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Curves in a Foliated Plane

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Dedicated to Vladimir Igorevich Arnold
on the occasion of his 70th birthday

Abstract—The paper is devoted to the classification of nonsingular and singular plane curve germs with respect to the group of local diffeomorphisms preserving the foliation of the plane by the phase curves of a fixed vector field, either nonsingular or singular. We define the multiplicity of a pair consisting of a plane curve and a vector field and prove an analog of the Tougeron theorem on finite determinacy. It leads, almost immediately, to a number of classification results; a part of them is contained in the work.

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1. INTRODUCTION

All objects below are germs at $0 \in \mathbb{K}^2$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. They belong to a fixed category that is either C^∞, real-analytic, or holomorphic. The paper is devoted to the classification of nonsingular and singular plane curves with respect to the group of local diffeomorphisms preserving the foliation of $\mathbb{K}^2$ by the phase curves of a fixed vector field ξ. The foliation may be either nonsingular ($\xi(0) \neq 0$) or singular ($\xi(0) = 0$). It will be denoted by (ξ) and the foliated plane by $(\mathbb{K}^2, \xi)$. A diffeomorphism preserving the foliation will be called a *symmetry* of this foliation.

Classification of curves in a foliated plane is a part of the local analysis of many objects on 2-manifolds, including differential 1-forms, affine modules of vector fields (= control systems), reversible vector fields (with respect to one or two involutions), and ODEs of the form $A(x)\dot{x} = F(x)$, $\det A(0) = 0$. The volume of this paper does not allow us to present new classification results for these objects that follow from the theorems given below; they will be published elsewhere.

Note that in the problem of classification of *functions* on a foliated plane there are functional moduli even in the case of nonsingular function germs f on $(\mathbb{K}^2, \partial/\partial x)$, $f(0) = 0$, when the order $\mathbf{s}$ of tangency between the curve $\{f = 0\}$ and the x-axis is ≥ 2. In this case f can be reduced by a symmetry of the foliation $(\partial/\partial x)$ to the classical normal form $f = y + a_1(y)x + \ldots + a_{\mathbf{s}-1}(y)x^{\mathbf{s}-1} \pm x^{\mathbf{s}+1}$, where $a_i(0) = 0$. It is worth noting that, although $a_i(y)$ are invariants up to a finite group of transformations, this normal form is "right" only in the case when the order $\mathbf{r}$ of vanishing of $\partial f/\partial x$ (the minimal $\mathbf{r}$ such that $j^{\mathbf{r}}(\partial f/\partial x) \neq 0$; if $a_1'(0) \neq 0$, then $\mathbf{r} = 1$) takes the maximal possible value $\mathbf{s}$, which is the case of codimension $\mathbf{s}(\mathbf{s}-1)/2$. Whatever is $\mathbf{s}$, it is easy to construct a "right" normal form (allowing one to calculate the Poincaré series of the moduli numbers, see [1]) which is parameterized by $\mathbf{r}$ functions of one variable.

As we show below, in the problem of classifying curves in a foliated plane there are no functional moduli, even for arbitrarily deep singularities in the case of singular curves and singular foliations.

The case of nonsingular curves and nonsingular foliations is a trivial part of this problem. Take local coordinates in which the foliation is $(\partial/\partial x)$ and a curve is transversal to the y-axis, i.e., can be described by the equation $y = f(x)$. If it has a finite order of tangency $\mu \geq 0$ with the x-axis, then

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$f(x) = x^{\mu+1}g(x)$, $g(0) \neq 0$, and a suitable symmetry of the foliation of the form $(x, y) \to (\phi(x), \pm y)$ brings the curve to $\{y = x^{\mu+1}\}$. Therefore, one has the following well-known theorem.

Theorem. *All nonsingular curves in the foliated plane* $(\mathbb{K}^2, \xi)$, $\xi(0) \neq 0$, *that have the same finite order of tangency* $\mu \geq 0$ *with the phase curve of the vector field* ξ *passing through* 0 *are equivalent with respect to the action of the group of symmetries of the foliation* (*to the curve* $\{y = x^{\mu+1}\}$ *if* $\xi = \partial/\partial x$).

In Section 2 we introduce the multiplicity of a pair consisting of a plane curve and a vector field, and in Section 3 we present an analog of the classical Tougeron theorem on the finite determinacy of a function germ. It allows us to classify arbitrary singularities of curves in $(\mathbb{K}^2, \xi)$ for any fixed vector field ξ, which is illustrated by almost immediate corollaries in the case of nonsingular curves and a singular foliation (Section 4), in the case of singular curves and a nonsingular foliation (Section 5), and in the case of singular curves and a singular foliation (Section 6). The results of these sections can be continued in various directions.

As one can expect, we deal with a slightly wider class of objects than the plane curve, namely, with 1-generated ideals (f), $f(0) = 0$, in the ring of function germs on $\mathbb{K}^2$. Such an ideal can be identified with the set of points $\gamma = \{f = 0\}$ if the function germ f has the *property of zeros*: any function vanishing at the points of γ belongs to the ideal generated by f. The identification means that the sets $\{f = 0\}$ and $\{\widetilde{f} = 0\}$ coincide if and only if $(f) = (\widetilde{f})$. In order to simplify the exposition, we will use the following convention.

Convention. In what follows a plane curve is a 1-generated ideal (f), $f(0) = 0$, in the ring of function germs on $\mathbb{K}^2$. The curve (f) is nonsingular if $df(0) \neq 0$ and singular if $df(0) = 0$. If f has the property of zeros, in particular if $df(0) \neq 0$, we will use, without mentioning, the identification $(f) \Leftrightarrow \{f = 0\}$ in all statements involving geometric characteristics (the order of tangency, strata of a curve, etc.).[1]

2. MULTIPLICITY OF A PAIR {PLANE CURVE, VECTOR FIELD}

By $\mathcal{F}$ we denote the ring of function germs $f\colon (\mathbb{K}^2, 0) \to \mathbb{K}$. An ideal in $\mathcal{F}$ generated by functions $f_1, \dots, f_s$ will be denoted by $(f_1, \dots, f_s)$. Given a vector field ξ and $f \in \mathcal{F}$, by $\xi(f)$ we denote the Lie derivative of f along ξ.

Definition 2.1. The *multiplicity of a pair consisting of a plane curve* (f) *and a vector field* ξ on $\mathbb{K}^2$, or the ξ*-multiplicity of a curve* (f), is the dimension of the space $\mathcal{F}/(f, \xi(f))$. It will be denoted by $\mu((f), \xi)$.

The multiplicity is well defined: if $(\widetilde{f}) = (f)$, then $\widetilde{f} = Hf$, $H(0) \neq 0$, and consequently $(\widetilde{f}, \xi(\widetilde{f})) = (Hf, \xi(H)f + H\xi(f)) = (f, \xi(f))$.

Proposition 2.2. *Let* ξ *be a vector field on* $\mathbb{K}^2$ *which is either nonsingular or with an algebraically isolated singularity at* 0. *Any two plane curves that can be brought to each other by a symmetry of the foliation* (ξ) *have the same* ξ*-multiplicity.*

Proof. We use the following division property: if a vector field ξ satisfies the assumption of Proposition 2.2 and $\xi \wedge \widetilde{\xi} \equiv 0$, then $\widetilde{\xi} = Q\xi$ for some function Q. (The proof of much more general division properties can be found, for example, in [6].) It follows that any symmetry of the foliation (ξ) preserves the vector field ξ up to multiplication by a nonvanishing function. Therefore, any symmetry of (ξ) that brings an ideal (f) to an ideal $(\widetilde{f})$ brings the ideal $(f, \xi(f))$ to the ideal $(\widetilde{f}, \xi(\widetilde{f}))$.

