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Relative Darboux Theorem for Singular Manifolds and Local Contact Algebra

Dedicated to the memory of my mother, Valentina Mikhaïlovna Borok

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Abstract. In 1999 V. Arnol'd introduced the local contact algebra: studying the problem of classification of singular curves in a contact space, he showed the existence of the ghost of the contact structure (invariants which are not related to the induced structure on the curve). Our main result implies that the only reason for existence of the local contact algebra and the ghost is the difference between the geometric and (defined in this paper) algebraic restriction of a 1-form to a singular submanifold. We prove that a germ of any subset N of a contact manifold is well defined, up to contactomorphisms, by the algebraic restriction to N of the contact structure. This is a generalization of the Darboux-Givental' theorem for smooth submanifolds of a contact manifold. Studying the difference between the geometric and the algebraic restrictions gives a powerful tool for classification of stratified submanifolds of a contact manifold. This is illustrated by complete solution of three classification problems, including a simple explanation of V. Arnold's results and further classification results for singular curves in a contact space. We also prove several results on the external geometry of a singular submanifold N in terms of the algebraic restriction of the contact structure to N. In particular, the algebraic restriction is zero if and only if N is contained in a smooth Legendrian submanifold of M.

0 Introduction

The main result of the present work, Theorem 2.1 in Section 2, is a generalization of the Darboux–Givental' theorem (Section 1) on smooth submanifolds N of a contact manifold M stating that the (geometric) restriction of the contact structure to N is the only invariant of the germ of N with respect to the group of *contactomorphisms* — diffeomorphisms preserving the contact structure. Theorem 2.1 allows N to have arbitrarily deep singularities, provided that the geometric restriction to N is replaced by the *algebraic restriction* defined in Section 2.

The starting point for the present work was the paper [Ar-1] in which V. Arnol'd explained the existence of local contact algebra: the problem of classifying singular curves in a contact space with respect to contactomorphisms is non-trivial even if the equivalence class with respect to diffeomorphisms is fixed and the geometric restriction of the contact structure to the regular part of the curve gives no invariants. In [Ar-1] V. Arnol'd showed the existence of "ghost" invariants in the classification,

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with respect to contactomorphisms, of integral curves in a contact space diffeomorphic to the cusp $(t^2, t^{2k+1}, 0, \dots, 0)$ and of non-integral curves diffeomorphic to the cusp $(t^2, t^3, 0, \dots, 0)$ and having a fixed order of tangency with the contact structure. In the introduction to [Ar-1] V. Arnol'd wrote that it would be interesting to describe the ghost of the contact structure algebraically.

Theorem 2.1 in the present work implies that *the only reason for existence of local contact algebra (and "ghost" invariants) is the difference between the geometric and the algebraic restriction of a* 1*-form to a singular submanifold*.

Studying this difference gives a powerful tool for classification of singular submanifolds N of a contact manifold M, especially when the dimension of N is small (for singular curves $\dim N = 1$). This is illustrated by complete solution of three classification problems, Sections 3–5.

The algebraic restriction of a contact structure to a singular manifold N is related to the ideal consisting of functions vanishing at points of N, but it also has a geometric meaning. In Section 2 we give several results showing that the algebraic restriction of a contact structure to a singular submanifold $N \subset M$ allows to describe *all singularities occurring when restricting the contact structure to any smooth submanifold of M containing N*. The main corollary is as follows: the algebraic restriction of the contact structure to N is zero if and only if N is contained in a smooth Legendrian submanifold of M.

In Section 3 we apply the results of Section 2 to the problem of classification of stratified submanifolds $N = N_1 \cup N_2$, where N_1 and N_2 are smooth 1-dimensional submanifolds of a contact manifold $\left(\mathbb{R}^{2n+1}, (\alpha)\right)$ with regular intersection at 0. We give a complete classification of such stratified submanifolds. We prove that if the contact structure (α) is transversal to one of the strata then its order of tangency with the other stratum is a complete invariant, and if the contact structure is tangent to each of the strata then, except the orders of tangency, there is a modulus $\lambda \geq 0$. This modulus expresses the difference between the algebraic restriction and the geometric restriction. We also give a complete classification in the case that N_1 and N_2 are integral 1-dimensional submanifolds.

In Section 4 we give an explanation, in terms of the algebraic restriction, of the results of V. Arnol'd in [Ar-1, Section 2] on classification, with respect to contactomorphisms, of integral curves in a contact space $\mathbb{R}^{2n+1}$ diffeomorphic to the curve $A_{2k} : (t^2, t^{2k+1}, 0, \dots, 0)$. Our method allows to give a simple proof of these results and explain the "ghost" invariants. Also, we construct an invariant μ (multiplicity) taking values in the set $\{0, 1, \dots, 2k\}$ and distinguishing non-contactomorphic integral curves diffeomorphic to A_{2k}. The invariant μ is constructed in canonical (coordinate-free) terms and therefore it can be easily calculated in any coordinate system.

In Section 5 we study non-integral curves in a contact space $\mathbb{R}^{2n+1}$ diffeomorphic to the cusp $(t^2, t^{2k+1}, 0, \dots, 0)$. Within such curves we classify all contact-simple curves, for any $k \geq 1$ and $n \geq 1$. The obtained singularity classes are described in canonical (coordinate-free) terms which gives a possibility to distinguish them in any local coordinate system.

The case $k = 1$ was studied by V. Arnol'd in [Ar-1, Section 3]. V. Arnol'd obtained, for the case $n \geq 2$, five contact-simple singularities; he denoted them by a^0,

b^1, c^2, e^3, f^4 ([Ar-1, Section 3, Theorem 2). Our results for the case $k = 1$ imply that in fact there are only four contact-simple singularities: e^3 and f^4 are the same singularity. It follows that there exists a contactomorphism sending f^4 to e^3. Such a contactomorphism is given explicitly.

In Section 6 we prove the main theorems given in Section 2. In Appendix A, we give an additional explanation why the classification results in the 3-dimensional case differ from those in the $(2n + 1)$-dimensional case, $n \geq 2$.

A similar method can be developed for local classification of singular submanifolds of a symplectic manifold. One can define the algebraic restriction of a symplectic structure to a singular submanifold, to explain, in terms of the algebraic restriction, the local symplectic algebra introduced by V. Arnol'd in [Ar-2] and to obtain a series of new classification results. A work on this topic, jointly with W. Domitrz and S. Janeczko, will be published elsewhere, but in Appendix B we present the main result and explain the difference between the contact and the symplectic cases, related to the relative Poincaré lemma property.

1 Darboux–Givental' Theorem

This section contains two equivalent formulations of the Darboux–Givental' theorem, also called the relative Darboux theorem. This is a reduction theorem for the following equivalent classification problems:

(a) local classification of smooth r-dimensional submanifolds of a fixed contact manifold with respect to the group of contactomorphisms (diffeomorphisms preserving the contact structure);

(b) local classification of contact structures on an odd-dimensional manifold with respect to the group of diffeomorphisms preserving a fixed smooth r-dimensional submanifold.

The classification problems (a) and (b) are equivalent by the classical Darboux theorem stating that any two contact structures on an odd-dimensional manifold are locally equivalent. The Darboux–Givental' theorem reduces these classification problems to the classification of Pfaff equations on $\mathbb{R}^r$.

Recall that a *Pfaff equation* on a manifold N is a differential 1-form on N defined up to multiplication by a non-vanishing function. Equivalently, it is a module of 1-forms on N over the ring of functions generated by a single 1-form.

A Pfaff equation generated by a 1-form α will be denoted by (α). A Pfaff equation (α_1) on a manifold N_1 is diffeomorphic to a Pfaff equation (α_2) on a manifold N_2 if there exists a diffeomorphism $\Phi\colon N_1 \to N_2$ such that $(\Phi^*\alpha_2) = (\alpha_1)$, that is $\Phi^*\alpha_2 = Q\alpha_1$, where Q is a non-vanishing function.

Any contact structure on a manifold M is a Pfaff equation on M (the contact 1-form is defined up to multiplication by a non-vanishing function). Therefore a contact structure will be denoted by (α), where α is a contact 1-form.

Definition The geometric restriction of a contact structure (α) on a manifold M to a smooth submanifold $N \subset M$ is a Pfaff equation (β) on N, where β is the restriction of α to TN.

Convention In what follows, all objects (manifolds, 1-forms, diffeomorphisms, *etc.*) are germs at a fixed point 0, and they belong to a fixed category — either C^∞ or real analytic.

Theorem 1.1 **(A. Givental', see [Ar-Gi])** *Let N be a smooth submanifold of an odd-dimensional manifold M. Any two contact structures on M having the same geometric restriction to N can be brought one to the other by a diffeomorphism Φ of M such that $\Phi(n) = n$ for any point $n \in N$.*

Definition Two subsets of a fixed contact manifold are called *contactomorphic* if they can be brought one to the other by a contactomorphism of M.

Theorem 1.2 **(Corollary of Theorem 1.1)** *Two smooth submanifolds N_1 and N_2 of the same dimension of a fixed contact manifold $\big(M, (\alpha)\big)$ are contactomorphic if and only if the geometric restrictions of the contact structure (α) to N_1 and N_2 can be brought one to the other by a diffeomorphism $\phi\colon N_1 \to N_2$.*

2 Algebraic Restriction. Main Theorems

Theorem 2.1 *The statements of the Theorems* 1.1 *and* 1.2 *remain true if N, N_1, N_2 are arbitrary subsets of M provided that geometric restrictions are replaced by algebraic restrictions defined below.*

The algebraic restriction of a contact structure on M to a subset N of M is related to the ideal consisting of smooth (analytic) functions vanishing at points of N, and the bigger is this ideal the more effective is Theorem 2.1. It is the most effective if N is a stratified submanifold of M of dimension 1. On the other hand, if N is a dense set then the ideal consists of the zero function only, in this case Theorem 2.1 becomes a statement like $1 = 1$.

Notation Given a subset $N \subset M$ denote by $\Lambda^0_N(M)$ the ideal of smooth (analytic) functions on M vanishing at any point of N, and by $\Lambda^1_N(M)$ the module of smooth (analytic) 1-forms on M vanishing at any point of N.

Definition of Algebraic Restriction of a 1-Form to a Subset Let N be a subset of a manifold M. Two 1-forms on M will be called *N-equivalent* if their difference has the form

$$\mu + dH, \quad \mu \in \Lambda^1_N(M),\ H \in \Lambda^0_N(M). \tag{1}$$

The class of N-equivalence of a 1-form α on M will be denoted $[\alpha]_N$ and called *algebraic restriction* of α to N.

Note that the set of 1-forms (1) is a module over the ring of functions (in fact, $QdH = d(QH) - HdQ$). If the ideal $\Lambda^0_N(M)$ is finitely generated,

$$\Lambda^0_N(M) = (H_1, \dots, H_p),$$

then this module consists of 1-forms of the form

$$H_1\theta_1 + \cdots + H_p\theta_p + f_1 dH_1 + \cdots + f_p dH_p,$$

where θ_i are arbitrary 1-forms and f_i are arbitrary functions, and it will be denoted by $(H_1, \ldots, H_p, dH_1, \ldots, dH_p)$.

Definition of Algebraic Restriction of a Pfaff Equation or Contact Structure to a Subset Let (α) be a Pfaff equation on a manifold M, and let N be a subset of M. The algebraic restriction of (α) to N, denoted $[(\alpha)]_N$, is the algebraic restriction $[\alpha]_N$ defined up to multiplication by a non-vanishing function. If M is a contact manifold then the algebraic restriction of the contact structure to N is $[(\alpha)]_N$, where α is a 1-form describing the contact structure.

The multiplication of $[\alpha]_N$ by a function Q on M is defined by the relation

$$Q[\alpha]_N = [Q\alpha]_N.$$

This definition is correct since the set (1) is a module over the ring of functions.

It is clear that if N is a smooth submanifold of M then the algebraic restriction of a 1-form α to N can be identified with the geometric restriction: $[\alpha]_N = [\tilde{\alpha}]_N$ if and only if $\alpha|_{TN} = \tilde{\alpha}|_{TN}$. For singular (stratified) submanifolds N the algebraic restriction is a stronger invariant than the restriction of α to TN^{reg}, where N^{reg} is the regular part of N, see Sections 3–5.

