f_2(t)\big)^1$	$2m+10$	$f_1(t)=1$ $f_2(t)=1$	$f_1(t)=1$ $f_2(t)=1+\mathbf{b}t^2$
$\big(t^6, t^{6s+16}f_1(t^2)+ t^{6s+17}f_2(t)\big)^1$	$6s+12$	$f_1(t)=1$ $f_2(t)=1$	$f_1(t)=1$ $f_2(t)=1+\mathbf{b}t^2$

TABLE B.2. Continuation of Table B.1

A set Q of Legen. curves	r	Normal form for j^rQ, $\delta \in \{\pm 1, 0\}$	Normal form for a generic jet in $j^{r+1}Q$ **b** is a modulus
$\big(t^6, t^{6s+16}f_1(t^2)+$ $t^{6s+19}f_2(t)\big)^1$	$6s+13$	$f_1(t)=1$ $f_2(t)=1$	$f_1(t)=1$ $f_2(t)=1+\mathbf{b}t$
$\big(t^6, t^{6s+16}f_1(t^2)+$ $t^{6s+15+2m}f_2(t)\big)^1$ $m\geq 3$	$6s+2m$ $+8$	$f_1(t)=1\pm\delta t^2$ $f_2(t)=0$	$f_1(t)=1\pm t^2$ $f_2(t)=\mathbf{b}$
$\Big(t^8, t^{8s+20}f_1(t^4)+$ $t^{8s+18+4m}f_2(t^2)+$ $t^{8s+17+4m+2n}f_3(t)\Big)^1$ $s=0, n\geq 1$ or $s\geq 1, n\in\{0,1,2\}$	$8s+4m+$ $2n+8$	$f_1(t)=1$ $f_2(t)=\pm 1$ $f_3(t)\to 0$	$f_1(t)=1$ $f_2(t)=\pm 1$ $f_3(t)\to\mathbf{b}$
$\big(t^5, t^{11}f(t)\big)^1$	8	$f_1(t)=1\pm\delta t^2$	$f_1(t)=1\pm t^2+\mathbf{b}t^3$
$\big(t^8, t^{18}f_1(t^2)+$ $t^{19}f_2(t)\big)^1$	12	$f_1(t)=1$ $f_2(t)=1$	$f_1(t)=1$ $f_2(t)=1+\mathbf{b}t^2$
$\big(t^8, t^{18}f_1(t^2)+$ $t^{21}f_2(t)\big)^1$	13	$f_1(t)=1$ $f_2(t)=1$	$f_1(t)=1$ $f_2(t)=1+\mathbf{b}t$
$\big(t^8, t^{18}f_1(t^2)+$ $t^{17+2m}f_2(t)\big)^1$ $m\geq 3$	$8+2m$	$f_1(t)=1\pm\delta t^2$ $f_2(t)=0$	$f_1(t)=1\pm t^2$ $f_2(t)=\mathbf{b}$

APPENDIX C

Critical, singular and rigid curves

We begin by recalling definitions. A critical curve in the Monster at level j is an integral curve whose projection to one of the lower Monsters $\mathbb{P}^i\mathbb{R}^2$, $j \geq i \geq 1$ is a constant curve. In section 2.2 we showed that any critical curve is either a vertical curve or the prolongation of a vertical curve. The latter are called tangency curves. An integral curve Γ for an arbitrary distribution $D \subset TM$ on a manifold M is called rigid if it admits a C^1 neighborhood such that every integral curve in this neighborhood which shares endpoints with Γ is a reparameterization of Γ. An integral curve Γ is called locally rigid if there exists an $\epsilon > 0$ such whenever $a < b$ are parameter values with $b - a \leq \epsilon$ then the arc $\Gamma|_{[a,b]}$ is rigid. Finally, a curve is called singular (or abnormal) if it is a critical point for the endpoint map. The endpoint map assigns to each integral curve its endpoint. With domain appropriately chosen, the endpoint map becomes a smooth map from a Hilbert manifold (modelled on $L^2([a,b],\mathbb{R}^2)$ in the case of a rank 2 distribution) to the underlying manifold. Consequently it makes sense for a curve to be singular for the endpoint map. Various equivalent characterizations of singular curves can be found in the book [**Mon2**] or the articles [**LS**], [**B**], [**Z4**].