[1]In fact, the assumptions of our theorems imply that f has a finite multiplicity. In the holomorphic category any such function germ has the property of zeros. In the C^∞ and real-analytic categories the property of zeros of a function germ f of finite multiplicity is violated only in the case $\{f = 0\} = \{0\}$, for example, $f = x^2 + y^{2k}$.

Example 2.3. $\mu((f),\xi)=0$ if and only if the curve (f) is nonsingular, the vector field ξ is nonsingular, and ξ is transversal to (f).

Example 2.4. The multiplicity of a pair consisting of a nonsingular curve (f) and a nonsingular vector field coincides with the order of tangency between (f) and the phase curve of the vector field passing through 0.

This follows from the theorem in the Introduction: it suffices to calculate $\mu((y-x^r g(x)),\partial/\partial x)=\dim\mathcal{F}/(y,x^{r-1})=r-1$, provided $g(0)\neq 0$.

Example 2.5. The curves (x^2-y^4), (y^2-x^4), and (y^2+x^2y) are diffeomorphic. Their $\partial/\partial x$-multiplicities are equal to 4, 6, and ∞, respectively.

Example 2.6. Let (f) be a nonsingular plane curve and let ξ be a singular vector field. Then (f) has ξ-multiplicity ≥ 2 if and only if the restriction of the function $\xi(f)$ to the curve (f) has zero 1-jet. This is equivalent to the condition $j^1(\xi(f))=\lambda j^1 f$, $\lambda\in\mathbb{K}$, which means that the tangent line to the curve (f) is an eigenline of the linear approximation of ξ. This proves the following statement.

Proposition 2.7. *A pair consisting of a nonsingular curve and a singular vector field has the minimal possible multiplicity* 1 *if and only if the curve is transversal to any eigenvector of the linear approximation of the vector field.*

If $\mathbb{K}=\mathbb{R}$ and the eigenvalues of ξ are not real, then any nonsingular curve has ξ-multiplicity 1. Proposition 2.7 implies that a generic nonsingular curve has ξ-multiplicity 1 unless the matrix of the linear approximation of ξ is scalar, i.e., has the form λI, $\lambda\in\mathbb{K}$. In the latter case any curve has ξ-multiplicity ≥ 2.

Example 2.8. Let $\xi=\xi_1\partial/\partial x+\xi_2\partial/\partial y$ be a vector field with a nilpotent singularity at 0 of minimal possible multiplicity 2. This means that ξ has a nonzero linear approximation with zero eigenvalues and $\dim\mathcal{F}/(\xi_1,\xi_2)=2$.

Proposition 2.9. *Let ξ be a vector field with a nilpotent singularity at* 0 *of multiplicity* 2. *Then the ξ-multiplicity of any nonsingular curve is either* 1 *or* 2.

The case when a vector field has a nilpotent singularity of multiplicity ≥ 3 is fundamentally different (see Subsection 4.3).

Proof. In suitable coordinates the 2-jet of ξ has the form $y\partial/\partial x+(ax^2+bxy)\partial/\partial y$, where $a\neq 0$ (see, for example, [2]). Let (f) be a nonsingular curve. If (f) is transversal to the x-axis, then its ξ-multiplicity is equal to 1 (see Proposition 2.7). If (f) is tangent to the x-axis, then we may assume that $f(x,y)=y-g(x)$, where $g(0)=g'(0)=0$. The ideal $(f,\xi(f))$ has the form $(y+\ldots,ax^2+\ldots)$. Since $a\neq 0$, it is diffeomorphic to (y,x^2) and consequently $\dim\mathcal{F}/(f,\xi(f))=2$.

Example 2.10. Let (f) be a curve in $\mathbb{C}^2$ with the Morse (A_1) singularity and let ξ be a singular vector field. Take coordinates in which $f=xy$ and (a_{ij}) is the matrix of the linear approximation of ξ. Then $(f,\xi(f))=(xy,a_{21}x^2+a_{12}y^2+\ldots)$. It follows that $\mu((f),\xi)\geq 4$, and $\mu((f),\xi)=4$ if and only if $a_{21}\neq 0$ and $a_{12}\neq 0$, which means that each of the strata of the curve (f) is transversal to each of the eigenlines of the linear approximation of ξ.

3. THEOREM ON FINITE DETERMINACY

Definition 3.1. Let ξ be a vector field on $\mathbb{K}^2$. A plane curve (f) is called k-determined with respect to the group of symmetries of the foliation (ξ) if for any function g with zero k-jet there exists a symmetry of (ξ) bringing $(f+g)$ to (f).

Example 3.2. If (f) is a nonsingular curve, ξ is a nonsingular vector field, and $\mu((f),\xi)=\mu<\infty$, then (f) is $(\mu+1)$-determined with respect to the group of symmetries of (ξ) and is not μ-determined (see the theorem in the Introduction and Example 2.4).

Theorem A. *Let (f) be any plane curve and ξ be any vector field on $\mathbb{K}^2$ such that at least one of them is singular and $\mu((f),\xi) = \mu < \infty$. Then (f) is μ-determined with respect to the group of symmetries of the foliation (ξ).*

In the case $\xi(0) = 0$ Theorem A is tied with Theorem B below. In what follows by a continuous (respectively, C^1) path of functions, diffeomorphisms, etc., we mean a family of the corresponding objects parameterized by $t \in [0,1]$ that is continuous (respectively, continuously differentiable) with respect to the parameter t.

Theorem B. *Let ξ be a singular vector field on $\mathbb{K}^2$ and let f_t be a C^1-path of functions with the same $(\mu-1)$-jet such that the curves (f_t) have the same ξ-multiplicity $\mu < \infty$. Then there exists a path of symmetries of the foliation (ξ) that brings (f_t) to (f_0).*

Theorems A and B are proved in Section 7. In the case of nonsingular curves and singular foliations these theorems lead to final classification results. To obtain final normal forms in the case of singular curves, one should combine Theorem A with a theorem formulated in Section 5.

4. THE CASE OF NONSINGULAR CURVES AND SINGULAR FOLIATIONS

Throughout this section ξ is a singular vector field on $\mathbb{K}^2$. By λ_1 and λ_2 we denote the eigenvalues of the linearization of ξ.

4.1. Nonsingular curves of ξ-multiplicity 1. Recall from Example 2.6 that such curves exist (and are generic) if and only if the matrix of the linear approximation of ξ is not of the form λI, $\lambda \in \mathbb{K}$.

Theorem 4.1. *Let $\xi(0) = 0$. In the holomorphic category any two nonsingular plane curves of multiplicity 1 can be brought to each other by a symmetry of the foliation (ξ). The same holds in the C^∞ and real-analytic categories if λ_1 and λ_2 are not real or $\lambda_1 = \lambda_2$. If λ_1 and λ_2 are real and distinct, then any nonsingular curve of ξ-multiplicity 1 can be brought by a symmetry of (ξ) to one of the curves $(y \pm x)$, where (x, y) are any coordinates such that $j^1\xi = \lambda_1 x\partial/\partial x + \lambda_2 y\partial/\partial y$.*

Remarks. 1. This theorem is also a corollary of the results obtained in [5]. In fact, the results in [5] imply a much stronger statement: in the holomorphic category not only all nonsingular curves of ξ-multiplicity 1 but also all involutions whose sets of fixed points are such curves can be brought to each other by a symmetry of the foliation (ξ).