The group of local diffeomorphisms acts in a natural way in the space of algebraic restrictions.

Definition Let N_1 and N_2 be subsets of a manifold M. Two algebraic restrictions $[\alpha_1]_{N_1}$ and $[\alpha_2]_{N_2}$ of 1-forms α_1 and α_2 on M are diffeomorphic if there exists a diffeomorphism Φ of M such that $\Phi(N_1) = N_2$ and $[\Phi^*\alpha_2]_{N_1} = [\alpha_1]_{N_1}$. The algebraic restrictions $[(\alpha_1)]_{N_1}$ and $[(\alpha_2)]_{N_2}$ of two Pfaff equations on M are diffeomorphic if the algebraic restriction $[\alpha_1]_{N_1}$ is diffeomorphic to the algebraic restriction $Q[(\alpha_2)]_{N_2}$ for some non-vanishing function Q.

Note that these definitions include the diffeomorphness of the sets N_1 and N_2. To check that the definitions are correct it suffices to note that a diffeomorphism Φ of M such that $\Phi(N_1) = N_2$ brings the set (1) with $N = N_2$ to the same set with $N = N_1$.

Theorem 2.1 is proved in Section 6. Though the algebraic restriction of a contact structure to a subset N is defined in algebraic way, it also has a geometric meaning — it allows us to describe all singularities of restrictions of the contact structure to smooth submanifolds containing N. Before formulating a general result, we state its main corollary.

Definition The algebraic restriction of a contact structure on a manifold M to a subset $N \subset M$ is zero if $[(\alpha)]_N = [(0)]_N$, where α is a contact 1-form.

Theorem 2.2 *Let M be a contact manifold, and let $N \subset M$ be an arbitrary subset. The following statements are equivalent:*

(i) *The algebraic restriction of the contact structure to N is zero;*
(ii) *N is contained in a smooth Legendrian submanifold of M.*

The implication (ii) $\to$ (i) follows from the following obvious statement: if N is contained in a smooth submanifold $S \subset M$ then the algebraic restriction to N of any Pfaff equation (α) on M can be identified with the algebraic restriction to N of the geometric restriction of (α) to S. In other words, two Pfaff equations on M have the same algebraic restriction to N if and only if their geometric restrictions to S have the same algebraic restrictions to N.

The starting point for the implication (i) $\to$ (ii) is the following theorem stating that if a subset N of a contact manifold M is contained in two equal-dimensional smooth submanifolds S_1, S_2 of M then, though the geometric restrictions of the contact structure to S_1, S_2 might have different singularities, the algebraic restrictions to N of these geometric restrictions have the same singularity.

Theorem 2.3 *Let N be a subset of a contact manifold M contained in two smooth equal-dimensional submanifolds $S_1, S_2 \subset M$. The geometric restrictions of the contact structure to S_1 and S_2 have diffeomorphic algebraic restrictions to N.*

Proof of Theorem 2.3 It is easy to see that the germs of any two equal-dimensional smooth submanifolds $S_1, S_2 \subset M$, not necessarily with regular intersection, can be brought one to the other by a local diffeomorphism $\Phi\colon M \to M$ preserving pointwise the intersection $S_1 \cap S_2$. Since $N \subset S_1 \cap S_2$ then Φ preserves pointwise N and consequently it brings the contact structure (α) to another contact structure $(\Phi^*\alpha)$ with the same algebraic restriction to N. Therefore the geometric restrictions of (α) and $\Phi^*(\alpha)$ to S_1 have the same algebraic restrictions to N. This is equivalent to the statement of Theorem 2.3. ■

Theorem 2.3 suggests that the algebraic restriction of the contact structure to N accounts for singularities of the geometric restriction of the contact structure to *all* smooth submanifolds S containing N. The following Theorem 2.4 says that this is so if S has minimal possible dimension.

Notation Given a subset N of a manifold M denote by $m = m(N)$ the minimal dimension of a smooth submanifold of M containing N.

For example, if N is the image of the curve $t \to (t^3, t^4, 0, \ldots, 0) + o(t^5)$ then $m(N) = 2$ since N is diffeomorphic to the image of the plane curve $(t^3, t^4, 0, \ldots, 0)$. If N is the image of the curve $t \to (t^3, t^4, t^5, 0, \ldots, 0) + o(t^5)$ then $m(N) = 3$ since N is diffeomorphic to the image of the space curve $(t^3, t^4, t^5, 0, \ldots, 0)$ and not diffeomorphic to the image of a plane curve, see [Gi-Ho], [Ar-3].

Theorem 2.4 *Let N be a subset of a contact manifold $\big(M, (\alpha)\big)$, and let S be any smooth m-dimensional submanifold of M containing N, where $m = m(N)$. Let (β) be the geometric restriction of the contact structure (α) to S. Let $(\hat{\beta})$ be another Pfaff equation on S. The following statements are equivalent:*

(i) *The algebraic restriction to N of the Pfaff equations (β) and $(\hat{\beta})$ are diffeomorphic;*

(ii) *there exists another smooth m-dimensional submanifold* $\hat{S} \subset M$ *containing* N *such that the geometric restriction of the contact structure to* $\hat{S}$ *is diffeomorphic to* $(\hat{\beta})$ *via a diffeomorphism* $\hat{S} \to S$ *preserving* N.

Theorem 2.4 is proved in Section 6. By this theorem in order to see what singularities appear when restricting the contact structure to smooth $m(N)$-dimensional submanifolds containing N it suffices to analyze the restriction to just one such submanifold. The deepest singularity of a Pfaff equation is the Pfaff equation generated by the zero 1-form. Taking $\hat{\beta} = 0$ in Theorem 2.4 we obtain the implication (i) $\to$ (ii) in Theorem 2.2. Moreover, we obtain a bit stronger result.

Theorem 2.5 *If the contact structure on a* $(2n+1)$*-dimensional manifold* M *has zero algebraic restriction to a subset* $N \subset M$ *then* $m(N) \leq n$ *and there exists a smooth isotropic* $m(N)$*-dimensional submanifold of* M *containing* N.

Remark The implication (ii) $\to$ (i) in Theorem 2.4 holds without assumption $m = m(N)$, but the implication (i) $\to$ (ii) does not. For example, if $N = \{0\}$ then any two 1-forms have the same algebraic restriction to N, whereas not any singularity of a Pfaff equation on, say, $\mathbb{R}^{2n}$ with $2n = \dim M - 1$, can be realized as the geometric restriction of a contact structure to a hypersurface.

We conclude this section with a realization theorem. Let N be a subset of $\mathbb{R}^{2n+1}$, and let (α) be a Pfaff equation on $\mathbb{R}^{2n+1}$. We will say that the algebraic restriction $[(\alpha)]_N$ is realizable by a contact structure if there exists a contact structure $(\tilde{\alpha})$ on $\mathbb{R}^{2n+1}$ such that $[(\alpha)]_N = [(\tilde{\alpha})]_N$.

Theorem 2.6 *Let* N *be a subset of* $\mathbb{R}^{2n+1}$*, and let* $m = m(N)$*. Let* α *be a Pfaff equation on* $\mathbb{R}^{2n+1}$.

(i) *If* $m \leq n$ *then the algebraic restriction* $[(\alpha)]_N$ *is realizable by a contact structure on* $\mathbb{R}^{2n+1}$ *for any* 1*-form* α.
(ii) *Let* $m \geq n+1$*. Take any smooth* m*-dimensional submanifold* $S \subset \mathbb{R}^{2n+1}$ *containing* N*. The algebraic restriction* $[(\alpha)]_N$ *is realizable by a contact structure on* $\mathbb{R}^{2n+1}$ *if and only if the* 1*-form* $\beta = \alpha|_{TS}$ *satisfies one of the following conditions:*

$$\beta \wedge (d\beta)^{m-n-1}(0) \neq 0 \text{ or } (d\beta)^{m-n}(0) \neq 0. \tag{2.1}$$

Example Let $N \subset \mathbb{R}^{2n+1}$ be the image of the curve $x_1 = t^2, x_2 = t^{2k+1}, x_{\geq 3} = 0$. Let $\alpha = \alpha_1(x)dx_1 + \cdots + \alpha_{2n+1}(x)dx_{2n+1}$. If $n \geq 2$ then the algebraic restriction $[(\alpha)]_N$ is realizable by a contact structure. If $n = 1$ then it is realizable if and only if the 1-form $\beta = \alpha_1(x_1, x_2, 0, \ldots, 0)dx_1 + \alpha_2(x_1, x_2, 0, \ldots, 0)dx_2$ satisfies one of conditions: $\beta(0) \neq 0$ or $d\beta(0) \neq 0$.

Proof of Theorem 2.6 By Theorem 2.4 $[(\alpha)]_N$ is realizable by a contact structure if and only if there exists a contact structure whose geometric restriction to S is (β). If $\dim S \leq n$ then this is true for any Pfaff equation (β) on S, and if $\dim S > n$ then this is true if and only if one of the conditions (2.1) holds, see [Ar-Gi].

3 The Union of Two Nonsingular Curves

In this section we show how the results of Section 2 work for the classification of stratified submanifolds $N = N_1 \cup N_2$, where N_1 and N_2 are smooth 1-dimensional submanifolds of a contact manifold $\left(\mathbb{R}^{2n+1}, (\alpha)\right)$ with regular intersection at 0. We give a complete classification of such stratified submanifolds. We prove that if the contact structure is transversal to one of the strata then its order of tangency with the other stratum is a complete invariant, and if the contact structure is tangent to each of the strata then, except of the orders of tangency, there is a modulus $\lambda \geq 0$. This modulus expresses the difference between the algebraic restriction and the geometric restriction. We also consider the case that N_1 and N_2 are *integral* 1-dimensional submanifolds. In this case our results in Section 2 imply that in the 3-dimensional case all $N = N_1 \cup N_2$ are contactomorphic, and if $n \geq 2$ then N is contactomorphic to one of two normal forms without parameters; the second one holds if and only if N is contained in a smooth Legendrian submanifold of $\mathbb{R}^{2n+1}$.

In suitable local coordinates N is given by the equations

$$N : uv = w = 0, \quad w = (w_1, \dots, w_{2n-1}).$$

Two 1-forms on $\mathbb{R}^{2n+1}$ are N-equivalent if their difference belongs to the module $(uv, udv + vdu, w_1, dw_1, \dots, w_{2n-1}, dw_{2n-1})$. Therefore

$$[(\alpha)]_N = \left[\left(a(u,v)du + b(u,v)dv \right) \right]_N = \left[\left(a(u,0) + \mu(v) \right) du + \left(\nu(u) + b(0,v) \right) dv \right]_N,$$

where $a(u,v)du + b(u,v)dv$ is the restriction of α to the tangent bundle to the 2-manifold $w = 0$,

$$\mu(v) = a(0,v) - a(0,0), \quad \nu(u) = b(u,0) - b(0,0).$$

Note that

$$v^2 du = v(udv + vdu) \bmod(uv), \quad u^2 dv = u(udv + vdu) \bmod(uv)$$

and consequently

$$[v^2 du]_N = [u^2 dv]_N = 0.$$

Note also that $[vdu]_N = -[udv]_N$. Therefore

$$[(\alpha)]_N = \left[\left(a(u,0)du + b(0,v)dv + \lambda v du \right) \right]_N, \quad \lambda \in \mathbb{R}. \tag{3.1}$$

Here the functions $a(u,0)$ and $b(0,v)$ define uniquely the geometric restriction of α to the regular part $N^{\text{reg}} = N - \{0\}$ of N, and the number λ expresses the difference between the algebraic and geometric restrictions.

