Normal forms and symmetries of (2,5) distributions: 100 years after Cartan

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Congratulations to
Professor Keizo Yamaguchi
Professor Reiko Miyaoka

- E. Cartan, Les systemes de Pfaff a cinque variables et les equations aux derivees partielles du second ordre, *Ann. Sci. Ecole Normale, 1910*

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- Most famous results concern $(2,3,5)$ distributions: $span(V_1, V_2)$ such that the vectors

$$V_1,\ V_2,\ [V_1, V_2],\ [V_1,[V_1,V_2]],\ [V_2,[V_1,V_2]]$$

  are linearly independent at any point (generic growth vector).

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- A distribution D is flat if and only if it admits a nilpotent $(2,3,5)$ basis V_1, V_2:

  $D = span(V_1, V_2)$: all length ≥ 4 Lie brackets of V_1, V_2 are equal to 0.

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- In the local classification of distributions with 7-dimensional or 6-dimensional group of symmetries there is exactly one modulus.

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- Given a $(2,3,5)$ distribution D Cartan constructs a map, by a long procedure: Cartan's method (way, road, approach?)

$$\mathbb{R}^5 \ni p \longrightarrow \; C^{(4)}_D(p; x_1, x_2) \text{ (Cartan's tensor at } p)$$

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where $C^{(4)}(x_1, x_2)$ is a homogeneous degree 4 polynomial of two variables satisfying the following:

- if the germ of D at p is diffeomorphic to the germ of $\widetilde{D}$ at $\widetilde{p}$ then $C_D^{(4)}(p; x_1, x_2)$ and $C_{\widetilde{D}}^{(4)}(\widetilde{p}; x_1, x_2)$ are linearly equivalent;

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- if D is the germ at $0 \in \mathbb{R}^5$ then D is flat if and only $C_D^{(4)}(p; x_1, x_2) = 0$ for all p close to 0.

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- otherwise much longer computation and weaker results.
- The normalization procedure, leading to exact normal form, consists of two steps (if one uses the usual filtration: 7 or 8 steps).

- The most important is the first step. Though it gives only "almost exact" normal form (exact up to a finite dimensional group of transformations),

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- Modulo quasi-homogeneity, the first step of the normalization procedure is not more than the classical "normalization by the principal part", for example the resonant normal form serving for all germs of singular vector field germs with a fixed linear approximation at 0 (Poincare, Dulac).

- The role of linear approximation: the nilpotent approximation of a $(2,3,5)$ distribution D which is the symbol of D (nilpotent graded (2,3,5) Lie algebra) expressed in terms of quasi-homogeneous degree -1 vector fields.

- The first step of the normalization procedure based on quasi-homogeneity is, to some extend, “diffeomorphic” to Tanaka prolongation of the nilpotent $(2, 3, 5)$ algebra, and probably some of results I will tell about can be obtained developing Tanaka prolongation, but in my opinion developing the method going back to Poincare and Dulak is simpler.

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- The generalized Cartan invariant allows to analyze the Lie algebras of all possible groups of symmetries preserving 0 (a very important subgroup of the whole symmetry group) and to prove
- Theorem. As an abstract Lie algebra, the algebra of infinitesimal symmetries vanishing at 0, of any non-flat $(2,3,5)$ distribution, is either $\{0\}$, or 1-dimensional, or 2-dimensional non-Abelian, or the 3-dimensional algebra $sl_2(\mathbb{R})$ (the last case seems to be new).

- Continuation of the theorem. As a Lie algebra of singular vector fields, in each of these cases it is linearizable, i.e. in suitable local coordinates it consists of linear vector fields, even though each of them is resonant.

  Any vanishing at 0 infinitesinmal symmetry of any non-flat (2,3,5) distribution has, in suitable coordinates, the form

$$V_A : \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \dot{x}_3 = (traceA)x_3,$$

$$\begin{pmatrix} \dot{x}_4 \\ \dot{x}_5 \end{pmatrix} = (A + traceA \cdot I) \begin{pmatrix} x_4 \\ \dot{x}_5 \end{pmatrix}$$

  where a constant 2×2 matrix A is one of the follows:

- $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$: vector fields $b^{\pm}_{1:1}$

  $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$: vector field $b_{0:1}$

  $\begin{pmatrix} p & 0 \\ 0 & -q \end{pmatrix}$: $1 \leq p < q$: vector field $b_{p:q}$

  $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$: vector field $b_{0:0}$

  Either one of the eigenvalues is 0 or their ratio is a negative rational number.