2. Whether or not $\pm$ can be reduced to $+$ in the normal form $(y \pm x)$ (if $\mathbb{K} = \mathbb{R}$) depends on the discrete part of the group of local symmetries of the foliation (ξ). For example, it is clear that $\pm$ can be reduced to $+$ if the vector field ξ is linearizable. One of examples in which $\pm$ cannot be reduced to $+$ is the resonant node singularity $\xi = x\partial/\partial x + (Ny + x^N)\partial/\partial y$, where N is an odd integer ≥ 3.

4.2. An elementary singular point: $(\lambda_1, \lambda_2) \neq (0, 0)$.

Theorem 4.2. *Let ξ be a singular vector field on $\mathbb{K}^2$ such that $(\lambda_1, \lambda_2) \neq (0, 0)$. Any C^1-path (f_t) of nonsingular plane curves of the same finite ξ-multiplicity can be brought to (f_0) by a path of symmetries of the foliation (ξ).*

In the case $\mu = 1$ this theorem is a direct corollary of Theorem B. To prove Theorem 4.2 and to obtain normal forms for nonsingular curves in the case $\mu \geq 2$, we will use the simplest classically known normal forms for vector fields (see, for example, [2]). First we consider the case

$$(\lambda_1, \lambda_2) \neq (0, 0), \qquad \lambda_1/\lambda_2, \lambda_2/\lambda_1 \notin \{1, 2, 3, \dots\}. \tag{4.1}$$

In this case any finite jet of ξ has in suitable coordinates the form

$$xa(x,y)\frac{\partial}{\partial x} + yb(x,y)\frac{\partial}{\partial y}, \qquad a(0) = \lambda_1, \quad b(0) = \lambda_2. \tag{4.2}$$

Consider curves of the form

$$\gamma_{x,r} = (f) = (y - cx^r + \text{h.o.t.}), \qquad \gamma_{y,r} = (g) = (x - cy^r + \text{h.o.t.}), \qquad c \neq 0. \tag{4.3}$$

Lemma 4.3. *Given a singular vector field ξ with eigenvalues satisfying* (4.1), *fix coordinates such that the r-jet of ξ has the form* (4.2). *Any curve of the form $\gamma_{x,r}$ or $\gamma_{y,r}$ has ξ-multiplicity r.*

Proof. The restriction of the function $\xi(f)$ (respectively, $\xi(g)$) to the curve $\gamma_{x,r}$ (respectively, $\gamma_{y,r}$) has the form δx^r + h.o.t. (respectively, δy^r + h.o.t.), where $\delta = c(\lambda_2 - r\lambda_1)$ (respectively, $\delta = c(\lambda_1 - r\lambda_2)$). By (4.1) one has $\delta \neq 0$, which implies the lemma.

Lemma 4.3 and Theorem B imply the following statement.

Theorem 4.4. *Let ξ be a singular vector field with eigenvalues satisfying* (4.1). *Let $\mu \geq 2$. Take a coordinate system in which the μ-jet of ξ has form* (4.2). *A nonsingular curve has ξ-multiplicity μ if and only if it has tangency of order $\mu - 1$ with either the x-axis or the y-axis. Any such curve can be brought by a symmetry of the foliation (ξ) to one of the curves $(y \pm x^\mu)$ or $(x \pm y^\mu)$ $(\pm \hookrightarrow +$ if $\mathbb{K} = \mathbb{C})$.*

Consider now the case

$$\lambda_1/\lambda_2 = N \qquad \text{or} \qquad \lambda_2/\lambda_1 = N, \qquad N \in \{1, 2, 3, \dots\}, \tag{4.4}$$

which means that ξ has a singularity of the resonant node type. In this case in suitable coordinates ξ has either the form

$$\lambda x\frac{\partial}{\partial x} + \big(N\lambda y + x^N\big)\frac{\partial}{\partial y}, \qquad \lambda \neq 0, \quad N \in \{1, 2, 3, \dots\}, \tag{4.5}$$

corresponding to the generic (within resonant nodes) case when ξ is not linearizable (diagonalizable if $N = 1$), or the form

$$\lambda x\frac{\partial}{\partial x} + N\lambda y\frac{\partial}{\partial y}, \qquad \lambda \neq 0, \quad N \in \{1, 2, 3, \dots\}. \tag{4.6}$$

Consider curves (4.3). Calculating the ξ-multiplicity in the same way as in the proof of Lemma 4.3, it is easy to prove the following statement.

Lemma 4.5. *In case* (4.5) *any curve of the form $\gamma_{y,r}$, $r \geq 1$, and any curve of the form $\gamma_{x,r}$, $r < N$, has ξ-multiplicity r; all curves of the form $\gamma_{x,r}$, $r \geq N$, and the curve $\gamma_{x,\infty} = (y)$ have ξ-multiplicity N. In case* (4.6) *with $N \geq 2$ any curve of the form $\gamma_{y,r}$, $r \geq 1$, and any curve of the form $\gamma_{x,r}$, $r \neq N$, has ξ-multiplicity r.*

In case (4.6) the ξ-multiplicity of a curve of the form $\gamma_{x,N}$ depends on the higher order terms. In this case we have the following obvious lemma.

Lemma 4.6. *In case* (4.6) *any transformation of the form $(x, y) \to (x, y + \alpha x^N)$ is a symmetry of the vector field ξ, and a suitable transformation of this form brings any curve of the form $\gamma_{x,N}$ to a curve of the form $(y + x^{N+1}g(x))$.*

Lemmas 4.5, 4.6 and Theorem B imply the following classification.

Theorem 4.7. *Let ξ be a vector field of the form* (4.5) *or* (4.6). *A nonsingular curve has ξ-multiplicity $\mu \geq 2$ if and only if it satisfies the condition given in the corresponding cell of the table below. All such curves can be brought by a symmetry of the foliation (ξ) to one of the curves given in the same cell.*

Remarks. 1. Theorem 4.7 and Theorem 4.4 for the case of finitely determined vector fields also follow from (rather involved) classification tables in [4].

Classification of nonsingular curves in $(\mathbb{K}^2, \xi)$; cases (4.5) and (4.6)

Case	$2 \le \mu < N$	$\mu = N \ge 2$	$\mu > N$
(4.5)	Tangency of order $\mu - 1$ with one of the axes x or y $(y \pm x^\mu)$, $(x \pm y^\mu)$	Tangency of order $N-1$ with the y-axis or tangency of order $\ge N-1$ with the x-axis $(x \pm y^N)$, (y)	Tangency of order $\mu - 1$ with the y-axis $(x \pm y^\mu)$
(4.6)	Tangency of order $\mu - 1$ with one of the axes x or y $(y + x^\mu)$, $(x + y^\mu)$	Tangency of order $N-1$ with the y-axis $(x + y^N)$	Tangency of order $\mu - 1$ with one of the axes x or y $(y + x^\mu)$, $(x + y^\mu)$

2. Of course, in the holomorphic category all $\pm$ can be replaced by $+$. It is easy to prove that if $\mathbb{K} = \mathbb{R}$, then in case (4.5) one can replace $\pm$ by $+$ in the normal form $(x \pm y^\mu)$ (respectively, $(y \pm x^\mu)$) if and only if the number $N\mu$ (respectively, $N + \mu + 1$) is even.