Assume that the contact structure (α) has finite orders of tangency q and p with the strata N_1 and N_2. This means that the Taylor series of the functions $a(u,0)$ and $b(0,v)$ are not zero, they start with with terms of order $q \geq 0$ and $p \geq 0$ respectively. A local diffeomorphism of the form $(u,v,w) \to \left(\phi(u), \psi(v), w \right)$ preserves N and brings the form $\lambda v du$ to $\tilde{\lambda} v du$ modulo (uv). It follows that the algebraic restriction (3.1) is diffeomorphic to

$$[\pm u^q du \pm v^p dv + \lambda v du]_N, \quad \lambda \in \mathbb{R},\ 0 \leq q \leq p. \tag{3.2}$$

To reach the condition $q \leq p$ one should apply, if necessary, the diffeomorphism $(u, v, w) \to (v, u, w)$ which preserves N and brings $\lambda v du$ to $-\lambda v du$ mod $udv + vdu$.

Theorem 2.6 implies the following statements on the realizability of the normal form (3.2):

- if $n \geq 2$ then (3.2) is realizable for any values of parameters q, p, λ;
- if $n = 1$ and $q = 0$ then (3.2) is realizable for any values of p and λ;
- if $n = 1$ and $q \geq 1$ then (3.2) is realizable if and only if $\lambda \neq 0$.

Now one can easily get a complete classification of the algebraic restrictions of contact structures to N.

Proposition 3.1 *If a contact structure (α) on $\mathbb{R}^{2n+1}$ has finite orders of tangency with the strata N_1, N_2 then the algebraic restriction of (α) to N is diffeomorphic to one of the normal forms*

$$[0, p] : [(du + v^p dv)]_N, \quad p \geq 0;$$
$$[1, 1] = \{[1, 1]_\lambda : [(udu \pm vdv + \lambda v du)]_N, \lambda \geq 0\};$$
$$[q, p] : [(u^q du \pm v^p dv + \delta v du)]_N, \quad 1 \leq q \leq p \geq 2,\ \delta \in \{1, 0\}.$$

In the normal form $[1, 1]$ the parameter $\lambda \geq 0$ is an invariant (the algebraic restrictions $[1, 1]_\lambda$ and $[1, 1]_{\tilde{\lambda}}$ are diffeomorphic only if $\lambda = \tilde{\lambda}$.) If $n \geq 2$ then all given algebraic restrictions are realizable by a contact structure, if $n = 1$ — all except $[1, 1]$ with $\lambda = 0$ and $[q, p]$ with $\delta = 0$.

The normal form $[0, p]$ corresponds to the case that the contact structure is transversal to one of the strata N_1, N_2 and has tangency of order p with the other stratum. The normal form $[1, 1]$ corresponds to the case that the contact structure has tangency of order 1 with each of the strata N_1, N_2, and the normal form $[q, p]$ to the case that the contact structure has tangency of order ≥ 1 with one of the strata and of order ≥ 2 with the other stratum. In these normal forms one may replace $\pm$ by + if and only if at least one of the numbers q, p is even.

The normal forms $[0, p]$, $[1, 1]$, and $[q, p]$ can be obtained from the normal form (3.2) as follows. To get normal form $[0, p]$ one has to reduce the both $\pm$ to + (changing, if necessary, u to $-u$ and/or multiplying (3.2) by -1), to divide the obtained 1-form $du + v^p dv + \lambda v du$ by $1 + \lambda v$, and to make a suitable change of coordinates of the form $(u, v, w) \to \big(u, \psi(v), w\big)$ which preserves N for any $\psi(v)$. To get normal forms $[1, 1]$ and $[q, p]$ one has to make a change of coordinates $(u, v) \to (k_1 u, k_2 v)$ and to multiply (3.2) by k_3, where k_1, k_2, k_2 are suitable non-zero numbers.

The fact that in $[1, 1]$ the parameter λ is an invariant can be easily proved algebraically. Another, geometric explanation of this invariant is as follows. Take any smooth 2-manifold S containing the stratified submanifold N. All such manifolds have the same tangent space L_2 at 0. Let β be the restriction of the contact 1-form to

TS. One can pass from β to a vector field X via a volume form on S. In the case that the contact structure is tangent to the strata N_1 and N_2 it is tangent to S and consequently $\beta(0) = X(0) = 0$. The eigenspaces $l_1, l_2 \subset L_2$ of the linear approximation j^1X of X depend on the choice of S: if A is the matrix of j^1X for some fixed S then for any other S the matrix of j^1X has the form $A + \mathrm{diag}(c, -c)$, $c \in \mathbb{R}$. If the contact structure has tangency of order 1 with each of the strata N_1 and N_2 then independently of the choice of S the eigenspaces l_1, l_2 either are transversal to the strata N_1, N_2 or are not real. We have four lines TN_1, TN_2, l_1, l_2 in the 2-plane L_2 (or its complexification), but the cross-ratio $\theta \in S^1$ is not an invariant since l_1 and l_2 depend on the choice of S. Nevertheless, we also have the ratio $r = \lambda_1/\lambda_2$ of the eigenvalues of X, defined up to transformation $r \to r^{-1}$. All possible changes of S lead to a 1-parameter group acting in the space of pairs (θ, r). It is easy to see that the orbits of this action are parameterized by $\lambda \geq 0$ in the normal form $[1, 1]$.

The modulus λ disappears in the normal form $[q, p]$ since in this case one of the lines l_1, l_2 coincides with one of the strata N_1, N_2 independently of the choice of S.

By Theorem 2.1 the obtained classification of the algebraic restrictions to N of all possible contact structures on $\mathbb{R}^{2n+1}$ is, de-facto, a classification, with respect to contactomorphisms, of all stratified submanifolds diffeomorphic to $\{uv = w = 0\}$ in any fixed contact space, for example the contact space

$$\left(\mathbb{R}^{2n+1}, (dz - y_1dx_1 - \cdots - y_ndx_n)\right). \tag{3.3}$$

Fix any contact 1-forms $\alpha_{0,p}$, $\alpha^{\lambda}_{1,1}$, $\alpha_{q,p}$ whose algebraic restriction to N are $[0, p]$, $[1, 1]_\lambda$, $[q, p]$ respectively. Also fix diffeomorphisms $\Phi_{0,p}$, $\Phi^{\lambda}_{1,1}$, $\Phi_{q,p}$ from $\mathbb{R}^{2n+1}(x_1, y_1, \ldots, x_n, y_n, z)$ to $\mathbb{R}^{2n+1}(u, v, w_1, \ldots, w_{2n-1})$ which bring these contact 1-forms to the contact 1-form (3.3). These diffeomorphisms bring the stratified submanifold $uv = w = 0$ to certain stratified submanifolds $N_{0,p}$, $N^{\lambda}_{1,1}$, $N_{q,p}$. By Theorem 2.1 we obtain the following classification result.

Corollary 3.2 *Let $N = N_1 \cup N_2$, where N_1 and N_2 are 1-dimensional submanifolds of the contact space (3.3) with regular intersection at 0. Assume that the contact structure has tangency of finite orders $0 \leq q \leq p$ with the strata N_1, N_2 (tangency of zero order means transversality). If $q = 0$ then N is contactomorphic to $N_{0,p}$. If $q = p = 1$ then N is contactomorphic to $N^{\lambda}_{1,1}$, where λ is an invariant. If $q \geq 1$ and $p \geq 2$ then N is contactomorphic to $N_{q,p}$.*

If $n = 1$ (the 3-dimensional case) then one can fix

$$\alpha_{0,p} = du + v^p dv + wdv, \quad \Phi_{0,p} : x = -v,\ y = w,\ z = u + \frac{v^{p+1}}{p+1};$$

$$\alpha^{\lambda}_{1,1} = udu \pm vdv + \lambda vdu + dw, \quad \Phi_{1,1} : x = -u,\ y = \lambda v,\ z = w + u^2/2 \pm v^2/2;$$

$$\alpha_{q,p} = u^q du \pm v^p dv + vdu + dw, \quad \Phi_{q,p} : x = -u,\ y = v,\ z = w + \frac{u^{q+1}}{q+1} \pm \frac{v^{p+1}}{p+1}.$$

Simplifying the obtained stratified manifolds $N_{0,p}$, $N^{\lambda}_{1,1}$, $N_{q,p}$ by a scale contactomorphism $(x, y, z) \to (k_1x, k_2y, k_1k_2z)$ with suitable non-zero k_1, k_2 we get the following normal forms

$$(0,p)_3: \ \{(x,y,z): x(z-x^{p+1}) = y = 0\}, \quad p \geq 0,$$

$$(1,1)_3^\lambda: \ \{(x,y,z): xy = \lambda z - x^2 \pm y^2 = 0\}, \quad \lambda > 0,$$

$$(q,p)_3: \ \{(x,y,z): xy = z - x^{q+1} \pm y^{p+1} = 0\}, \quad 1 \leq q \leq p \geq 2.$$

Corollary 3.3 *Let $N = N_1 \cup N_2$, where N_1 and N_2 are 1-dimensional submanifolds of the contact space $\left(\mathbb{R}^3, (dz - ydx)\right)$ with regular intersection at 0. Assume that the contact structure has tangency of finite order ≥ 0 with each of the strata N_1, N_2. Then N is contactomorphic to one of the stratified manifolds $(0,p)_3$, $(1,1)_3^\lambda$, $(q,p)_3$. Two different normal forms are not contactomorphic up to reduction of $\pm$ to $+$ in $(q,p)_3$ in the case that at least one of the numbers q,p is even. In particular, q, p and $\lambda > 0$ are invariants.*

In the same way one can obtain similar normal forms in the contact manifold (3.3) with $n \geq 2$ corresponding to the algebraic restrictions $[0,p]$, $[1,1]^\lambda$ with $\lambda > 0$ and $[q,p]$ with $\delta = 1$:

$$(0,p)_{\geq 5}: \ \{x_1(z - x_1^{p+1}) = y_{\geq 1} = x_{\geq 2} = 0\}, \quad p \geq 0;$$

$$(1,1)_{\geq 5}^\lambda: \ \{x_1 y_1 = \lambda z - x_1^2 \pm y_1^2 = x_{\geq 2} = y_{\geq 2} = 0\}, \quad \lambda > 0;$$

$$(q,p)_{\geq 5}: \ \{xy = z - x_1^{q+1} \pm y_1^{p+1} = x_{\geq 2} = y_{\geq 2} = 0\}, \quad 1 \leq q \leq p \geq 2;$$

If the dimension of a contact manifold is ≥ 5 then the case $\lambda = 0$ in $[1,1]$ and the case $\delta = 0$ in $[q,p]$ are realizable. In these cases the algebraic restrictions can be realized by the contact 1-forms $udu \pm vdv + R$ and $u^q du \pm v^p dv + R$, where $R = dw_1 + w_2 du + w_3 dv + w_4 dw_5 + w_6 dw_7 + \cdots + w_{2n-2} dw_{2n-1}$. Bringing these 1-forms to the Darboux normal form (3.3) by a diffeomorphism and applying the same diffeomorphism to the manifold $\{uv = w_1 = \cdots = w_{2n-1} = 0\}$ we obtain the following normal forms:

$$(1,1)_{\geq 5}^0: \ \{x_1 x_2 = (z - x_1^2 \pm x_2^2) = x_{\geq 3} = y_{\geq 1} = 0\};$$

$$(q,p)_{\geq 5}^0: \ \{x_1 x_2 = z - x_1^{q+1} \pm x_2^{p+1} = x_{\geq 3} = y_{\geq 1} = 0\}, \quad 1 \leq q \leq p \geq 2.$$

Corollary 3.4 *Let $N = N_1 \cup N_2$, where N_1 and N_2 are 1-dimensional submanifolds of the contact space (3.3), $n \geq 2$, with regular intersection at 0. Assume that the contact structure has tangency of finite order ≥ 0 with each of the strata N_1, N_2. Then N is contactomorphic to one of the stratified manifolds $(0,p)_{\geq 5}$, $(1,1)_{\geq 5}^\lambda$, $(1,1)_{\geq 5}^0$, $(q,p)_{\geq 5}$, $(q,p)_{\geq 5}^0$. Two different normal forms are not contactomorphic up to reduction of $\pm$ to $+$ in $(q,p)_{\geq 5}$ and $(q,p)_{\geq 5}^0$ in the case that at least one of the numbers q,p is even. In particular, q, p and $\lambda > 0$ are invariants.*

The obtained normal forms give a complete classification of stratified submanifolds of the contact space (3.3) which are the union $N = N_1 \cup N_2$ of two 1-dimensional strata N_1, N_2 intersecting regularly at 0 and having finite order of tangency with the contact structure. The latter assumption excludes an infinitely degenerate case. For any p the singularity $N_{0,p}$ is *simple* in the following sense: if N is diffeomorphic to $N_{0,p}$ and sufficiently close to $N_{0,p}$ then N is contactomorphic to one of the stratified manifolds of a *finite* tuple $N_{0,0}, N_{0,1}, \ldots, N_{0,p}$. All other singularities are not simple because of the modulus λ in the normal form $[1,1]$ and the adjaciences $[1,1] \leftarrow [q,p]$ for any $q \geq 1$.