- $b^+_{1:1} = x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2} + x_5\frac{\partial}{\partial x_4} - x_4\frac{\partial}{\partial x_5}$

  $b^-_{1:1} = x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_4} - x_5\frac{\partial}{\partial x_5}$

  $b_{1:0} = x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial x_4} + 2x_5\frac{\partial}{\partial x_5}$

  $b_{p:q} = px_1\frac{\partial}{\partial x_1} - qx_2\frac{\partial}{\partial x_2} + (p-q)x_3\frac{\partial}{\partial x_3} +$

  $+(2p-q)x_4\frac{\partial}{\partial x_4} + (q-2p)x_5\frac{\partial}{\partial x_5}$

  $b_{0:0} = x_1\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_5}.$

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- Lie algebras of infinitesimal symmetries vanishing at 0 of any non-flat distribution:

  - 1-dim (any of these vector field)
  - 2-dim non-Abelian $span(b_{1:0}, b_{0:0}$
  - $sl_2(\mathbb{R})$: $span(b^{+}_{1:1}, b^{-}_{1:1}, b_{0:0}) =$
    $=$ all linear traceless vector fields.

Additional important statement:

The almost exact normal form and the exact normal form are parameterized by a function $C(x_1, ..., x_5)$ in a certain ideal in the ring of function germs. They hold in the same coordinates as the coordinates in which all symmetries vanishing at 0 are linear. These symmetries annihilate $C(x_1, .., x_5)$.

The information about the Lie algebra of infinitesimal symmetries vanishing at 0, up to diffeomorphisms rather than isomorphisms, allows to classify (easily) all possible complete symmetry algebras (including infinitesimal symmetries not vanishing at 0), and for each of them to classify distributions with this symmetry algebra.

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- The function $C(x)$ in the almost exact normal form starts with quasi-degree ≥ 4 and satisfies $b_{1:0}(C(x) = 0$. Consequently

$$C = C(x_1) = \pm x_1^m + r_1 x_1^{m+1} + r_2 x_1^{m+2} + h.o.t.(quasi) \quad m \geq 4$$

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- and 4 linearly independent infinitesimal symmetries which do not vanish at 0.

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- From the data we have we obtain that if the distribution is homogeneous and non-flat then a part of structure equations in the 7-dim algebra is as follows:

  $[a_1, a_2] = a_3,\ [a_1, a_3] = a_4,\ [a_2, a_3] = a_5$

  $[a_1, b_{1:0}] = 0,\ [a_1, b_{0:0}] = a_2,\ [a_2, b_{1:0}] = a_2,\ [a_2, b_{0:0}] = 0$

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- and (as an exercise on Jacobi identity) we obtain that the algebra is isomorphic to one with the structure equations above and the structure equations:

  $[a_1, a_4] = Ba_3 - Cb_{0:0},\ \ [a_1, a_5] = [a_2, a_4] = 0,\ \ [a_2, a_5] = 0.$

  (remaining structure equations follow from Jacobi identity).

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- The couple (B, C) is an invariant modulo scaling

  $B \to k^2 B, \; C \to k^4 C$

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- $\implies$ the $(2, 3, 5)$ distribution with the above 7-dim symmetry algebra and $\lambda = 100/9$ is flat.

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- the normal form:

$$\omega_1 = dx_1 + \frac{7}{3} I x_3 dx_2 + x_4 dx_3 - \left(\frac{1}{2}x_4^2 + \frac{2}{3} I x_3^2 - \frac{1}{2}(1 + I^2 - I'')x_2^2\right) dx_5,$$

$$\omega_2 = dx_2 - x_3 dx_5, \quad \omega_3 = -dx_3 + x_4 dx_5.$$

- EXAMPLE: homogeneous non-flat distributions with the symmetry

  $$b^{+}_{1:1} = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + x_5 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_5}$$

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- The function $C(x)$ in the almost exact normal form satisfies

  $b^+_{1:1}(C(x)) = 0$

  and starts with terms of quasi-degree ≥ 4. It follows

  $C(x) =$
  $\pm(x_1^2 + x_2^2)^m + r_1(x_1^2 + x_2^2)^{m+1} + r_2 x_3 (x_1^2 + x_2^2)^2 + h.o.t(quasi)$

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- The second step normalization gives exact normal form: the same with $r_2 = 0$:

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- If it is so then $m = 2$:

  $C(x) = \pm(x_1^2 + x_2^2)^2 + r(x_1^2 + x_2^2)^3 + \text{higher quasi-degree}$

  r can be arbitrary and h.o.t. are uniquely determined by r.