Proof of Theorem 4.2. If $\mu = 1$, then Theorem 4.2 is a direct corollary of Theorem B. If $\mu \ge 2$, then Theorem 4.2 follows from Theorem B and Lemmas 4.3, 4.5, and 4.6. In fact, let γ_t be a C^1-path of nonsingular curves of the same ξ-multiplicity $\mu \ge 2$. Lemmas 4.3 and 4.5 imply that the curves γ_t can be described by functions with the same $(\mu-1)$-jet unless one has case (4.6) with $\mu = N$. In the latter case Lemma 4.6 implies that there exists a C^1-path of symmetries of ξ reducing the path γ_t to a path of curves with the same $(N-1)$-jet.

4.3. The case of nilpotent linear approximation. Consider now the case $\lambda_1 = \lambda_2 = 0$, $j^1\xi \ne 0$, i.e., ξ has a nilpotent singularity at 0. The Taylor series of ξ can be reduced by a change of coordinates to the form

$$y\frac{\partial}{\partial x} + \big((a_2x^2 + a_3x^3 + \ldots) + y(b_1x + b_2x^2 + \ldots)\big)\frac{\partial}{\partial y} \tag{4.7}$$

(see [2]). In what follows, distinguishing or normalizing curves of ξ-multiplicity μ, we use any coordinate system in which the μ-jet of ξ has this normal form. Propositions 2.7 and 2.9 and Theorem A imply

Theorem 4.8. *If $a_2 \ne 0$, then any nonsingular curve has ξ-multiplicity 1 or 2 and can be reduced by a symmetry of (ξ) to one of the curves (x) or (y).*

The case $a_2 = 0$ is different. The case of ξ-multiplicity 1 is completely covered by Proposition 2.7 and Theorem 4.1: it holds if and only if the curve is transversal to the x-axis and if and only if it can be brought by a symmetry of the foliation to the curve (x). Consider a nonsingular curve of ξ-multiplicity >1. It has the form

$$(y - \alpha_2x^2 - \alpha_3x^3 - \ldots). \tag{4.8}$$

It is easy to calculate that the ξ-multiplicity of this curve depends on the numbers

$$\theta_3 = a_3 + \alpha_2(b_1 - 2\alpha_2), \qquad \theta_m = a_m + \sum_{i=1}^{m-2}\big(b_i - (i+1)\alpha_{i+1}\big)\alpha_{m-i}, \quad m \ge 4.$$

Proposition 4.9. *Let $a_2 = 0$. The ξ-multiplicity of the curve (4.8) is greater than or equal to 3. It is equal to 3 if $\theta_3 \ne 0$, and it is equal to $\mu \ge 4$ if $\theta_3 = \ldots = \theta_{\mu-1} = 0$ and $\theta_\mu \ne 0$.*

Note that if we fix α_2 such that $\theta_3 = 0$, i.e., α_2 is a root of the polynomial

$$P(\alpha) = a_3 + b_1\alpha - 2\alpha^2,$$

then the conditions $\theta_4 = \theta_5 = \ldots = 0$ define unique values of $\alpha_3, \alpha_4, \ldots$, provided that $\alpha_2 \neq b_1/(i+1)$, $i = 4, \ldots, m$. Such numbers are not the roots of $P(\alpha)$ if

$$(m+1)^2 a_3 + (m-1)b_1^2 \neq 0, \qquad m \in \{4, 5, 6, \ldots\}. \tag{4.9}$$

We obtain the following corollary of Proposition 4.9 and Theorem B.

Theorem 4.10. *Let $a_2 = 0$ and let a_3 and b_1 satisfy* (4.9). *Consider a nonsingular curve of ξ-multiplicity μ.*

1. *In the holomorphic category the realizable values for μ are $\{1, 3, 4, 5, \ldots, \infty\}$. The same holds if $\mathbb{K} = \mathbb{R}$ provided that $b_1^2 + 8a_3 \geq 0$. If $\mathbb{K} = \mathbb{R}$ and $b_1^2 + 8a_3 < 0$, then the realizable values for μ are* 1 *and* 3 *only.*[2]

2. *The case $\mu = 3$ holds if and only if the curve can be reduced by a symmetry of* (ξ) *to the normal form $(y - \mathbf{r}x^2)$, where $\mathbf{r}$ is not a root of the polynomial $P(\alpha)$.*

3. *The case $4 \leq \mu < \infty$ holds if and only if the curve can be reduced by a symmetry of* (ξ) *to the normal form $(y - \alpha_2^* x^2 - \alpha_3^* x^3 - \ldots - \alpha_{\mu-2}^* x^{\mu-2} - \mathbf{r}x^{\mu-1})$, where α_2^* is one of the roots of the polynomial $P(\alpha)$, $\alpha_3^*, \ldots, \alpha_{\mu-2}^*$ are uniquely defined by α_2^* and the conditions $\theta_4 = \ldots = \theta_{\mu-1} = 0$, and $\mathbf{r}$ is a parameter such that $\theta_\mu \neq 0$.*

The case when condition (4.9) is violated can also be analyzed. Consider, for example, the case $a_2 = a_3 = b_1 = 0$, $a_4 \neq 0$. In this case $\theta_3 = -2\alpha_2^2$ and $\theta_4 = a_4 + b_2\alpha_2 - 5\alpha_2\alpha_3$. Therefore, $\theta_3 = 0$ if and only if $\alpha_2 = 0$, and the case $\theta_3 = \theta_4 = 0$ is impossible. We obtain the following result.

Theorem 4.11. *In the case $a_2 = a_3 = b_1 = 0$, $a_4 \neq 0$ the ξ-multiplicity of any nonsingular curve is either* 1, 3, *or* 4. *It is equal to* 3 (*respectively,* 4) *if and only if the curve can be brought by a symmetry of the foliation* (ξ) *to the normal form* $(y - \mathbf{r}x^2)$, $\mathbf{r} \neq 0$ (*respectively,* $(y - \mathbf{r}x^3)$).

One can prove that in all the normal forms obtained the parameter $\mathbf{r}$ is a modulus under certain genericity assumptions on some nonfixed parameters in (4.7).

4.4. The case when a vector field has zero 1-jet. Consider a singular vector field of the form

$$\xi = \big(P_m(x,y) + \ldots\big)\frac{\partial}{\partial x} + \big(Q_m(x,y) + \ldots\big)\frac{\partial}{\partial y}, \qquad m \geq 2, \tag{4.10}$$

where P_m and Q_m are homogeneous degree m polynomials and the dots denote functions with the zero m-jet. It is clear that the ξ-multiplicity of any nonsingular curve $(ax + by + \ldots)$ is $\geq m$, and it is easy to show that the ξ-multiplicity equals m if and only if the straight line $\{ax + by = 0\}$ is not an invariant line of the homogeneous vector field $P_m(x,y)\partial/\partial x + Q_m(x,y)\partial/\partial y$, i.e., $aP_m(b,-a) + bQ_m(b,-a) \neq 0$. Note that if $yP_m(x,y) - xQ_m(x,y) \not\equiv 0$, then this homogeneous vector field has not more than $m+1$ invariant straight lines, and consequently a generic nonsingular curve has ξ-multiplicity m. By Theorem B we obtain the following result.