Finally, let us consider the case that $N = N_1 \cup N_2$ is an *integral* submanifold, which means that each of the strata N_1, N_2 is everywhere tangent to the contact structure (α). In terms of normal form (3.1) this means that $a(u,0) \equiv 0$, $b(0,v) \equiv 0$. Therefore the algebraic restriction of the contact structure to N is equal to $[(\alpha)]_N = [(\lambda v du)]_N$. In the 3-dimensional case $\lambda \neq 0$ by the realization Theorem 2.6, and therefore the algebraic restriction is diffeomorphic to a fixed algebraic restriction $[(vdu)]_N$. If $n \geq 2$ then the case $\lambda = 0$ is not excluded, therefore either the algebraic restriction is diffeomorphic to $[(vdu)]_N$ or it is equal to zero. It is clear that these two cases can be distinguished as follows: in the first case the differential $d\alpha$ of the contact 1-form α does not annihilate the 2-plane $L_2 = T_0N_1 + T_0N_2$, in the second case it does. Therefore Theorem 2.1, 2.2, and 2.5 imply the following result.

Corollary 3.5 *Let N_1 and N_2 be smooth integral 1-dimensional submanifolds of the contact space $\left(\mathbb{R}^{2n+1}, (\alpha)\right)$, $\alpha = dz - y_1 dx_1 - \cdots - y_n dx_n$, with regular intersection at 0. In the 3-dimensional case the stratified submanifold $N = N_1 \cup N_2$ is contactomorphic to the submanifold $N_3^* = \{(x,y,z) : xy = z = 0\}$. If $n \geq 2$ then N is contactomorphic to one of the submanifolds*

$$N^*_{\geq 5} = \{x_1 y_1 = z = x_{\geq 2} = y_{\geq 2} = 0\}, \quad N^{**}_{\geq 5} = \{x_1 x_2 = z = x_{\geq 3} = y_{\geq 1} = 0\}.$$

The set N is contained in a smooth isotropic 2-dimensional submanifold (or, equivalently, in a smooth Legendrian submanifold) if and only if N is contactomorphic to $N^{**}_{\geq 5}$. This is so if and only if $d\alpha$ annihilates the 2-plane $L_2 = T_0N_1 + T_0N_2$.

4 Integral Curves Diffeomorphic to A_{2k}

In what follows by a curve in a contact space $\left(\mathbb{R}^{2n+1}, (\alpha)\right)$ we will understand a map $\gamma\colon (\mathbb{R},0) \to (\mathbb{R}^{2n+1},0)$. Two curves are diffeomorphic if their images can be brought one to the other by a local diffeomorphism, and contactomorphic if they can be brought one to the other by a local contactomorphism. A curve γ is called *integral* if $\gamma^*\alpha = 0$, *i.e.*, the vector $\dot{\gamma}(t)$ is tangent to the contact hyperplane at $\gamma(t)$ for all t. By Theorems 1.1 and 1.2 all nonsingular (immersed) integral curves are contactomorphic.

In [Ar-1, Section 2] V. Arnol'd classified, with respect to contactomorphisms, all integral curves diffeomorphic to the curve

$$A_{2k} : \; x_1 = t^2, \; x_2 = t^{2k+1}, \; x_{\geq 3} = 0.$$

It turned out that the classification is nontrivial, though the geometric restriction of the contact structure to the regular part of any integral curve gives no invariants and the equivalence class with respect to diffeomorphisms is fixed. V. Arnol'd proved that if $n \geq 2$ then there are, up to contactomorphisms, exactly $2k+1$ integral curves diffeomorphic to A_{2k}; if $n = 1$ then all integral curves diffeomorphic to A_{2k} are contactomorphic. In this section we give a simple proof of this result using the algebraic restrictions and Theorem 2.1. Results of Section 2 also allow us to construct an invariant μ (multiplicity) taking values in the set $\{0, 1, \ldots, 2k\}$ and distinguishing non-contactomorphic integral curves diffeomorphic to A_{2k}. The invariant μ can be easily calculated in any coordinate system.

In what follows the notation A_{2k} will be used both for the curve and its image. If the algebraic restriction to A_{2k} of a 1-form α is zero then $A_{2k}^*\alpha = 0$. The inverse is not true. Take the 1-form

$$\theta = (2k+1)x_2dx_1 - 2x_1dx_2. \tag{4.1}$$

It is easy to see that the algebraic restriction of θ to A_{2k} is not zero, though $A_{2k}^*\theta = 0$.

Proposition 4.1 *Let α be any 1-form on $\mathbb{R}^p(x_1, \ldots, x_p)$, $p \geq 2$, such that $A_{2k}^*\alpha = 0$. Then*

$$[\alpha]_{A_{2k}} = [(c_0 + c_1x_1 + c_2x_1^2 + \cdots + c_{2k-1}x_1^{2k-1})\theta]_{A_{2k}}, \quad c_0, c_1, \ldots, c_{2k-1} \in \mathbb{R},$$

where θ is the 1-form (4.1). The algebraic restriction of α to A_{2k} is zero if and only if $c_0 = \cdots = c_{2k-1} = 0$. Consequently, the set of algebraic restrictions to A_{2k} of all 1-forms α such that $A_{2k}^*\alpha = 0$ is a $(2k)$-dimensional vector space.

It follows that the set of algebraic restrictions to A_{2k} of all Pfaff equations (α) on $\mathbb{R}^p(x_1, \ldots, x_p)$ such that $A_{2k}^*\alpha = 0$ consists of $(2k+1)$ elements

$$[\theta]_{A_{2k}}, [x_1\theta]_{A_{2k}}, [x_1^2\theta]_{A_{2k}}, \ldots, [x_1^{2k-1}\theta]_{A_{2k}}, [0]_{A_{2k}}. \tag{4.2}$$

Corollary 4.2 *The algebraic restriction to A_{2k} of any Pfaff equation (α) on $\mathbb{R}^p(x_1, \ldots, x_p)$, $p \geq 2$, such that $A_{2k}^*\alpha = 0$ is equal to one of $2k+1$ algebraic restrictions (4.2). These algebraic restrictions are different.*

To describe the algebraic restrictions to A_{2k} of all contact structures on $\mathbb{R}^{2n+1}$ we use the realization Theorem 2.6. By this theorem any of the algebraic restrictions (4.2) is realizable by a local contact structure on $\mathbb{R}^{2n+1}$, $n \geq 2$, and only the first of the algebraic restrictions (4.2) is realizable by a local contact structure on $\mathbb{R}^3$.

Corollary 4.3 *The set of algebraic restrictions to A_{2k} of all contact structures on $\mathbb{R}^{2n+1}$ with respect to which the curve A_{2k} is integral consists of $(2k+1)$ points (4.2) if $n \geq 2$ and of a single point $[\theta]_{A_{2k}}$ if $n = 1$.*

Proof of Proposition 4.1 The curve $A_{2k} : x_1^{2k+1} = x_2^2$, $x_{\geq 3} = 0$ is contained in a smooth 2-manifold $S : x_{\geq 3} = 0$, therefore the algebraic restriction of any 1-form α to A_{2k} can be represented by a 1-form $a(x_1, x_2)dx_1 + b(x_1, x_2)dx_2$. Let $H = x_1^{2k+1} - x_2^2$.

By the division theorem (see [Ar-Va-Gu]) any function on S has the form $g(x_1) + x_2 h(x_1) \bmod(H)$, therefore

$$[\alpha]_{A_{2k}} = \left[\left(g_1(x_1) + x_2 h_1(x_1) \right) dx_1 + \left(g_2(x_1) + x_2 h_2(x_1) \right) dx_2 \right]_{A_{2k}}.$$

The integrability of A_{2k} implies that

$$2t\left(g_1(t^2) + t^{2k+1} h_1(t^2) \right) + (2k+1)t^{2k}\left(g_2(t^2) + t^{2k+1} h_2(t^2) \right) \equiv 0.$$

Distinguishing the odd and the even part of this relation, one gets

$$2g_1(x_1) = -(2k+1)x_1^{2k} h_2(x_1), \quad (2k+1)g_2(x_1) = -2x_1 h_1(x_1).$$

These relations imply that $[\alpha]_{A_{2k}} = [f(x_1)\theta + q(x_1)dH]_{A_{2k}}$ for some functions $f(x_1)$ and $q(x_1)$. Since $[q(x_1)dH]_{A_{2k}} = [0]_{A_{2k}}$ then $[\alpha]_{A_{2k}} = [f(x_1)\theta]_{A_{2k}}$.

It remains to show that $[f(x_1)\theta]_{A_{2k}} = [0]_{A_{2k}}$ if and only if $f(x_1) = o(x_1^{2k-1})$. The relation $[f(x_1)\theta]_{A_{2k}} = [0]_{A_{2k}}$ means that $\left(f(x_1)\theta + H\gamma \right) \wedge dH = 0$ for some 1-form γ. Since $\theta \wedge dH = 2(2k+1)H dx_1 \wedge dx_2$ it follows that $f(x_1)$ belongs to the ideal generated by the partial derivatives of H. This is so if and only if $f(x_1) = o(x_1^{2k-1})$. ∎

The following statement completes the classification of algebraic restrictions.

Proposition 4.4 *No two different algebraic restrictions in the set* (4.2) *are diffeomorphic.*

This statement is proved below by constructing an invariant distinguishing non-diffeomorphic algebraic restrictions to A_{2k} of Pfaff equations (α) such that $A_{2k}^* \alpha = 0$. This invariant takes values in the set $\{0, 1, \ldots, 2k\}$. We call it multiplicity.

Theorem 2.1 allows to transfer the normal forms (4.2) to normal forms for integral curves diffeomorphic to A_{2k} in any fixed contact space, for example

$$\left(\mathbb{R}^{2n+1}, (\alpha) \right), \quad \alpha = dz - y_1 dx_1 - \cdots - y_n dx_n. \tag{4.3}$$

Take any contact 1-forms α_i whose algebraic restrictions to A_{2k} are equal to $x_1^i \theta$, $i = 0, 1, \ldots, 2k-1$, and take a contact 1-form α_{2k} with the zero algebraic restriction to A_{2k}. A diffeomorphism Φ_i bringing α_i to the contact 1-form α in (4.3) brings A_{2k} to a certain curve A_{2k}^i. By Theorems 2.1, 2.2 and 2.5 we obtain the following corollary.