- The theorem on the all possible Lie algebras of vanishing at 0 infinitesimal symmetries implies that $b^+_{1:1}$ is the only such symmetry. Therefore the max dimension of the symmetry group is 6. If it is 6 the distribution is homogeneous.

- If it is so then $m = 2$:

  $C(x) = \pm(x_1^2 + x_2^2)^2 + r(x_1^2 + x_2^2)^3 +$ higher quasi-degree

  r can be arbitrary and h.o.t. are uniquely determined by r.

- By a simple work with Jacobi identity the 6-dim symmetry algebra is defined by the equations

  $[a_1, a_2] = a_3,\ [a_1, a_3] = a_4,\ [a_2, a_3] = a_5$

  $[a_1, b^+_{1:1}] = a_2,\ \ [a_2, b^+_{1:1}] = -a_1$

  $[a_1, a_4] = [a_2, a_5] = Ba_3 + Cb^+_{1:1},\ \ [a_1, a_5] = [a_2, a_4] = 0$

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- A distribution with transitively acting symmetry group of dimension 5 or more is uniquely determined by a generating 2-plane in the symmetry algebra.

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- Cartan has these results in Ch. XI. For the case $so_3(\mathbb{R}) \oplus so_3(\mathbb{R})$ he has a normal form with parameters $m, n \in \mathbb{R}$ whose ratio m/n is an invariant, with exceptional value 81.

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- The equation $\left(\frac{\mu^2+1}{\mu}\right)^2 = \frac{100}{9}$ has two positive solutions $\mu = 3$ and $1/3$ which gives one more proof of

Theorem (R. Bryant). The distribution is flat if and only if the ratio of the radii is 3 (or $1/3$).

Recent works where this theorem was proved + very interesting related math is explained:

A.Agrachev, Rolling Balls and Octonions, 2007

G.Bor, R. Montgomery, G_2 and the rolling distribution, 2009

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- Except the zero vector field there are no quasi-homogeneous degree ≤ -4 vector fields. The quasi-homogeneous vector fields of degree -3 have the form $a\frac{\partial}{\partial x_3} + b\frac{\partial}{\partial x_4}$
- It follows that any length ≥ 4 bracket of quasi-homogeneous degree -1 vector fields is 0.
- Consequently, any $(2, 3, 5)$ distribution spanned by quasi-homogeneous degree -1 vector fields is flat.

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  Proposition. Fix any $(2,3,5)$ distribution spanned by quasi-homogeneous degree -1 vector fields $\mathcal{N}_1, \mathcal{N}_2$. Any $(2,3,5)$ distribution D is diffeomorphic to a distribution spanned by vector fields of the form

$$V_1 = \mathcal{N}_1 + \cdots\,, \;\; V_2 = \mathcal{N}_2 + \cdots\,,$$

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- It is the symbol of D (graded nilpotent $(2,3,5)$ Lie algebra; all such symbols are isomorphic) represented by vector fields.

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- $$L_{\mathcal{N}} : \quad (Z, h) \ \rightarrow \ [Z, \mathcal{N}] + h\mathcal{N}.$$

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Dimension of $kerL_{\mathcal{N}}^{(i)}$:

$i \geq 4$: 0

- $i=-1$: 2 $i=1$: 2
  $i=-2$: 1 $i=0$: 4 $i=2$: 1
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- The last two pages are "isomorphic" to g_2 in terms of Tanaka prolongation.

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- Like in all local classification problems, the first normalization step gives the preliminary normal form

$$V = \mathcal{N} + V^{(0)} + V^{(1)} + \cdots ,$$
$$V^{(i)} = (V_1, V_2) \in W^{(i)}$$

serving for all $(2,3,5)$ distributions with a fixed nilpotent approximation $\mathcal{N}$.

- It is easy to compute that

$$W^{(0)} = W^{(1)} = W^{(2)} = \{0\}, \quad W^{(3)} \neq 0.$$

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- Claim. If $d = 3$ then the generalized Cartan invariant is exactly the Cartan invariant at $0 \in \mathbb{R}^5$.