Theorem 4.12. *Let $m \geq 2$ and let ξ be a vector field with the m-jet $P_m(x,y)\partial/\partial x + Q_m(x,y)\partial/\partial y$, where P_m and Q_m are homogeneous degree m polynomials. Any curve $(f) = (ax + by + \ldots)$ such that $aP_m(b,-a) + bQ_m(b,-a) \neq 0$ can be brought by a symmetry of the foliation* (ξ) *to the curve* $(j^{m-1}f)$.

[2] We see that the analysis of realizable values of the ξ-multiplicity of nonsingular curves can be used for distinguishing different types of phase portraits. The condition $D = b_1^2 + 8a_3 \geq 0$ can be expressed in terms of the blow-up $(x,y) \to (x^2, y)$ of the vector field $y\partial/\partial x + (a_3x^3 + b_1xy)\partial/\partial y$, which gives xV, $V = 2y\partial/\partial x + (a_3x + b_1y)\partial/\partial y$. The linear vector field V has real eigenvalues if and only if $D \geq 0$. A nonsingular curve tangent to the x-axis has the minimal possible ξ-multiplicity 3 if and only if it is transversal to the eigenvectors of V.

In particular, if $m = 2$ and $yP_2(x,y) - xQ_2(x,y) \not\equiv 0$, then a generic nonsingular curve can be brought by a symmetry of (ξ) to the normal form $(\cos\theta x + \sin\theta y)$. It is clear that if P_2 and Q_2 are generic, then there are no linear transformations, except the identity, preserving the vector field $P_2\partial/\partial x + Q_2\partial/\partial y$ up to multiplication by a number, and then the parameter $\theta \in S^1$ is an invariant.

5. THE CASE OF SINGULAR CURVES AND NONSINGULAR FOLIATIONS

Using Theorem A it is easy to determine and classify all *simple* singular curves in a plane endowed with a nonsingular foliation.

Definition 5.1. Let ξ be a nonsingular vector field on $\mathbb{K}^2$. A curve in the foliated plane $(\mathbb{K}^2, \xi)$ is simple if the pair consisting of this curve and the foliation (ξ) is simple with respect to the action of the group of diffeomorphisms $(\mathbb{K}^2, 0) \to (\mathbb{K}^2, 0)$.

We use the standard notation A_k for the class of functions that are R-equivalent to $x^2 \pm y^{k+1}$, as well as for curves (f), $f \in A_k$.

Theorem 5.2. *A singular curve (f) in a foliated plane $(\mathbb{K}^2, \xi)$, $\xi(0) \neq 0$, of ξ-multiplicity μ is simple if and only if it satisfies one of the following conditions*:

(a) $f \in A_1$, $\mu < \infty$;

(b) $f \in A_2$;

(c) $f \in A_k$, $k \geq 3$, $\mu = k + 1$.

Take coordinates (x, y) such that $\xi = \partial/\partial x$. Any curve satisfying one of these conditions can be reduced by a symmetry of the foliation $(\partial/\partial x)$ to one of the curves

$$A_1^k = (xy + x^k), \qquad A_k^{k+1} = (x^2 \pm y^{k+1}), \qquad A_2^4 = (y^2 + x^3), \qquad k \geq 2,$$

where the upper index in the notation of the normal forms is the $\partial/\partial x$-multiplicity.

If $\mathbb{K} = \mathbb{C}$ or if k is even, then $\pm$ should be replaced by $+$. By Theorem 5.2 *the ξ-multiplicity of any curve of the class A_2 is either* 3 *or* 4. This statement is a part of the following statement.

Theorem 5.3. *The multiplicity of a pair consisting of a curve of the class A_k and a nonsingular vector field is not less than $k + 1$. If k is an even number, then it is not greater than $2k$. It may be arbitrarily large (including ∞) if k is odd.*

Theorems 5.2 and 5.3 are proved below. The hierarchy of A-singularities in a plane endowed with a nonsingular foliation starts with

$$\begin{tikzcd}
(\mathbf{A}_1^2)^0 & (\mathbf{A}_2^3)^1 \arrow[l] & (\mathbf{A}_3^4)^2 \arrow[l] & (\mathbf{A}_4^5)^3 \arrow[l] & \cdots \arrow[l] \\
(\mathbf{A}_1^3)^1 \arrow[u] & (\mathbf{A}_2^4)^2 \arrow[l] \arrow[u] & (A_3^6)^3 \arrow[l] \arrow[u] \arrow[dll] & (A_4^7)^4 \arrow[l] \arrow[u] & \cdots \arrow[l] \\
(\mathbf{A}_1^4)^2 \arrow[u] & & (A_3^7)^4 \arrow[u] \arrow[dll] & & \\
(\mathbf{A}_1^5)^3 \arrow[u] & & (A_3^8)^5 \arrow[u] \arrow[dll] & & \\
\cdots \arrow[u] & & \cdots \arrow[u] & &
\end{tikzcd}$$

Here $\mathbf{A}_k^{k+1}$, $\mathbf{A}_1^k$, and $\mathbf{A}_2^4$ are the simple singularities determined in Theorem 5.2 (they are marked by bold face), and the other singularities are unimodal[3] and as follows (in the coordinates x, y such

[3] I classified all unimodal A-singularities; the diagram contains not all of them.

that the foliation is $(\partial/\partial x)$; the case $\mathbb{K}=\mathbb{C}$):

$$A_3^6 = (y^2 + x^2y + \mathbf{r}x^4), \qquad \mathbf{r} \notin \{0, 1/4\};$$
$$A_3^{k+2} = (y^2 + x^2y + x^k), \qquad k \in \{5,6,7,\dots\};$$
$$A_{k-1}^{k+2} = ((y + x^2)^2 + x^k), \qquad k \in \{5,6,7,\dots\}.$$

The parameter $\mathbf{r}$ in A_3^6 is a modulus. The diagram contains all singularities that are irremovable in 4-parameter families of curves. It shows all adjacencies between the presented singularities. The index k in the notation $(A_k^\mu)^c$ means that the singularity belongs to the class A_k (with respect to the group of all diffeomorphisms); μ is the $\partial/\partial x$-multiplicity; and c is the codimension, i.e., the minimal number c of parameters such that the singularity is irremovable in c-parameter families of curves.

Proof of Theorem 5.3. We may assume that $\xi = \partial/\partial x$. Consider a curve $(f) = (ax^2 + bxy + cy^2 + \dots) \in A_k$. If $a \neq 0$, then the curve can be reduced to $x^2 \pm y^{k+1}$ by a local diffeomorphism $(x, y) \to (\Phi(x, y), \phi(y))$, which preserves $(\partial/\partial x)$. The $\partial/\partial x$-multiplicity of this curve is equal to $k+1$. Therefore, the first statement holds for generic and consequently for all curves in the class A_k.

To prove the second statement, consider the case $a = 0$. Then $b = 0$ (otherwise the curve belongs to the class A_1) and $c \neq 0$. We may assume $c = 1$. Then on the level of Taylor series $(f) = (y^2 + g_0(x) + g_1(x)y)$. A curve of this form is diffeomorphic to the curve $(y^2 + g_0(x) - g_1^2(x)/4)$. If it belongs to a class $A_{2\ell}$, then $j^{2\ell+1}g_0 \neq 0$. It follows that the $\partial/\partial x$-multiplicity of any curve of the class $A_{2\ell}$ is not greater than the $\partial/\partial x$-multiplicity of the curves $(y^2 + x^{2\ell+1} + \text{h.o.t.})$. All such curves have the same $\partial/\partial x$-multiplicity $\dim \mathcal{F}/(y^2, x^{2\ell}) = 4\ell$.