Corollary 4.5 *Any integral curve γ in the contact space* (4.3) *which is diffeomorphic to A_{2k} is contactomorphic to the curve A_{2k}^0 if $n = 1$ and to one and only one of the curves $A_{2k}^0, \ldots, A_{2k}^{2k}$ if $n \geq 2$. The image of γ is contained in a smooth isotropic* 2*-dimensional submanifold (or, equivalently, in a smooth Legendrian submanifold) if and only if γ is contactomorphic to A_{2k}^{2k}.*

Example Let $n = 2$. One can take

$$\alpha_i = x_1^i \theta + x_3 dx_1 + x_4 dx_2 + dx_5, \quad i = 0, \ldots, 2k-1,$$
$$\alpha_{2k} = x_3 dx_1 + x_4 dx_2 + dx_5.$$

These contact 1-forms can be brought to $\alpha = dz - y_1dx_1 - y_2dx_2$ by the diffeomorphisms

$$\Phi_i\colon (x_1, x_2, y_1, y_2, z) \to (x_1, x_2, -x_3 - (2k+1)x_1^i x_2, -x_4 + 2x_1^{i+1}, x_5),$$
$$\Phi_{2k}\colon (x_1, x_2, y_1, y_2, z) \to (x_1, x_2, -x_3, -x_4, x_5).$$

We obtain:

$$A_{2k}^i:\ x_1 = t^2,\ x_2 = t^{2k+1},\ y_1 = -(2k+1)t^{2k+2i+1},\ y_2 = 2t^{2i+2},\ z = 0,$$
$$A_{2k}^{2k}:\ x_1 = t^2,\ x_2 = t^{2k+1},\ y_1 = y_2 = z = 0.$$

Now we will construct an invariant

$$\mu\colon\ \mathrm{Int}(A_{2k}, \mathbb{R}^{2n+1}, \alpha) \to \{0, 1, \ldots, 2k\},$$

where $\mathrm{Int}(A_{2k}, \mathbb{R}^{2n+1}, \alpha)$ denotes the set of all integral curves γ in a fixed contact space $\left(\mathbb{R}^{2n+1}, (\alpha)\right)$ which are diffeomorphic to A_{2k}. Take any smooth 2-manifold S containing the image of γ. The algebraic restriction of (α) to the image of γ can be represented by a Pfaff equation (β) on S. Let H be a generator of the ideal consisting of functions on S vanishing at points of γ. The integrability of γ implies

$$\beta \wedge dH = FH\Omega,$$

where Ω is a non-degenerate volume form and F is some function. Let β_1, β_2 be the coefficients of β in any coordinate system (x_1, x_2) on S. Consider the ideal

$$J = \left(\beta_1, \beta_2, F, \frac{\partial H}{\partial x_1}, \frac{\partial H}{\partial x_2}\right)$$

generated by the coefficients of β, the function F, and the partial derivatives of H. Consider the factor space $R[[x_1, x_2]]/J$, where $R[[x_1, x_2]]$ is the ring of all formal series.

Definition We will say that the number

$$\mu = \dim R[[x_1, x_2]]/J$$

is the multiplicity of the curve $\gamma \in \mathrm{Int}(A_{2k}, \mathbb{R}^{2n+1}, \alpha)$.

Proposition 4.6 *The multiplicity μ of a curve $\gamma \in \mathrm{Int}(A_{2k}, \mathbb{R}^{2n+1}, \alpha)$ is an invariant of the algebraic restriction of the contact structure (α) to the image of γ. It is well-defined: μ does not depend on the choice of S, a coordinate system on S, a 1-form β representing the algebraic restriction, a generator H, and a volume form Ω.*

Proof We use Theorem 2.3. It suffices to prove that μ is a well-defined invariant of the algebraic restriction if S is fixed. Then, by Theorem 2.3, the choice of S is irrelevant.

The choice of a coordinate system on S is irrelevant because the function F is constructed in a coordinate-free way, the ideal generated by the coefficients of β and the ideal generated by the partial derivatives of H also can be constructed in a coordinate-free way: the first ideal consists of functions of the form $v \rfloor \beta$, and the second one consists of functions of the form $v \rfloor dH$, where v is an arbitrary vector field.

Multiplication of H by a non-vanishing function does not change the ideal generated by the partial derivatives of H since $H \in (\frac{\partial H}{\partial x_1}, \frac{\partial H}{\partial x_2})$. It changes the function F to a function $\tilde{F}$ such that $\tilde{F} - F \in (\beta_1, \beta_2)$. Therefore the ideal J remains the same, *i.e.*, the choice of a generator H is irrelevant.

It is clear that the choice of Ω is also irrelevant and that the ideal J remains the same under multiplication of β by a non-vanishing function.

It remains to check that the ideal J does not change when replacing β to another 1-form $\tilde{\beta}$ with the same algebraic restriction to A_{2k}. The latter means that $\tilde{\beta} = \beta + H\gamma + f dH$, where γ is a 1-form and f is a function. Replacing β by $\tilde{\beta}$ we do not change the ideal $(\beta_1, \beta_2, \frac{\partial H}{\partial x_1}, \frac{\partial H}{\partial x_2})$, and the function F remains the same modulo the ideal $(\frac{\partial H}{\partial x_1}, \frac{\partial H}{\partial x_2})$. Therefore the ideal J remains the same. ■

Since the multiplicity of the singularity of the function $H = x_1^{2k+1} - x_2^2$ is equal to $\dim R[[x_1, x_2]]/(\frac{\partial H}{\partial x_1}, \frac{\partial H}{\partial x_2}) = 2k$ then the multiplicity of any algebraic restriction to A_{2k} does not exceed $2k$. It might be equal to any integer $\in \{0, 1, \dots, 2k\}$.

Example The multiplicity of the algebraic restriction $[(x_1^i \theta)]_{A_{2k}}$ is equal to $\min(i, 2k)$, $i \geq 0$.

In fact, taking $H = x_1^{2k+1} - x_2^2$ we have $\theta \wedge dH = 2(2k+1)H$ and consequently $F = x_1^i$ up to multiplication by a non-vanishing function. Therefore the ideal J is generated by the functions x_2, x_1^i, x_1^{2k} and $\mu = \min(i, 2k)$.

The multiplicity of the zero algebraic restriction is equal to $2k$. The given example and Proposition 4.6 prove Proposition 4.4. We also obtain the following corollary.

Corollary 4.7

(i) *All curves in* $\mathrm{Int}(A_{2k}, \mathbb{R}^3, \alpha)$ *are contactomorphic. Any curve in* $\mathrm{Int}(A_{2k}, \mathbb{R}^3, \alpha)$ *has multiplicity* 0.

(ii) *In the set* $\mathrm{Int}(A_{2k}, \mathbb{R}^{2n+1}, \alpha)$ *with* $n \geq 2$ *there are, up to contactomorphisms, exactly* $(2k+1)$ *integral curves. The multiplicity of any such curve takes values in the set* $\{0, 1, \dots, 2k\}$. *Two curves in* $\mathrm{Int}(A_{2k}, \mathbb{R}^{2n+1}, \alpha)$ *are contactomorphic if and only if they have the same multiplicity.*

(iii) *The multiplicity of a curve* $\gamma \in \mathrm{Int}(A_{2k}, \mathbb{R}^{2n+1}, \alpha)$, $n \geq 2$ *takes the maximal possible value* $2k$ *if and only if the image of* γ *is contained in a smooth isotropic* 2*-dimensional submanifold of the contact space (or, equivalently, in a smooth Legendrian submanifold).*

Example The curves

$$\gamma_s: \ x_1 = t^2, \ x_2 = t^{2k+1}, \ y_1 = t^{2k+2s+1}, \ y_{\geq 2} = x_{\geq 3} = 0, \ z = 2\kappa t^{2k+2s+3}, \quad s \geq 0,$$
$$\kappa = 1/(2k+2s+3),$$

are integral curves in the contact space (4.3). The image of any of these curves is contained in the smooth 2-manifold

$$S: \ y_1 = x_1^s x_2, \ y_{\geq 2} = x_{\geq 3} = 0, \ z = 2\kappa x_1^{s+1} x_2.$$

The restriction of the contact structure to S is a Pfaff equation of $S(x_1, x_2)$ generated by the 1-form $\beta = \kappa x_1^s \theta$, where θ is the 1-form (4.1). The ideal of functions on S vanishing at points of A_{2k} is generated by $H(x_1, x_2) = x_1^{2k+1} - x_2^2$. The function $F = (\beta \wedge dH)/H\Omega$ is equal to x_1^s up to multiplication by a non-vanishing function. Therefore the ideal J is generated by x_2, x_1^s, x_1^{2k} and the multiplicity of the curve γ_s is equal to $\min(s, 2k)$.

Corollary 4.8 *Any integral curve in the contact space* (4.3) *of dimension* ≥ 5 *which is diffeomorphic to* A_{2k} *is contactomorphic to one and only one of the curves* γ_s, $s \in \{0, 1, \dots, 2k\}$.

Remark Another invariant of a curve $\gamma \in \mathrm{Int}(A_{2k}, \mathbb{R}^{2n+1}, \alpha)$ is

$$\mu' = \dim R[[x_1, x_2]] \Big/ \left(\beta_1, \beta_2, \frac{\partial H}{x_1}, \frac{\partial H}{x_2} \right),$$

where β_1, β_2 and H are the same as in the definition of the multiplicity μ. It is easy to see that

$$\mu' = \min(\mu + 1, 2k).$$

This means that μ' distinguishes all curves $\gamma \in \mathrm{Int}(A_{2k}, \mathbb{R}^{2n+1}, \alpha)$ such that the algebraic restriction of the contact structure (α) to the image of γ is not zero (*i.e.*, integral curves whose image does not belong to a smooth Legendrain submanifold). Nevertheless, μ' does not distinguish curves with multiplicities $2k - 1$ and $2k$.

Remark The difference $2k - \mu$ shows how far the image of a curve $\gamma \in \mathrm{Int}(A_{2k}, \mathbb{R}^{2n+1}, \alpha)$ is from the nearest smooth isotropic 2-manifold, *cf.* [Ar-1, Section 4]. This can be expressed as follows. Let us say that a smooth 2-manifold S is s-jet-isotropic if the restriction of the contact form to S is a 1-form with zero s-jet. Theorem 2.4 and results in this section imply that if the multiplicity of γ is equal to $\mu < 2k$ then the image of γ is contained in some smooth μ-jet-isotropic submanifold and is not contained in any smooth $(\mu + 1)$-jet-isotropic submanifold.

5 Non-Integral Curves Diffeomorphic to A_{2k}

In this section we use results of Section 2 to start the classification of arbitrary curves $\gamma: \mathbb{R} \to \big(\mathbb{R}^{2n+1}, (\alpha)\big)$ which are diffeomorphic to the curve

$$A_{2k}: \ x_1 = t^2, \ x_2 = t^{2k+1}, \ x_{\geq 3} = 0. \tag{5.1}$$

We will use the notation $(A_{2k}, \mathbb{R}^{2n+1}, \alpha)$ for the space of all such curves. Note that the set $\mathrm{Int}(A_{2k}, \mathbb{R}^{2n+1}, \alpha)$ considered in Section 4 has infinite codimension in $(A_{2k}, \mathbb{R}^{2n+1}, \alpha)$.

The case $k = 1$ was studied by V. Arnol'd in [Ar-1, Section 3]. Theorem 2 in [Ar-1, Section 3] states that if $n \geq 2$ then there are exactly 5 simple singularities in $(A_2, \mathbb{R}^{2n+1}, \alpha)$ called a^0, b^1, c^2, e^3, f^4. Our results imply that e^3 and f^4 are the same singularity, *i.e.*, there exists a contactomorphism sending f^4 to e^3. Such a contactomorphism is constructed explicitly, see example after Corollary 5.4.

In this section we distinguish all simple singularities in $(A_{2k}, \mathbb{R}^{2n+1}, \alpha)$ (for any $k \geq 1, n \geq 1$) in canonical terms. This involves the order of tangency of a curve $\gamma \in (A_{2k}, \mathbb{R}^{2n+1}, \alpha)$ with the contact structure, the limit tangent line l_1 to γ and the 2-plane $L_2 \subset T_0\mathbb{R}^{2n+1}$ which is tangent to any smooth 2-dimensional submanifold containing the image of γ.

Definition A curve $\gamma \in (A_{2k}, \mathbb{R}^{2n+1}, \alpha)$ is simple if there exists $l \geq 0$ and curves $\gamma_1, \ldots, \gamma_p \in (A_{2k}, \mathbb{R}^{2n+1}, \alpha)$ such that any curve $\tilde{\gamma} \in (A_{2k}, \mathbb{R}^{2n+1}, \alpha)$ with the l-jet sufficiently close to the l-jet of γ is contactomorphic to one of the curves $\gamma_1, \ldots, \gamma_p$.