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- Finding W reduces to finding a complementary space to the image of the linear operator

$$T: \ (f_1, f_2, g) \ \to \ \begin{pmatrix} \mathcal{N}_1(f_1) + g & \mathcal{N}_1(f_2) \\ \mathcal{N}_2(f_1) & \mathcal{N}_2(f_2) + g \end{pmatrix}$$

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where f_1, f_2, g are function germs:

- if U is a complementary space to the image of T in the space $Mat_{2\times 2}(x)$ then

$$\left\{ A(x) \begin{pmatrix} [N_1, [\mathcal{N}_1, \mathcal{N}_2]] \\ [N_2, [\mathcal{N}_1, \mathcal{N}_2]] \end{pmatrix}, \quad A(x) \in U \right\}$$

is a complementary space to the image of $L_{\mathcal{N}}$.

- Choice of nilpotent approximation $\mathcal{N} = span(\mathcal{N}_1, \mathcal{N}_2)$:

$$\mathcal{N}_1 = \frac{\partial}{\partial x_1} + x_2\left(\frac{\partial}{\partial x_3} + x_1\frac{\partial}{\partial x_4} + x_2\frac{\partial}{\partial x_5}\right)$$

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$$\mathcal{N}_2 = \frac{\partial}{\partial x_2} - x_1\left(\frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_4} + x_2 \frac{\partial}{\partial x_5}\right)$$

- Advantage: Any quasi-linear symmetry of $span(\mathcal{N}_1, \mathcal{N}_2)$ (= symmetry of quasi-degree 0) is linear.

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- It has the form

$$g_Q : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to Q\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \;\; x_3 \to detQ\cdot x_3, \;\; \begin{pmatrix} x_4 \\ x_5 \end{pmatrix} \to detQ\cdot Q\begin{pmatrix} x_4 \\ x_5 \end{pmatrix}$$

where Q is a non-singular 2×2 matrix.

- Asymptotically exact normal form

$$V = \mathcal{N} + C(x) \begin{pmatrix} x_1 x_2 & x_2^2 \\ -x_1^2 & -x_1 x_2 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_4} \\ \frac{\partial}{\partial x_5} \end{pmatrix}$$

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- Almost exact normal form: same with $C(x)$ in the ideal generated by the monomials

Monomials generating the ideal **I**	Quasi-degree
$x_1^i x_2^j,\ i+j=4$	4
$x_1^i x_2^j x_3,\ i+j=3$	5
$x_1^i x_2^j x_3^2,\ i+j=2$	6
$x_1x_3\theta,\ x_2x_3\theta$	7
θ^2	8

$$\theta = x_1x_4 - x_2x_5$$

- $$V = \mathcal{N} + C(x)\begin{pmatrix} x_1x_2 & x_2^2 \\ -x_1^2 & -x_1x_2 \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial x_4} \\ \frac{\partial}{\partial x_5} \end{pmatrix}, \quad C(x) \in \text{the ideal.}$$

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- The expansion of $C(x)$ by quasi-homogeneous terms:

$$C(x) = C^{(d+1)}(x) + C^{(d+2)}(x) + \cdots, \ d \geq 3$$

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- $C^{(4)}(x) = C^{(4)}(x_1, x_2)$, therefore if $d = 3$ (minimal possible degree of non-flatness) then the generalized invariant is the classical Cartan invariant.

If we replace $0 \in \mathbb{R}^5$ by as point nearby, the Cartan tensor (defined in this way) changes in very involved way, but its infinitesimal change is simple. This allows to give a simple proof of Cartan theorem that the distribution is flat if and only if the Cartan tensor vanishes identically reducing it to Frobenius theorem.

- If b is an infinitesimal symmetry vanishing at 0 then its quasi-linear part V_A annihilates the generalized Cartan invariant $C^{(d+1)}(x)$. Since $C^{(d+1)}(x)$ is a non-zero polynomial, it is possible only if V_A is a resonant vector field: either one of the eigenvalues is 0 or their ratio is a negative integer number. This is the main point for all theorems on infinitesimal symmetries vanishing at 0.

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- The proof of linearization theorems for such symmetries follows from the second normalization step in which the almost exact normal is "corrected" using the 5-dimensional "positive" part of g_2.

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- And thanks to the AUDIENCE!

  If you had no time to read all what is written in some pages and are interested to return to them, this file will be published soon in my homepage (google Technion, Math).