To prove the last statement, consider the curve $(y^2 + xy^\ell + x^p)$, where $p \geq 2\ell + 1$. It belongs to the class $A_{2\ell+1}$, and its $\partial/\partial x$-multiplicity tends to ∞ as $p \to \infty$. The curve $(y^2 + xy^\ell)$ belongs to the class $A_{2\ell+1}$ and has infinite $\partial/\partial x$-multiplicity.

Proof of Theorem 5.2. We may assume that $\xi = \partial/\partial x$. Theorem 5.2 is a corollary of the following statements.

1. If $j^2 f = 0$, then the curve (f) is not simple. In fact, the dimension of the space of homogeneous degree 3 polynomials is equal to 4, whereas the dimension of the group of linear transformations preserving the x-axis is equal to 3.

2. It is clear that a curve of the class A_k of the form $(x^2 + bxy + cy^2 + \text{h.o.t.})$ can be reduced to the curve $(x^2 \pm y^{k+1})$ by a local diffeomorphism $(x, y) \to (\Phi(x, y), \phi(y))$, which preserves $(\partial/\partial x)$. The $\partial/\partial x$-multiplicity of this curve equals $k + 1$.

3. A curve of the form $(f) = (xy + cy^2 + \text{h.o.t.})$ belongs to the class A_1. The Taylor series of such a curve can be reduced by a change of coordinates of the form $(x, y) \to (\Phi(x, y), y)$ to the form $(xy + g(x))$. The ideal $(xy + g(x), y + g'(x))$ is diffeomorphic to the ideal $(y, g(x) - xg'(x))$. It follows that $g(x) = x^\mu \widetilde{g}(x)$, $\widetilde{g}(0) \neq 0$. By Theorem A the curve (f) can be reduced by a symmetry of $(\partial/\partial x)$ to the curve $(xy + cx^\mu)$, $c \neq 0$. The parameter c can be scaled to 1.

4. Let $(f) = (y^2 + \text{h.o.t.})$. Then on the level of Taylor series one has $(f) = (y^2 + g_0(x) + g_1(x)y)$. We will need the following representation for the 4-jet:

$$(f) = (y^2 + a_1x^3 + b_1x^2y + a_2x^4 + b_2x^3y + \text{h.o.t.}). \tag{5.1}$$

If $a_1 \neq 0$, then a_1 can be scaled to 1 and b_1, a_2, and b_2 can be reduced to 0 by a change of coordinates of the form $(x, y) \to (\Phi(x, y), y)$. The curve $(y^2 + x^3)$ has $\partial/\partial x$-multiplicity 4. Therefore, by Theorem A the curve (f) can be reduced to $(y^2 + x^3)$ by a symmetry of $(\partial/\partial x)$.

5. Assume now that in (5.1) one has $a_1 = 0$ and $b_1 \neq 0$. Then b_1 can be reduced to 1 and a_2 to 0. In the obtained normal form $(y^2 + x^2y + \mathbf{r}x^4 + \text{h.o.t.})$ the parameter $\mathbf{r}$ is a modulus not only with respect to the group of symmetries of $(\partial/\partial x)$ but also with respect to the group of diffeomorphisms preserving the x-axis. It is the same as the modulus in the normal form J_{10} for function germs, i.e., the modulus in the classification of the 2-jets of triples consisting of a nonsingular curve and two parabolas $(\{y + r_ix^2 = 0\}$, $r_{1,2} = 1/2 \pm \sqrt{1/4 - \mathbf{r}})$ tangent to this curve.

To classify singular curves in $(\mathbb{K}^2, \xi)$, $\xi(0) \neq 0$, of modality ≥ 1, one should combine Theorem A with the following theorem.

Theorem C. *Let f and g be function germs on $\mathbb{K}^2$ such that $j^{k-1}g = 0$. Let ξ be a vector field on $\mathbb{K}^2$. Assume that there exist functions h_1 and h_2 such that*

$$h_1(0) = 0, \qquad j^1(h_2\xi) = 0, \qquad j^k(h_1 f + h_2\xi(f) + g) = 0.$$

Then there exists a symmetry of the foliation (ξ) bringing the curve $(f + g)$ to a curve of the form $(f + \widetilde{g})$, where $j^k\widetilde{g} = 0$.

This theorem is proved in Section 7, along with Theorems A and B. We also need it for classifying singular curves in a plane endowed with a singular foliation.

6. THE CASE OF SINGULAR CURVES AND SINGULAR FOLIATIONS

We present two examples for singular curves in $(\mathbb{C}^2, \xi)$, $\xi(0) = 0$. In the first example the curves belong to the class A_1, and we restrict ourselves to the case when the eigenvalues λ_1 and λ_2 of the linearization of ξ satisfy the condition

$$(\lambda_1, \lambda_2) \neq (0, 0), \qquad \lambda_1/\lambda_2, \lambda_2/\lambda_1 \notin \{1, 2, 3, 4, \dots\}. \tag{6.1}$$

In the second example the curves belong to the class A_2, and

$$\lambda_1 \neq \lambda_2, \qquad \lambda_1/\lambda_2, \lambda_2/\lambda_1 \neq 3/2. \tag{6.2}$$

Note that it is clear a priori that precisely these constraints on the eigenvalues simplify the classification.

Theorem 6.1. *Let ξ be a singular vector field on $\mathbb{C}^2$ whose eigenvalues satisfy* (6.1). *Any curve (f) in $\mathbb{C}^2$ of the class A_1 has ξ-multiplicity $\mu \geq 4$. If $\mu < \infty$, then it can be reduced by a symmetry of the foliation (ξ) to one of the curves*

$$(x^2 + y^2) \quad (\mu = 4); \qquad (xy + x^n + \mathbf{r}y^m), \quad \mathbf{r} \neq 0 \quad (\mu = m + n + 2 \geq 4),$$

where (x, y) is a coordinate system in which $j^\mu\xi = xf_1(x, y)\partial/\partial x + yf_2(x, y)\partial/\partial y$.

Remark. Under condition (6.1) the required coordinate system exists for any μ (see [2]). In the second normal form in the generic case $(n, m) = (2, 2)$ the parameter $\mathbf{r}$ is a modulus. If $n + m \geq 5$, then $\mathbf{r}$ can be reduced to 1 under certain conditions on ξ. The simplest sufficient condition for this reduction is the case when ξ is linearizable. If ξ is a finitely determined vector field, then Theorem 6.1 can also be obtained from the classification tables in [4].

Theorem 6.2. *Let ξ be a singular vector field on $\mathbb{C}^2$ whose eigenvalues satisfy* (6.2). *The ξ-multiplicity of any curve (f) of the class A_2 is either* 5 *or* 6. *Fix in this case any coordinate system in which the* 1*-jet of ξ has the form $\lambda_1 x\partial/\partial x + \lambda_2 y\partial/\partial y$. The curve (f) can be reduced by a symmetry of (ξ) to one of the curves*

$$((x + y)^2 + \mathbf{r}x^3), \quad \mathbf{r} \neq 0 \quad (\textit{if} \ \ \mu = 5); \qquad (x^2 + \mathbf{r}xy^2 + y^3), \quad (y^2 + \mathbf{r}x^2y + x^3) \quad (\textit{if} \ \ \mu = 6).$$

Remark. If ξ is linearizable, then in all these normal forms $\mathbf{r}$ can be reduced to 1 provided that in the last two normal forms $\mathbf{r} \neq 0$.