It is well-known that locally rigid curves must be singular. It is also well-known that the vertical curves in $\mathbb{P}^2\mathbb{R}^2$ are locally rigid. Somewhat less well-known is the fact that all vertical curves in all Monsters are locally rigid. (We will recall the proof below.) The main result of this appendix is:

Theorem C.1. *For a C^1* **immersed** *integral curve in the Monster at level i, $i > 1$ the following are equivalent:*

(i) *The curve is critical;*

(ii) *The curve is singular;*

(iii) *The curve is locally rigid.*

Remark. We are no longer in the category of analytic curves in this theorem, but rather in the category of smooth immersed curves. Indeed we only need the curves of the theorem to be C^1. This change is possible because prolongation is well-defined on immersed C^1 curves, and because the projection (or deprolongation) of an integral curve increases its smoothness.

Proof. In section C.1 we prove the implication $(i) \implies (iii)$. The implication $(iii) \implies (ii)$ is well-known. In section C.2 we prove $(ii) \implies (i)$, completing the circle.

C.1. Critical $\Longrightarrow$ locally rigid

In Theorem 2.20 we proved that any analytic critical curve is the prolongation of an analytic vertical curve. That result also holds in the C^1 immersed category, with the same proof. Thus, any C^1 immersed critical curve is the prolongation of an immersed vertical curve which is C^{r+1}, where r is the difference between the levels of the critical curve and the vertical curve. Therefore to prove that "critical $\Longrightarrow$ locally rigid" it suffices to prove that "vertical and immersed $\Longrightarrow$ locally rigid" (subsection C.1.1) and that the prolongation of an immersed locally rigid curve is also locally rigid (subsection C.1.2).

C.1.1. Vertical $\Longrightarrow$ locally rigid. We use the criterion of Bryant-Hsu [**B**]. (See also Liu-Sussmann [**LS**].) The vertical foliation V of $\mathbb{P}^i\mathbb{R}^2$ satisfies $[V, D^2] \subset D^2$ where D^2 is the square $[D, D]$ of the Goursat distribution, $D = \Delta^i$. Also $D^2 \neq D^3$ where $D^3 = [D, D^2]$. It follows from a fairly easy computation then, that each leaf of V, that is to say each vertical curve, is the projection of a regular characteristic for $D^{2\perp} \subset T^*\mathbb{P}^i\mathbb{R}^2$. (These regular characteristics are called "regular abnormal extremals" in [**LS**]). The main theorem of Bryant-Hsu in [**B**] asserts that any such projection is locally rigid.

In the final section of this appendix, in section C.3 we will give an alternative self-contained proof of vertical $\Longrightarrow$ locally rigid in a slightly more general context.

C.1.2. The prolongation of an immersed locally rigid curve is locally rigid. Let ψ be locally rigid curve defined on an interval I and $\Gamma = \psi^{(1)}$ its prolongation. For $t_0 \in I$ choose δ small so that $I_\delta = [t_0 - \delta, t_0 + \delta] \subset I$ and so that the restriction ψ_δ of ψ to I_δ is rigid. Write Γ_δ for the restriction of Γ to I_δ. Let $\widetilde{\Gamma}_\delta$ be any other integral arc defined on I_δ connecting the endpoints of Γ_δ, and let $\widetilde{\psi}_\delta$ be its one step projection. Then $\widetilde{\psi}_\delta$ is an integral arc connecting the endpoints of ψ_δ. As $\widetilde{\Gamma}_\delta$ tends to Γ_δ in the C^1-topology, $\widetilde{\psi}_\delta$ tends to ψ_δ in the C^1-topology. (Indeed, the $\widetilde{\psi}_\delta$ will converge in the C^2 topology, because prolongation adds a derivative.) Rigidity of ψ_δ implies for $\widetilde{\Gamma}_\delta$ sufficiently C^1-close to Γ we have that $\widetilde{\psi}_\delta$ is a reparameterization of ψ, and is immersed. Prolongation is well-defined and unique on immersed curves, so that the prolongation $(\widetilde{\psi}_\delta)^1 = \widetilde{\Gamma}_\delta$ must be a reparameterization of $\psi^1_\delta = \Gamma_\delta$. This establishes the the local rigidity of Γ.