Theorem 5.1 *A curve in $(A_{2k}, \mathbb{R}^{2n+1}, \alpha)$ is simple if and only if it has tangency with the contact structure (α) of order ≤ 3. If $n = 1$ (resp. $n \geq 2$) then there are, up to contactomorphisms, exactly* 3 *(resp.* 4*) simple curves in $(A_{2k}, \mathbb{R}^{2n+1}, \alpha)$.*

The order of tangency of any curve γ with any contact structure (α) can be defined as follows. Take the 1-form $\gamma^*\alpha$ on $\mathbb{R}^1$. It has the form $a(t)dt$. The order of tangency of γ and (α) is equal to r if the Taylor series of $a(t)$ starts with a term of order r. The zero order of tangency (the case $a(0) \neq 0$) means transversality. If the function $a(t)$ has zero Taylor series then the order of tangency is ∞ (this is so for integral curves, and in the analytic category — for integral curves only). For example, the order of tangency of the curve $x = t^{r_1}$, $y = t^{r_2}$, $z = t^{r_3}$, $r_1, r_2, r_3 \geq 1$, with the contact structure $(dz - ydx)$ is equal to $\min(r_3 - 1, r_1 + r_2 - 1)$.

Notation The order of tangency of a curve $\gamma \in (A_{2k}, \mathbb{R}^{2n+1}, \alpha)$ with the contact structure (α) will be denoted $\mathrm{ord} = \mathrm{ord}(\gamma, \alpha)$.

No singular curve, in particular A_{2k}, can be transversal to a contact structure, therefore $\mathrm{ord}(A_{2k}, \alpha) \geq 1$. The minimal order of tangency $\mathrm{ord}(A_{2k}, \alpha) = 1$ has the following geometric meaning. The image of $\gamma \in (A_{2k}, \mathbb{R}^{2n+1}, \alpha)$ is contained in a smooth 2-dimensional manifold S. Such a manifold is not unique, but all such manifolds have the same tangent 2-plane at 0. Denote this tangent 2-plane by $L_2 \subset T_0\mathbb{R}^{2n+1}$. The curve γ has *unique limit tangent line at* 0 — a 1-dimensional subspace $l_1 \subset L_2$. For example, in the coordinates of normal form (5.1) the plane L_2 is spanned by $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial x_2}$, and the line l_1 is spanned by $\frac{\partial}{\partial x_1}$.

Restrict the contact 1-form α to the plane $S = (x_1, x_2)$ containing the image of A_{2k}:

$$\alpha|_{TS} = \alpha_1(x_1, x_2)dx_1 + \alpha_2(x_1, x_2)dx_2. \tag{5.2}$$

Then $\mathrm{ord}(A_{2k}, \alpha)$ is the order of zero of the 1-form

$$\left(2t\alpha_1(t^2, t^{2k+1}) + (2k+1)t^{2k}\alpha_2(t^2, t^{2k+1})\right) dt. \tag{5.3}$$

We see that $\operatorname{ord}(\gamma, \alpha)$ is equal to 1 if and only if $\alpha_1(0) \neq 0$. This is equivalent to the condition that the contact hyperplane at 0 is transversal to the line $l_1 = \operatorname{span}\left(\frac{\partial}{\partial x_1}\right)$. Therefore $\operatorname{ord}(\gamma, \alpha) = 1$ *if and only if the contact hyperplane at* 0 *is transversal to the line* l_1.

The plane L_2 and the line l_1 allow to define the singularity classes of curves $\gamma \in (A_{2k}, \mathbb{R}^{2n+1}, \alpha)$ distinguished by the following conditions on a curve γ:

C_1. The contact hyperplane at 0 is transversal to the line l_1.
C_2. The contact hyperplane at 0 contains the line l_1, but transversal to the 2-plane L_2.
C_3. The contact hyperplane at 0 contains the 2-plane L_2. The 2-form $d\alpha$ does not annihilate the 2-plane L_2.
C_4. The contact hyperplane at 0 contains the plane L_2. The 2-form $d\alpha$ annihilates the 2-plane L_2.

These singularity classes are well-defined — the condition $d\alpha|_{L_2} \neq 0$ does not depend on the choice of α describing a fixed contact structure provided that the contact hyperplane at 0 contains the 2-plane L_2.

If $n \geq 2$ then the classes C_1, C_2, C_3, C_4 are realizable (not empty), but if $n = 1$ then only C_1, C_2, C_3 are realizable. In fact, if (α) is a contact structure on $\mathbb{R}^3$ tangent to a 2-plane $L_2 \subset T_0\mathbb{R}^3$ then $L_2 = \ker \alpha(0)$ and consequently $d\alpha|_{L_2} \neq 0$, *i.e.*, the case C_4 is impossible.

Theorem 5.2 *The three simple singularities of curves $\gamma \in (A_{2k}, \mathbb{R}^3, \alpha)$ correspond to the singularity classes $C_i \cap \{\operatorname{ord} \leq 3\}$, $i = 1, 2, 3$. The four simple singularities of curves $\gamma \in (A_{2k}, \mathbb{R}^{2n+1}, \alpha)$, $n \geq 2$ correspond to the singularity classes $C_i \cap \{\operatorname{ord} \leq 3\}$, $i = 1, 2, 3, 4$.*

The intersection $C_i \cap \{\operatorname{ord} \leq 3\}$ can easily be analyzed using (5.2) and (5.3). As we already explained, $C_1 = \{\operatorname{ord} = 1\}$, therefore $C_1 \cap \{\operatorname{ord} \leq 3\} = C_1$. For $i \geq 2$ the intersection $C_i \cap \{\operatorname{ord} \leq 3\}$ expresses in different terms in the cases $k = 1$ (the curve A_2) and $k \geq 2$ (the curves $A_4, A_6, \dots$).

Let $k = 1$. The 1-form (5.3) takes the form $\left(2t\alpha_1(t^2, t^3) + 3t^2\alpha_2(t^2, t^3)\right) dt$. The class C_2 corresponds to the case $\alpha_1(0) = 0$, $\alpha_2(0) \neq 0$. In this case ord $= 2$. The inverse is also true: if $\operatorname{ord}(\gamma, \alpha) = 2$ then $\alpha_1(0) = 0$, $\alpha_2(0) \neq 0$, which means that $\gamma \in C_2$. Therefore $C_2 = \{\operatorname{ord} = 2\}$ and consequently $C_2 \cap \{\operatorname{ord} \leq 3\} = C_2$.

In the case $k = 1$ we also see that the class $C_3 \cup C_4$ corresponds to the condition $\alpha_1(0) = \alpha_2(0) = 0$, and the class $\{\operatorname{ord} = 3\}$ corresponds to the condition $\alpha_1(0) = \alpha_2(0) = 0$, $\frac{\partial \alpha_1}{\partial x_1}(0) \neq 0$. Therefore $\{\operatorname{ord} = 3\} \subset C_3 \cup C_4$. If $n = 1$ then C_4 is empty and consequently $C_3 \cap \{\operatorname{ord} \leq 3\} = \{\operatorname{ord} \leq 3\}$.

Therefore in the case $k = 1$ Theorem 5.2 can be reformulated as follows.

Corollary 5.3 *The three simple singularities of curves $\gamma \in (A_2, \mathbb{R}^3, \alpha)$ are distinguished by* $\operatorname{ord}(\gamma, \alpha)$*: there is exactly one singularity such that* $\operatorname{ord}(\gamma, \alpha) = 1$*, exactly one singularity such that* $\operatorname{ord}(\gamma, \alpha) = 2$*, and exactly one singularity such that* $\operatorname{ord}(\gamma, \alpha) = 3$*. The four simple singularities of curves $\gamma \in (A_2, \mathbb{R}^{2n+1}, \alpha)$, $n \geq 2$ are as follows: there is exactly one singularity such that* $\operatorname{ord}(\gamma, \alpha) = 1$*, exactly one singularity such that* $\operatorname{ord}(\gamma, \alpha) = 2$*, and exactly two singularities such that* $\operatorname{ord}(\gamma, \alpha) = 3$*. The latter two singularities of order* 3 *correspond to the cases $d\alpha|_{L_2} \neq 0$ and $d\alpha|_{L_2} = 0$, where L_2 is the* *tangent* 2*-plane at* 0 *to some (and then any) smooth* 2*-manifold containing the image of* γ.

Now we consider the case $k \geq 2$. In this case the class $\{\text{ord} = 2\}$ is empty — for any curve $\gamma \in (A_{2k}, \mathbb{R}^{2n+1}, \alpha)$ either $\text{ord}(\gamma, \alpha) = 1$ (if and only if $\gamma \in C_1$) or $\text{ord}(\gamma, \alpha) \geq 3$. The class C_2 corresponds to the condition $\alpha_1(0) = 0, \alpha_2(0) \neq 0$, the class $C_3 \cup C_4$ — to the condition $\alpha_1(0) = \alpha_2(0) = 0$, and the class $\{\text{ord} = 3\}$ — to the condition $\alpha_1(0) = 0$, $\frac{\partial \alpha_1}{\partial x_1}(0) \neq 0$.

Corollary 5.4 *The three simple singularities of curves $\gamma \in (A_{2k}, \mathbb{R}^3, \alpha)$, $k \geq 2$, are as follows: there is exactly one singularity such that* $\text{ord}(\gamma, \alpha) = 1$*, exactly one singularity such that* $\text{ord}(\gamma, \alpha) = 3$ *and the contact structure is transversal to the* 2*-plane* L_2*, and exactly one singularity such that* $\text{ord}(\gamma, \alpha) = 3$ *and the contact structure is tangent to* L_2*. The four simple singularities of curves $\gamma \in (A_{2k}, \mathbb{R}^{2n+1}, \alpha), k \geq 2, n \geq 2$ are as follows: there is exactly one singularity such that* $\text{ord}(\gamma, \alpha) = 1$*, exactly one singularity such that* $\text{ord}(\gamma, \alpha) = 3$ *and the contact structure is transversal to* L_2*, and exactly two singularities such that* $\text{ord}(\gamma, \alpha) = 3$ *and the contact structure is tangent to* L_2*. The latter two singularities correspond to the cases $d\alpha|_{L_2} \neq 0$ and $d\alpha|_{L_2} = 0$.*

Example In [Ar-1] V. Arnol'd obtained 5 singularities of curves diffeomorphic to A_2 in the contact space

$$(\mathbb{R}^{2n+1}, \alpha), \quad \alpha = dz - y_1 dx_1 - \cdots - y_n dx_n. \tag{5.4}$$

represented by the curves

$$a^0 : \; z = t^2, \; x_1 = t^3, \; y_{\geq 1} = 0, \; x_{\geq 2} = 0,$$

$$b^1 : \; z = t^3, \; x_1 = t^2, \; y_{\geq 1} = 0, \; x_{\geq 2} = 0,$$

$$c^2 : \; z = t^4, \; x_1 = t^2, \; y_1 = t^3, \; y_{\geq 2} = x_{\geq 2} = 0,$$

$$e^3 : \; z = t^4, \; x_1 = t^2, \; x_2 = t^3, \; y_1 = t^5, \; y_{\geq 2} = x_{\geq 3} = 0,$$

$$f^4 : \; z = t^4, \; x_1 = t^2, \; x_2 = t^3, \; y_{\geq 1} = 0, \; x_{\geq 2} = 0$$

(notations are taken from [Ar -1]). Let us show that e^3 and f^4 *are the same singularity*, *i.e.*, the curves e^3 and f^4 are contactomorphic. The curves e^3 and f^4 have the same order of tangency 3 with the contact structure. The image of the curve e^3 belongs to the 2-manifold $z = x_1^2, y_1 = x_1 x_2, y_{\geq 2} = x_{\geq 3} = 0$, and the image of f^4 belongs to the 2-manifold $z = x_1^2$, $y_{\geq 1} = 0, x_{\geq 3} = 0$. The 2-plane L_2 for e^3 and f^4 is the same: $L_2 = \text{span}(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$. The restriction of $d\alpha$ to L_2 is zero. Therefore the curves e^3 and f^4 belong to the class $C_4 \cap \{\text{ord} \leq 3\}$ and by Theorem 5.2 (or Corollary 5.3) they are contactomorphic.