Proof of Theorem 6.1. Let $(f) \in A_1$, $f = P(x,y) + \text{h.o.t.}$, where $P(x,y) = ax^2 + bxy + cy^2$. Calculate the ideal $(P, \xi(P))$. The assumption $\lambda_1 \neq \lambda_2$ implies

$$(P, \xi(P)) = \bigl(ax^2 + bxy + cy^2,\ 2by^2 + cxy + \text{h.o.t.}\bigr). \tag{6.3}$$

First we consider the degenerate but simple case $b = 0$. In this case, since $P \in A_1$, one has $a, c \neq 0$ and then (6.3) is diffeomorphic to the ideal $(xy, x^2 + y^2)$. It follows that the ξ-multiplicity of (f) is equal to 4. By Theorem C any finite jet of (f), in particular the 4-jet, can be reduced to $(ax^2 + cy^2)$ by a symmetry of (ξ). By Theorem A the whole curve (f) can be reduced to this normal form by a symmetry of (ξ). The parameters a and c can be scaled to 1 using the flow of ξ (it has the form $(x,y) \to (e^{t\lambda_1}x, e^{t\lambda_2}y) + \text{h.o.t.}$) and the assumption $\lambda_1 \neq \lambda_2$.

Now we assume $b \neq 0$. In this case, for any homogeneous polynomial $h(x,y)$ of degree $k \geq 3$, one has $h(x,y) = \alpha_1 x^k + \alpha_2 y^k + j^k \widetilde{h}(x,y)$, where $\widetilde{h}$ belongs to the ideal (6.3). (To see this in the case $c \neq 0$, it is enough to take the second generator of the ideal. If $c = 0$, then $b \neq 0$ and it is enough to take the first generator.) Theorem C implies that (f) can be reduced by a symmetry of (ξ) to the form $(xy + g_1(x) + g_2(y))$ on the level of Taylor series. Since $\xi(x) \in (x)$ and $\xi(y) \in (y)$, the ξ-multiplicity of (f) is finite if and only if $j^\infty g_1 \neq 0$ and $j^\infty g_2 \neq 0$. Therefore, we have the normal form

$$\bigl(xy + (a_n x^n + a_{n+1}x^{n+1} + \ldots) + (b_m y^m + b_{m+1} y^{m+1} + \ldots)\bigr), \qquad a_n \neq 0, \quad b_m \neq 0.$$

One of the parameters a_n or b_m can be reduced to 1 using the flow of ξ and the assumptions $\lambda_2 \neq (n-1)\lambda_1$ and $\lambda_1 \neq (m-1)\lambda_2$. Now Theorem 6.1 follows from Theorems C and A and the following statements (each of them can be easily checked).

1. Let $Q(x,y) = xy + x^2 + ry^2$, $r \neq 0$. The ideal $(Q, \xi(Q))$ is diffeomorphic to the ideal $(xy, x^2 + y^2)$ provided that $\lambda_1 \neq \lambda_2$ and $Q(x,y)$ is a nondegenerate quadratic form (cf. Example 2.10). In this case one has $(Q, \xi(Q)) \supset \mathcal{M}^3$ and $\dim \mathcal{F}/(Q, \xi(Q)) = 4$, where $\mathcal{M}$ is the maximal ideal in the ring of function germs.

2. Let $Q(x,y) = xy + x^n$. Then $(Q, \xi(Q)) \supset x\mathcal{M}^n$ provided that $\lambda_2 \neq (n-1)\lambda_1$. Similarly, if $Q(x,y) = xy + y^m$, then $(Q, \xi(Q)) \supset y\mathcal{M}^m$ provided that $\lambda_1 \neq (m-1)\lambda_2$.

3. Let $Q(x,y) = xy + r_1 x^n + r_2 y^m$, where $r_1, r_2 \neq 0$, $n, m \geq 2$, and $n + m \geq 5$. If $\lambda_2 \neq (n-1)\lambda_1$ and $\lambda_1 \neq (m-1)\lambda_2$, then

$$(Q, \xi(Q)) \supset \mathcal{M}^k, \qquad k = \max(n,m) + 1, \qquad \dim \mathcal{F}/(Q, \xi(Q)) = n + m + 2.$$

Proof of Theorem 6.2. Let $(f) = (S(x,y) + \text{h.o.t.})$, $S(x,y) = (ax + by)^2$. First assume that $a \neq 0$ and $b \neq 0$. Since $\lambda_1 \neq \lambda_2$, a and b can be reduced to 1 by the flow of ξ. One has $(S, \xi(S)) = (x + y)\mathcal{M}$. By Theorem C the curve (f) can be brought by a symmetry of (ξ) to the form $(\widetilde{f})$, where $j^3\widetilde{f} = Q = (x+y)^2 + rx^3$. It is easy to prove that the ideal $(Q, \xi(Q))$ is diffeomorphic to $(xy, x^2 + y^3)$ (for any r provided $\lambda_1 \neq \lambda_2$). Therefore, $(Q, \xi(Q)) \supset \mathcal{M}^4$ and consequently $(f, \xi(f)) \supset \mathcal{M}^4$. Theorems C and A imply that (f) can be reduced to (Q) by a symmetry of the foliation (ξ). The ξ-multiplicity of (f) is equal to $\dim \mathcal{F}/(xy, x^2 + y^3) = 5$.

The assumptions $\lambda_1/\lambda_2 \notin \{2/3, 3/2\}$ are essential only in the remaining cases $(f) = (x^2 + \text{h.o.t.})$ and $(f) = (y^2 + \text{h.o.t.})$. We will assume $(f) = (x^2 + \text{h.o.t.})$; the other case is symmetric. One has $(x^2 + \text{h.o.t.}) = (x^2 + rxy^2 + r_1 y^3 + \text{h.o.t.})$. Since $f \in A_2$, it follows that $r_1 \neq 0$. The assumption $2\lambda_1 \neq 3\lambda_2$ implies that r_1 can be reduced to 1 by the flow of ξ. Let $T = x^2 + rxy^2 \pm y^3$. Calculating the ideal $(T, \xi(T))$, we see that under the same assumption $2\lambda_1 \neq 3\lambda_2$ it is diffeomorphic to the ideal (x^2, y^3). Therefore, $(T, \xi(T)) \supset \mathcal{M}^4$ and consequently $(f, \xi(f)) \supset \mathcal{M}^4$. By Theorems C and A the curve (f) can be reduced to (T) by a symmetry of (ξ). The ξ-multiplicity of (f) is equal to $\dim \mathcal{F}/(x^2, y^3) = 6$.

7. PROOFS OF THEOREMS A–C

The homotopy method and the observation that for any vector field ξ and for any function germ h the vector field $h\xi$ is an infinitesimal symmetry of the foliation (ξ) lead to Proposition 7.1. Theorem C is an almost direct corollary of this proposition (Subsection 7.2). Theorems A and B follow from Proposition 7.1 and several auxiliary statements given in Subsections 7.3 and 7.4.