C.2. Singular $\Longrightarrow$ critical

Let Γ be non-critical. We must show it is regular. Being non-critical Γ cannot lie in the singular locus for all time. For if the point $\Gamma(t_0)$ of the curve lies in the singular locus then its projection γ to $\mathbb{P}^1\mathbb{R}^2$ satisfied $\frac{d\gamma}{dt}(t_0) = 0$. So if $\Gamma(t)$ lies in the singular locus for all t then its projection γ is the constant curve, and so Γ would be critical. It follows that there must an open interval along which γ lies in the open RVT class RR...R.

A curve is regular if its restriction to some subarc of itself is regular. So it suffices to show that any immersed integral curve lying in the open class RR...R. Such a curve is is the prolongation of an immersed Legendrian curve. In a neighborhood of a point of such a curve, the Monster $\mathbb{P}^i\mathbb{R}^2$ with its Goursat distribution is diffeomorphic to the space $J^i(\mathbb{R}, \mathbb{R})$ of i-jets of functions $y = y(x)$ with its canonical distribution, in such a way that the curve is mapped to the i-jet of the zero-function $y = 0$. Under this diffeomorphism, the KR coordinates u_j have the meaning of $d^j y/dx^j$. Consider the perturbation of our curve $y = 0$ defined by taking the i-jet of the function $y_\epsilon = \epsilon x^j/j$. For this perturbation $u_j = \epsilon$, $u_r = 0$, $r > j$, and the values of $u_r, r < j$ will not be important. For completeness, we set $u_0 = y$ and the formulae will all still hold. Differentiating with respect to ϵ we obtain a variation vector of the form $\frac{\partial}{\partial u_j} + (*)\frac{\partial}{\partial u_{j-1}} + \cdots (*)\frac{\partial}{\partial u_1} + (*)\frac{\partial}{\partial y}$, where the $(*)$s are coefficients whose precise values are unimportant. Each of these variation vectors is in the image of the differential of the endpoint map. Let j vary. We obtain $i+1$ vectors which, when $\frac{\partial}{\partial x}$ is added to them, form a basis for the entire tangent space of $J^i(\mathbb{R}, \mathbb{R})$. The vector $\frac{\partial}{\partial x}$ can be realized as lying in the differential of the endpoint map by varying the parameterization of the initial curve $x = t, y = 0$. We have established that the differential of the endpoint map is onto, so that the initial curve is regular.

C.3. Another proof that vertical curves are rigid

Here we will give a self-contained proof that vertical curves are locally rigid. The proof holds in the more general context of an n-dimensional manifold M^n $(n > 2)$ endowed with a rank 2 distribution D. Let $\mathbb{P}^1 M^n$ be the manifold obtained from M in the same way as the $(n+1)$-st level of the Monster is obtained from the n-th level. As a manifold, $\mathbb{P}^1 M^n$ is the total space of the projectivization of the vector bundle $D \to M$. $\mathbb{P}^1 M^n$ is endowed with a canonical rank 2 distribution constructed in a manner identical to the way we constructed the distribution on the Monster.

Theorem C.2. *An immersed vertical curve in $\mathbb{P}^1 M^n$ is C^1-rigid provided that* $\operatorname{rank}(D^2) = 3$.

The condition $rank(D^2) = 3$ cannot be removed. It is used in the following lemma on which the proof relies.