The curve a^0 has tangency of order 1 with the contact structure, and the curve b^1 — tangency of order 2. The curve c^2 has tangency of order 3 with the contact structure, its image is contained in the 2-manifold $z = x_1^2$, $y_{\geq 2} = x_{\geq 2} = 0$, therefore for c^2 the plane L_2 is spanned by $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial y_1}$. The restriction of the contact structure to L_2 is not zero. Therefore by Theorem 5.2 (or Corollary 5.3) any simple curve in $(A_2, \mathbb{R}^{2n+1}, \alpha)$ is contactomorphic to one and only one of the curves a^0, b^1, c^2, e^3 if $n \geq 2$ (e^3 can be replaced by f^4), and to one and only one of the curves a^0, b^1, c^2 if $n = 1$.

Remark It is not so easy to find explicitly a contactomotphism sending e^3 to f^4. Here is one such contactomorphism:

$$z = \tilde{z} + (\tilde{z} - \tilde{x}_1^2)\tilde{x}_2/2,\ x_i = \tilde{x}_i \quad (1 \leq i \leq n),$$

$$y_1 = \tilde{y}_1(1 + \tilde{x}_2/2) - \tilde{x}_1\tilde{x}_2,$$

$$y_2 = \tilde{y}_2(1 + \tilde{x}_2/2) + (\tilde{z} - \tilde{x}_1^2)/2,$$

$$y_i = (1 + \tilde{x}_2/2)\tilde{y}_i \quad (3 \leq i \leq n).$$

Example Consider the following curves diffeomorphic to A_{2k} in the contact space (5.4):

$$\mathcal{C}_1 :\ z = t^2,\ y_1 = t^{2k+1},\ y_{\geq 2} = x_{\geq 1} = 0,$$

$$\mathcal{C}_2 :\ z = t^{2k+1},\ y_1 = x_1 = t^2,\ y_{\geq 2} = x_{\geq 2} = 0,$$

$$\mathcal{C}_3 :\ z = t^4,\ y_1 = t^2,\ x_1 = t^{2k+1},\ y_{\geq 2} = x_{\geq 2} = 0,$$

$$\mathcal{C}_4 :\ z = t^4,\ y_1 = t^2,\ y_2 = t^{2k+1},\ y_{\geq 3} = x_{\geq 1} = 0.$$

It is easy to check that the curve $\mathcal{C}_i$ belongs to the singularity class $C_i \cap \{\text{ord} \leq 3\}$, $i = 1, 2, 3, 4$. Therefore by Theorem 5.2 any simple curve in $(A_{2k}, \mathbb{R}^{2n+1}, \alpha)$ is contactomorphic to one and only one of the curves $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$ if $n \geq 2$ and to one and only one of the curves $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ if $n = 1$. This is so for any $k \geq 1$. If $k = 1$ then the normal form C_2 can be simplified — the curve $\mathcal{C}_2$ is contactomorphic to the curve $z = t^3, x_1 = t^2, y_{\geq 1} = x_{\geq 2} = 0$.

The proofs of Theorems 5.1 and 5.2 are based on the classification of the algebraic restrictions to A_{2k} of all possible contact structures on $\mathbb{R}^{2n+1}(x_1, \ldots, x_{2n+1})$.

Proposition 5.5 *If A_{2k} belongs to the singularity class $C_i \cap \{\text{ord} \leq 3\}$, then the algebraic restriction of the contact structure to the image of A_{2k} is diffeomorphic to the algebraic restriction*

$$[(dx_1)]_{A_{2k}} \ \textit{if}\ i = 1; \qquad [(dx_2 + x_1 dx_1)]_{A_{2k}} \ \textit{if}\ i = 2;$$

$$[(x_1(dx_1 + dx_2))]_{A_{2k}} \ \textit{if}\ i = 3; \qquad [(x_1 dx_1)]_{A_{2k}} \ \textit{if}\ i = 4.$$

Proof We will restrict ourself to the proof for the case $i = 4$ (the proof for the other three cases is similar). Take coordinates in which A_{2k} has form (5.1). The restriction of the contact structure to A_{2k} can be represented by a 1-form $\alpha = \alpha_1(x_1, x_2)dx_1 + \alpha_2(x_1, x_2)dx_2$. The relations

$$[2x_2 dx_2]_{A_{2k}} = [(2k+1)x_1^{2k} dx_1]_{A_{2k}},$$

$$[2x_1^{2k+1} dx_2]_{A_{2k}} = [2x_2^2 dx_2]_{A_{2k}} = [(2k+1)x_1^{2k} x_2 dx_1]_{A_{2k}}$$

imply that if $j^{2k}\alpha_2(x_1, 0) = 0$ then $[\alpha]_{A_{2k}} = [a(x_1, x_2)dx_1]_{A_{2k}}$ for some function $a(x_1, x_2)$. We will show below that the condition $j^{2k}\alpha_2(x_1, 0) = 0$ can be reached by a diffeomorphism preserving A_{2k}.

The function $a(x_1, x_2)$ can be expressed in the form $a_1(x_1) + x_2a_2(x_1)$ modulo $(x_1^{2k+1} - x_2^2)$. Therefore $[a(x_1, x_2)dx_1]_{A_{2k}} = \left[\left(a_1(x_1)+x_2a_2(x_1)\right)dx_1\right]_{A_{2k}}$. The assumption $A_{2k} \in C_4 \cap \{\text{ord} \leq 3\}$ implies that $a_1(0) = a_2(0) = 0, a_1'(0) \neq 0$. Consequently the function $a_1(x_1) + x_2a_2(x_1)$ has the form $Q(x_1, x_2)x_1$, where $Q(0) \neq 0$, and we get the normal form $[(x_1dx_1)]_{A_{2k}}$.

A diffeomorphism of the plane $\mathbb{R}^2(x_1, x_2)$ preserving $A_{2k} : x_1 = t^2, x_2 = t^{2k+1}$ and reducing to zero the $2k$-jet of the function $\alpha_2(x_1, 0)$ can be constructed as follows. Take a vector field of the form

$$X = (c_0 + c_1x_1 + \cdots + c_{2k_1}x_1^{2k})\left(2x_2\frac{\partial}{\partial x_1} + (2k + 1)x_1^{2k}\frac{\partial}{\partial x_2}\right).$$

Let X^t be the flow of X. Since X is tangent to A_{2k} then the diffeomorphism X^1 preserves A_{2k}. It is easy to see that the assumptions $\alpha_1(0) = \alpha_2(0) = 0, \partial\alpha_1\partial x_1(0) \neq 0$ imply that for suitable $c_0, c_1, \ldots, c_{2k}$ the 1-form $(X^1)^*\alpha$ has the form $a(x_1, x_2)dx_1 + b(x_1, x_2)dx_2$ with $j^{2k}b(x_1, 0) = 0$. ∎

Proposition 5.5 and Theorem 2.1 imply that all curves $\gamma \in (A_{2k}, \mathbb{R}^{2n+1}, \alpha)$ belonging to the singularity class $C_i \cap \{\text{ord} \leq 3\}$ with fixed $i \in \{1, 2, 3, 4\}$ are contactomorphic. To prove Theorems 5.1 and 5.2 it remains to prove that there are no simple curves in the singularity class $\{\text{ord} \geq 4\} \subset (A_{2k}, \mathbb{R}^{2n+1}, \alpha)$. To prove this we will show that if $\text{ord}(A_{2k}, \alpha) \geq 4$ then the algebraic restriction $[(\alpha)]_{A_{2k}}$ is not simple within algebraic restrictions $[(\tilde{\alpha})]_{A_{2k}}$, where $(\tilde{\alpha})$ is a contact structure close to (α).

Assume that $\text{ord}(A_{2k}, \alpha) \geq 4$ and that the contact hyperplane at 0 is tangent to the 2-plane L_2 (if $k = 1$ then the first assumption implies the second one). Then the algebraic restriction of (α) to A_{2k} is represented by 1-form $\alpha = \alpha_1(x_1, x_2)dx_1 + \alpha_2(x_1, x_2)dx_2$, where the 1-jet of α has the form $j^1\alpha = a_1x_1dx_2 + a_2x_2dx_1 + a_3x_2dx_2$. Transfer α to a vector field X on the (x_1, x_2)-plane via a volume form. The ratio of the eigenvalues of the linearization of X is $\lambda = \lambda_\alpha = a_2/a_1$. If $\hat{\alpha}$ is a 1-form close to α and the Pfaff equations $(\hat{\alpha})$ and (α) are diffeomorphic then $\lambda_{\hat{\alpha}} = \lambda_\alpha$ (provided the genericity assumptions $a_1 \neq 0, a_1 \neq a_2$). The modulus λ does not change when replacing α by another 1-form $\tilde{\alpha}$ with the same algebraic restriction to A_{2k}. In fact, $\tilde{\alpha} = \alpha + H\mu + fdH$, where $H = x_1^{2k+1} - x_2^2$, μ is a 1-form, f is a function, therefore $j^1\tilde{\alpha} = a_1x_1dx_2 + a_2x_2dx_1 + \tilde{a}_3x_2dx_2$ and $\lambda_{\tilde{\alpha}} = \lambda_\alpha$. Consequently λ is a modulus in the classification of algebraic restrictions to A_{2k} of contact structures close to (α).

Assume now that $\text{ord}(A_{2k}, \alpha) \geq 4$ and the contact hyperplane at 0 is transversal to the 2-plane L_2. This is possible if $k \geq 2$ only. In this case a modulus appears in the classification of 3-jets of algebraic restrictions. The algebraic restriction can be represented by 1-form $dx_2 + f(x_1, x_2)dx_1$, where $f(0) = 0$, $\frac{\partial f}{\partial x_1}(0) = 0$. Using the group of diffeomorphisms preserving the image of A_{2k} one can reduce the 3-jet of $f(x_1, x_2)$ to $ax_1^2 + bx_1^3$. If $a \neq 0$ then a can be reduced to 1, and we get the family $dx_2 + (x_1^2 + \lambda x_1^3)dx_1$ of the 3-jets of algebraic restrictions. It is not hard to prove that in this normal form λ is a modulus.

6 Proofs of Theorems 2.1 and 2.4

The implication (ii) $\to$ (i) in Theorem 2.4 follows from Theorem 2.3. The implication (i) $\to$ (ii) is based on the following proposition which will also be used in the proof of Theorem 2.1. Recall that by $m(n)$ we denote the minimal dimension of a smooth submanifold containing the set N.

Proposition 6.1 *Let N be a subset of a contact manifold (M, α). Let S be a smooth m-dimensional submanifold of M containing N, where $m = m(N)$. Let β be the geometric restriction of α to S. If $\hat{\beta}$ is a 1-form on S such that β and $\hat{\beta}$ have the same algebraic restriction to N then there exists a 1-form $\hat{\alpha}$ on M whose geometric restriction to S is equal to $\hat{\beta}$ and such that $\hat{\alpha}(0) = \alpha(0)$, $d\hat{\alpha}(0) = d\alpha(0)$. Consequently, $\hat{\alpha}$ is a contact 1-form at 0.*

Proof The equality of the algebraic restrictions means that $\beta - \hat{\beta} = \mu + dH$, where μ and H are a 1-form and a function on S respectively vanishing at any point of N. If $d\mu(0) \neq 0$ or $dH(0) \neq 0$ then the ideal of functions on S vanishing at any point of N contains a function g with non-vanishing differential. Then N is contained in the smooth hypersurface of S given by the equation $g = 0$. This contradicts to the definition of $m(N)$. Therefore $d\mu(0) = 0$ and $dH(0) = 0$. Consequently $\hat{\beta}(0) = \beta(0)$ and $d\hat{\beta}(0) = d\beta(0)$.