7.1. The homotopy method. Given a C^1-path $f_t \in \mathcal{F}$ and a vector field ξ on $\mathbb{K}^2$, consider the following equation with respect to a couple of paths $h_{1,t}, h_{2,t} \in \mathcal{F}$:

$$W_t = h_{1,t} f_t + h_{2,t}\xi(f_t) + \frac{df_t}{dt} = 0. \tag{7.1}$$

Proposition 7.1. *If there exist C^0-paths $h_{1,t}$ and $h_{2,t}$ such that $(h_{2,t}\xi)(0) = 0$ and $W_t = 0$ (respectively, $j^k W_t = 0$), then there exists a C^1-path of symmetries of (ξ) bringing the ideal (f_t) to the ideal (f_0) (respectively, to the ideal $(f_0 + \widetilde{f}_t)$, where $j^k \widetilde{f}_t = 0$).*

Proof. Define a C^1-path of maps $\Phi_t\colon (\mathbb{K}^2, 0) \to (\mathbb{K}^2, 0)$ and a C^1-path of functions H_t by the following system of ODEs and the initial conditions:

$$\frac{d\Phi_t}{dt} = (h_{2,t}\xi)(\Phi_t), \qquad \Phi_0 = \mathrm{id}, \qquad \frac{dH_t}{dt} = h_{1,t}(\Phi_t)H_t, \qquad H_0 \equiv 1. \tag{7.2}$$

It is easy to prove that Φ_t is a local diffeomorphism for any t along the path. It moves any point p along the phase curve of ξ containing p. Therefore, it is a symmetry of the foliation (ξ). Differentiating the family $A_t = H_t f_t(\Phi_t)$ with respect to t and using (7.1) and (7.2), we obtain $A'(t) \equiv 0$. It follows that $H_t f_t(\Phi_t) = f_0$. Since the second ODE in (7.2) is linear, $H_t(0) \neq 0$ and consequently $(f_t(\Phi_t)) = (f_0)$. If one has $j^k W_t = 0$ instead of (7.1), then we obtain $(j^k A(t))' \equiv 0$, which implies $j^k(H_t f_t(\Phi_t)) = j^k f_0$. In this case one has $(f_t(\Phi_t)) = (f_0 + \widetilde{f}_t)$, where $j^k \widetilde{f}_t = 0$.

7.2. Proof of Theorem C. Let f, g, h_1, and h_2 be functions as in Theorem C. Let $f_t = f + tg$. Since $j^k(h_1 g) = j^k(h_2\xi(g)) = 0$, the equation $j^k W_t = 0$ has a solution $h_{1,t} \equiv h_1$, $h_{2,t} \equiv h_2$ and Theorem C follows from Proposition 7.1.

7.3. Auxiliary lemmas.

Lemma 7.2. *Let $\mathcal{I}$ be an ideal in the ring $\mathcal{F}$ such that $\dim \mathcal{F}/\mathcal{I} = \mu < \infty$. Then $\mathcal{I}$ contains any function germ with the zero $(\mu - 1)$-jet.*

A simple and nice proof can be found in [3, Section 5.5] for the case when $\mathcal{I}$ is the gradient ideal of a function; in the general case the proof is exactly the same.

Lemma 7.3. *Let f be a singular function germ ($df(0) = 0$) and let ξ be a nonsingular vector field such that $\dim \mathcal{F}/(f, \xi(f)) = \mu < \infty$. If $j^{\mu-1}(Af + B\xi(f)) = 0$ for some functions A and B, then $B(0) = 0$.*

Proof. Take local coordinates x, y such that $\xi = \partial/\partial x$. Assume that $B(0) \neq 0$. Then $\partial f/\partial x = Cf + g$ for some functions C and g such that $j^{\mu-1} g = 0$. It follows that $j^\mu f = a_1 y + a_2 y^2 + \ldots + a_\mu y^\mu$. Since $df(0) = 0$, we have $a_1 = 0$. Therefore, the classes of functions $1, y, x, x^2, \ldots, x^{\mu-1}$ in the factor space $\mathcal{F}(f, \xi(f))$ are linearly independent, and consequently the dimension of this factor space is not less than $\mu + 1$, a contradiction.

7.4. Auxiliary proposition. Combining Lemmas 7.2 and 7.3 with Proposition 7.1, we obtain the following statement towards the proof of Theorems A and B.

Proposition 7.4. *Let ξ be a vector field on $\mathbb{K}^2$ and let f_t be a C^1-path of functions with the same $(\mu - 1)$-jet such that either $df_t(0) = 0$ or $\xi(0) = 0$. Let $\dim \mathcal{F}/(f_0, \xi(f_0)) = \mu < \infty$. Assume that*

$$\begin{pmatrix} f_t \\ \xi(f_t) \end{pmatrix} = M_t \begin{pmatrix} f \\ \xi(f) \end{pmatrix} \tag{7.3}$$

for some C^0-path M_t of matrix-valued functions such that $\det M_t(0) \neq 0$ *for any* $t \in [0,1]$. *Then there exists a C^1-path of diffeomorphisms preserving the foliation* (ξ) *and bringing the ideal* (f_t) *to the ideal* (f_0).

Proof. By Proposition 7.1 it suffices to prove the existence of C^0-paths $h_{1,t}$ and $h_{2,t}$ satisfying (7.1) and such that $h_{2,t}(0) = 0$ if $\xi(0) \neq 0$. If $\xi(0) \neq 0$, then, as we have assumed, $df_t(0) = 0$ and Lemma 7.3 implies that the requirement $h_{2,t}(0) = 0$ is a corollary of (7.1). Since $\det M_t(0) \neq 0$ and $t \in [0,1]$, the solvability of (7.1) follows from that of the equation $\widetilde{h}_{1,t} f_0 + \widetilde{h}_{2,t}\xi(f_0) + (d/dt)f_t = 0$. The solvability of this equation with respect to the C^0-paths $\widetilde{h}_{1,t}$ and $\widetilde{h}_{2,t}$ follows from Lemma 7.2.

7.5. Proof of Theorem A. Let $g \in \mathcal{F}$, $j^\mu g = 0$, and let $f_t = f + tg$, $t \in [0,1]$. By Proposition 7.4 it suffices to prove that one has (7.3) with a C^0-path M_t of matrix-valued functions such that $\det M_t(0) \neq 0$. By Lemma 7.2 one has

$$g = h_{11} f + h_{12}\xi(f), \qquad \xi(g) = h_{21} f + h_{22}\xi(f), \tag{7.4}$$

where $h_{11}(0) = h_{12}(0) = 0$ and if $\xi(0) = 0$, then also $h_{21}(0) = h_{22}(0) = 0$. Therefore, in the case $\xi(0) = 0$ one has (7.3) with a C^∞-path M_t such that $M_t(0) = I$. Recall that in Theorem A either $\xi(0) = 0$ or $df(0) = 0$. In the case $\xi(0) \neq 0$, $df(0) = 0$ the second relation in (7.4) and Lemma 7.3 imply $h_{22}(0) = 0$. Therefore, (7.3) holds with a path M_t such that $M_t(0)$ is a triangular nonsingular matrix.

7.6. Proof of Theorem B. The functions f_t have the same $(\mu - 1)$-jet. Since $\xi(0) = 0$, the functions $\xi(f_t)$ also have the same $(\mu - 1)$-jet. Therefore, by Lemma 7.2 one has (7.3) with some C^0-path M_t, and consequently $(f_t, \xi(f_t)) \subseteq (f_0, \xi(f_0))$. It is easy to prove that these inclusions and the assumption $\dim \mathcal{F}/(f_t, \xi(f_t)) = \dim \mathcal{F}/(f_0, \xi(f_0)) < \infty$ imply that the ideals $(f_t, \xi(f_t))$ and $(f_0, \xi(f_0))$ coincide and $\det M_t(0) \neq 0$. Now Theorem B follows from Proposition 7.4.

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