Lemma C.3. *Let D be the germ of a 2-distribution on M^n at a point m_0 such that* $\operatorname{rank}(D^2) = 3$. *Let X be any vector field tangent to D and not vanishing at m_0. Then there exists a vector field Y and a function f such that $\{X, Y\}$ frame D near m_0 and such that*

$$f(m_0) = 0,\ X(f) \equiv 0,\ Y(f)(m_0) = 0,\ X\big(Y(f)\big) \equiv 1,\ Y\big(Y(f)\big) \equiv 0. \tag{C.1}$$

Proof. Take any vector field $\widetilde{Y} \in D$ for which $\{X, \widetilde{Y}\}$ frame D. Since $rank(D^2) = 3$ then the vector fields X and $[X, \widetilde{Y}]$ are linearly independent and consequently there exists a function germ f such that $f(m_0) = 0$, $X(f) \equiv 0$, $Y(f)(m_0) = 0$, and $[X, \widetilde{Y}](f) \equiv 1$. Let $h = \widetilde{Y}\big(\widetilde{Y}(f)\big)$. Set $Y = \widetilde{Y} - hX$. Then $\{X, Y\}$ frame D near m_0 and f satisfies (C.1). □

Proof of Theorem C.2. Let

$$\Gamma : t \to (m_0, \ell(t)),\ \ell'(t) \neq 0, \quad t \in [0, 1]$$

be the immersed vertical curve. Choose a vector field X tangent to D with $X(m_0)$ spanning the line $\ell(0)$ at $m_0 \in M$. By Lemma C.3 we can find another vector field Y and a function f, both defined in a neighborhood U of m_0 such that $\{X, Y\}$ frame D and f satisfies (C.1) in this neighbourhood.

The frame $\{X, Y\}$ define fiber coordinates for $\mathbb{P}^1 M$ over U in the usual way. Take any line $\ell \in D(m)$, $m \in U$ and express ℓ as the span of $AX(m) + BY(m)$, $(A, B) \neq (0, 0)$. Then $w = A/B$ is an affine coordinate for the fibers over U, well-defined on a neighbourhood $N \cong U \times (\mathbb{P}^1 \setminus \{[0, 1]\}) \subset \mathbb{P}^1 M$ which contains ℓ. (It contains ℓ since $\ell = span(X(m_0).)$ If $\widetilde{\Gamma}$ is any integral curve in $\mathbb{P}^1 M$ which lies in the neighborhood N then we can expand its derivative out as:

$$\widetilde{\Gamma}'(t) = a_1(t)\partial/\partial w + a_2(t)(X + wY).$$

Differentiate the functions

$$g = Y(f), \; F = f - g^2 \cdot w/2$$

along the curve $\widetilde{\Gamma}$ and use (C.1) to obtain

$$\frac{d}{dt} F\big(\widetilde{\Gamma}(t)\big) = -a_1(t) \cdot g^2(t)/2, \;\; \frac{d}{dt} g\big((\widetilde{\Gamma}(t)\big) = a_2(t). \tag{C.2}$$

Now let Γ_δ be the restriction of our vertical arc Γ to $[0, \delta]$ where δ is chosen small enough so that Γ_δ also lies in V. Along Γ_δ we have $a_2(t) \equiv 0$ because it is vertical and $a_1(t) \neq 0, t \in [0, \delta]$ because it is immersed. If an integral curve $\widetilde{\Gamma}$ is also defined on $[0, \delta]$ and shares the end points with Γ then by (C.1) one has $F(0) = F(\delta) = 0$. If $\widetilde{\Gamma}$ is sufficiently C^1-close to Γ then along $\widetilde{\Gamma}$ one has $a_1(t) \neq 0$. By continuity, the sign of $a_1(t)$ cannot change. Now the first relation in (C.2) imply $g(t) \equiv 0$. Then the second relation in (C.2) implies $a_2(t) \equiv 0$ which means that $\widetilde{\Gamma}$ is a reparameterization of Γ_δ.

Remark. Lemma C.3 is a coordinate-free version of the following well-known coordinate normal form for a frame for a rank 2 distribution D with $\mathrm{rank}(D^2) = 3$:

$$X_1 = \partial/\partial x_1, \quad X_2 = \partial/\partial x_2 + x_1 \partial/\partial x_3 + f_4(x)\partial/\partial x_4 + \cdots + f_n(x)\partial/\partial x_n.$$

The conclusion of Lemma C.3 holds with $X = X_1, Y = X_2$ and $f = x_3$. This coordinate normal form can be found in [**B**] , [**LS**], [**Z4**] among other places. The idea of the formulation of Theorem C.2 is taken from [**B**] and the idea of its proof is taken from [**Mon3**], [**LS**], [**Z4**].

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