Take a projection $\pi \colon M \to S$ (a nonsingular map preserving S pointwise). Set $\hat{\alpha} = \pi^*\hat{\beta} + \alpha - \pi^*\beta$. It is clear that the geometric restriction of $\hat{\alpha}$ to S is $\hat{\beta}$. Since $\beta(0) = \hat{\beta}(0)$ and $d\beta(0) = d\hat{\beta}(0)$ then $\alpha(0) = \hat{\alpha}(0)$ and $d\alpha(0) = d\hat{\alpha}(0)$. ■

Proof of Theorem 2.4 *Implication* (i) $\to$ (ii): We will prove this implication using Theorem 2.1 and Proposition 6.1. One can assume that 1-forms β and $\hat{\beta}$ have the same algebraic restriction to N. By Proposition 6.1 there exists another contact structure $(\hat{\alpha})$ on M whose geometric restriction to S is equal to $(\hat{\beta})$. The contact structures (α) and $(\hat{\alpha})$ have the same algebraic restriction to N. By Theorem 2.1 there exists a diffeomorphism Φ of M preserving N and sending the contact structure $(\hat{\alpha})$ to the contact structure (α). This diffeomorphism sends the manifold S to some manifold $\hat{S}$ containing N. Let ϕ be the restriction of Φ^{-1} to $\hat{S}$. Then $\phi^*(\hat{\beta}) = (\alpha|_{T\hat{S}})$. ■

Proof of Theorem 2.1 To prove Theorem 2.1 we have to show that if α_0 and α are contact 1-forms on M with the same algebraic restriction to a subset $N \subset M$ then there exists a local diffeomorphism preserving N pointwise and sending the contact structure (α) to the contact structure (α_0).

Let $m = m(N)$ be the minimal dimension of a smooth submanifold of M containing N. Let S be one of such smooth m-dimensional submanifolds of M. (It is possible that $m = \dim M$, then $S = M$). By Proposition 6.1 there exists a contact 1-form α_1 whose geometric restriction to S coincides with that of α and such that the difference of α_1 and α_0 vanishes at the origin along with its differential. By Darboux–Givental' Theorem 1.1 there exists a diffeomorphism preserving pointwise S (in particular, N) and sending (α) to (α_1). Therefore to prove Theorem 2.1 it suffices to prove that the contact structures (α_0) and (α_1) can be brought one to the other by a diffeomorphism preserving N pointwise. The advantage of this reduction is as follows: since the difference of α_1 and α_0 vanishes at 0 along with its differential then *the* 1*-form* $\alpha_t = \alpha_0 + t(\alpha_1 - \alpha_0)$ *is contact for all* t. This allows use of the homotopy method.

By the homotopy method, it suffices to prove the solvability of the equation

$$L_{Z_t}\alpha_t + h_t\alpha_t = \alpha_0 - \alpha_1 \tag{6.1}$$

with respect to a family Z_t of vector fields such that $Z_t(n) = 0$ for any $n \in N$ and $t \in [0, 1]$, and a family of functions h_t. Here L_{Z_t} is the Lie derivative along the vector field Z_t.

Since α_0 and α_1 have the same algebraic restriction to N then $\alpha_0 - \alpha_1 = dH + \mu$, where H is a function vanishing at any point of N and μ is a 1-form vanishing at any point of N. The 1-form α_t is contact, therefore $\alpha_t(0) \neq 0$ and there exists a family Y_t of vector fields such that $Y_t \rfloor \alpha_t = 1$, $t \in [0, 1]$. The vector field HY_t vanishes at any point of N, for all t, and satisfies the relation $L_{HY_t}\alpha_t = dH + HY_t \rfloor d\alpha_t$. Introduce, instead of the unknown family of vector fields Z_t in (6.1), another family V_t such that $Z_t = HY_t + V_t$. One gets the equation

$$L_{V_t}\alpha_t + h_t\alpha_t = \theta_t = \mu - HY_t \rfloor d\alpha_t, \tag{6.2}$$

with respect to (V_t, h_t) such that $V_t(n) = 0$, $n \in N$, $t \in [0, 1]$. Note that the 1-form θ_t vanishes at any point of N. To solve (6.2) we use the following lemma.

Lemma 6.2 *Let α_t be a family of contact* 1*-forms on a* $(2k+1)$*-space. If ν_t is a family of* 1*-forms such that $\alpha_t \wedge (d\alpha_t)^{k-1} \wedge \nu_t = 0$ then $\nu_t = h_t\alpha_t$ for some family h_t of functions.*

Let us show how one can use Lemma 6.2 to prove the solvability of equation (6.2). Let $\dim M = 2k+1$. Since the 1-forms α_t in (6.2) are contact then Lemma 6.2 allows to replace equation (6.2) by the equation

$$(L_{V_t}\alpha_t) \wedge \alpha_t \wedge (d\alpha_t)^{k-1} = \theta_t \wedge \alpha_t \wedge (d\alpha_t)^{k-1}, \tag{6.3}$$

with respect to a family V_t of vector fields vanishing at any point of N. Let us show that the family V_t defined by the relation

$$V_t \rfloor \big(\alpha_t \wedge (d\alpha_t)^k\big) = k\theta_t \wedge \alpha_t \wedge (d\alpha_t)^{k-1} \tag{6.4}$$

is a required solution of (6.3). At first note that V_t is well defined since $\alpha_t \wedge (d\alpha_t)^k$ is a family of non-degenerate volume forms. Note also that V_t vanishes at any point of N since so does the 1-form θ_t. The external multiplication of (6.4) by α_t leads to the relation $V_t \rfloor \alpha_t = 0$. It follows that $L_{V_t}\alpha_t = V_t \rfloor d\alpha_t$ and $(L_{V_t}\alpha_t) \wedge \alpha_t \wedge (d\alpha_t)^{k-1} = (V_t \rfloor (\alpha_t \wedge (d\alpha_t)^k))\big/ k$. By (6.4) V_t satisfies equation (6.3).

Proof of Lemma 6.2 Since α_t is a family of non-vanishing 1-forms then it suffices to prove that $\alpha_t \wedge \nu_t = 0$. If $k = 1$ there is nothing to prove. Assume that $k \geq 2$. Let p be a point of the $(2k + 1)$-space. Denote by $K_{p,t}$ the kernel of the form α_t at the point p, and by $\delta_{p,t}$ the restriction of $d\alpha_t(p)$ to the $2k$-space $K_{p,t}$. Assume that $(\alpha_t \wedge \nu_t)(p) \neq 0$. Then the set of vectors of the space $K_{p,t}$ annihilated by $\nu_t(p)$ is a hyperplane in $K_{p,t}$. Denote it by $W_{p,t}$. Since α_t is contact then $\delta_{p,t}$ is a non-degenerate 2-form. Since $k \geq 2$ then $\dim W_{p,t} = 2k - 1 > k/2$, therefore $\delta_{p,t}$ cannot vanish when being restricted to $W_{p,t}$. This contradicts to the relation $\alpha_t \wedge (d\alpha_t)^{k-1} \wedge \nu_t = 0$. The contradiction shows that $(\alpha_t \wedge \nu_t)(p) = 0$ at any point p of the $(2k + 1)$-space.

A Why is the 3-Dimensional Case Different?

The results in Sections 3–5 show that the classification of singular curves in a contact 3-manifold is much simpler than the classification of singular curves in contact manifolds of dimension ≥ 5. This is so because of the realization Theorem 2.6. The thing is that in Sections 3–5 we studied stratified 1-dimensional submanifolds N of a contact manifold which are contained in a smooth 2-dimensional submanifold. In notation of Section 2 one has $m(N) = 2$. For such $N \subset \mathbb{R}^{2n+1}$ the realization Theorem 2.6 implies that if $n \geq 2$ then the algebraic restriction to N of any Pfaff equation on $\mathbb{R}^{2n+1}$ is realizable by a contact structure, but if $n = 1$ then there is a big set of singular non-realizable algebraic restrictions to N.

By the results of Sections 3 and 4 any *integral* stratified submanifold of a contact 3-space which is diffeomorphic to $N = \{x_1x_2 = x_3 = 0\}$ or to $N = \{x_1^3 - x_2^2 = x_3 = 0\}$ is contactomorphic to N. In these cases N is a singular Legendrian submanifold. There are much more general results on classification of singular Legendrian submanifolds. In [Gi], A. Givental' proved the following result on structural stability of stratified integral Legendrian submanifolds of a contact manifold within perturbations preserving the equivalence class with respect to diffeomorphisms: if N_1 and N_2 are Legendrian stratified submanifolds of a fixed contact manifold whose germs at 0 are diffeomorphic and sufficiently close one to the other then the germ of N_1 at 0 is contactomorphic to the germ of N_2 at some point p close to 0.

The possibility to replace in this statement the point p by 0 and to remove the assumption of closeness of the germs was studied in [Zh-2] and [Is] for the case where N_1 and N_2 are the images of singular Legendrian curves. In [Zh-2] it is proved that almost any two diffeomorphic germs at a fixed point of Legendrian curves in contact 3-space are contactomorphic. In this statement a curve is the image of a map $(\mathbb{R}, 0) \to (\mathbb{R}^3, 0)$, and the words "almost any" exclude a certain set of Legendrain curves of infinite codimension within the space of all Legendrain curves. In [Is], G. Ishikawa proved (independently and by a different method) that in the complex case (holomorphic curves $(\mathbb{C}, 0) \to (\mathbb{C}^3, 0)$) *any* two diffeomorphic germs at a fixed point of Legendrian curves are contactomorphic. These statements do not hold for isotropic singular curves if the dimension of the contact manifold is ≥ 5, see [Ar-1] and results of Sections 3–4 of the present paper.

B Relative Darboux Theorem for Singular Submanifolds of a Symplectic Manifold

The Darboux–Givental' theorem given in Section 2 also holds in the symplectic case: the only local invariant of a *smooth* submanifold N of a symplectic space $(\mathbb{R}^{2n}, \omega)$ is the closed 2-form on N which is the geometric restriction to N of the symplectic 2-form ω, see [Ar-Gi]. Therefore it is natural to ask whether there is, in the symplectic case, an analogous of the algebraic restriction and Theorem 2.1 stating that the algebraic restriction is a complete invariant for *any* subset N of a symplectic space.

There are two ways to define the algebraic restriction of a closed 2-form on $\mathbb{R}^{2n}$ to a subset $N \subset \mathbb{R}^{2n}$:

(a) two germs of closed 2-forms ω_1, ω_2 have the same algebraic restriction to N

if $\omega_1 - \omega_2 = d\alpha$, where α is a 1-form such that $\alpha(p) = 0$ for any $p \in N$ (*i.e.,* the coefficients of α in some and then any coordinate system vanish at any point of N);

(b) two germs of closed 2-forms ω_1, ω_2 have the same algebraic restriction to N if $\omega_1 - \omega_2 = \omega + d\alpha$, where ω is a closed 2-form and α is a 1-form such that $\omega(p) = \alpha(p) = 0$ for any point $p \in N$ (*i.e.* the coefficients of ω and α in some and then any coordinate system vanish at any point of N).

W. Domitrz proved [Do] that if the algebraic restriction of a symplectic structure to N is defined by (a) then Theorem 2.1 holds in the symplectic case (the proof is similar to the proof in the contact case). Nevertheless, definition (b) is much more natural, and namely definition (b) leads to applications including explanation of the local symplectic algebra introduced by V. Arnol'd in [Ar-2] and obtaining further classification results.

If N is a smooth submanifold then (a) and (b) are the same definitions: it is easy to prove that if ω is a germ of a closed 2-form such that $\omega(p) = 0$ for any point $p \in N$ then $\omega = d\alpha$, where α is a germ of a 1-form such that $\alpha(p) = 0$ for any point $p \in N$ (the simplest version of the relative Poincaré lemma). This statement remains true if N is a stratified submanifold provided that the singularities of N are not too deep — a certain type of local quasi-homogeneity of N is required (see, for example, [Gi]).

Recently W. Domitrz, S. Janeczko, and the author obtained a series of new results [Do-Ja-Zh-1] relating the Poincaré lemma property of a stratified submanifold N, the quasi-homogeneity of N, and the algebra of vector fields tangent to N. These results allow to distinguish many new cases where N has the Poincaré lemma property. Under the Poincaré lemma property definitions (a) and (b) are the same. In this case one can construct a method [Do-Ja-Zh-2], parallel to the method of the present paper, allowing to explain and develop the local symplectic algebra introduced in [Ar-2]. If the Poincaré property does not hold (which is the case where N has rather deep singularities) then the classification, with respect to symplectomorphisms, of submanifolds diffeomorphic to N leads to additional invariants on top of the algebraic restriction defined by (b); this follows from the results in the work [Is-Ja]. The work [Do-Ja-Zh-2] will be published elsewhere.

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