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Fully simple singularities of plane and space curves

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Abstract

In this work we introduce the definition of fully simple singularities of parameterized curves and explain that this definition is more natural than the definition of simple singularities. The set of fully simple singularities is much smaller than the set of simple ones. We determine and classify all fully simple singularities of plane and space curves, with any number of components. Our classification results imply that any fully simple singularity of a plane or a space curve is quasi-homogeneous (whereas there is a number of non-quasi-homogeneous simple singularities). Another outcome of our classification results is a one-to-one correspondence between the fully simple singularities of plane curves and the classical A-D-E singularities of functions.

1. *Introduction and main results*

1.1. *Singularities of parameterized curves: simple and fully simple singularities*

We deal with parameterized curves. In what follows by a curve in $\mathbb{R}^n$ we mean a map $\Gamma : (a, b) \to \mathbb{R}^n$. All objects are assumed to be of the class C^∞. The purpose of this work is to determine and classify singularities of Γ at a point p of its image satisfying the following condition:

C: *All singularities of curves sufficiently close to* Γ *at points of their images sufficiently close to* p *are exhausted by a finite number of singularities.*

Singularities satisfying **C** will be called *fully simple*. Precise definition is given below. The reason for this terminology is as follows: a *simple singularity* of a curve does not need to be fully simple.

In order to explain this claim one should start with a definition of the singularity of a curve $\Gamma : (a, b) \to \mathbb{R}^n$ at a point p in the image of Γ. Let $t_1^0, \dots, t_d^0 \in (a, b)$ be the inverse images of the point p. Here $d \geqslant 1$. The local structure of Γ at p can be described by a *multigerm* with d components.

Definition. A multigerm of a curve in $\mathbb{R}^n$ is a collection $\gamma = (\gamma_1, \dots, \gamma_d)$, where $\gamma_i : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0)$ are map germs; they are called the *components* of γ. The collection is defined up to the order of the components.

To describe the local structure of Γ at p by a multigerm $\gamma = (\gamma_1, \dots, \gamma_d)$ take local coordinates t_i centered at t_i^0 and a local coordinate system x centered at p. Then γ_i is the germ of Γ at the point t_i^0 expressed in the local coordinates t_i and x, $i = 1, \dots, d$.

Changing the local coordinates t_i and x we obtain another multigerm $\tilde{\gamma} = (\tilde{\gamma}_1, \dots, \tilde{\gamma}_d)$ which is RL-*equivalent* to γ. This means that there exist local diffeomorphisms $\phi_i : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ (parameterizations of the components) and a local diffeomorphism $\Phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$

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such that $\Phi \circ \gamma_i = \tilde{\gamma}_i \circ \phi_i,\ i = 1, \ldots, d,$ up to numeration of the components of one of the multi-germs.

DEFINITION. The singularity of a curve Γ in $\mathbb{R}^n$ at a point p in the image of Γ is the RL-equivalence class of some (and then any) multigerm γ describing the local structure of Γ at p in the space of all multigerms with d components, where $d \geqslant 1$ is the number of the inverse images of the point p.

In most classification problems the first task is to determine and classify *simple singularities*. The simple singularities with one component of curves in $\mathbb{R}^n$ were determined and classified by Bruce and Gaffney in [**3**] for $n = 2$, by Gibbson and Hobbs in [**4**] for $n = 3$ and by Arnol'd in [**1**] for an arbitrary n. These results were continued in [**5**, **6**], where Kolgushkin and Sadykov classified all simple singularities of curves in $\mathbb{R}^n$, with any number of components and for any n. The works [**5**, **6**] contain almost 150 normal forms. The definition of simple germ or multigerm used in all cited above works is as follows.

DEFINITION. A multigerm $\gamma = (\gamma_1, \ldots, \gamma_d)$ (and a singularity defined by γ) is called simple if there exists $k < \infty$ such that the singularities defined by multigerms with d components and the k-jet sufficiently close to the k-jet of γ are exhausted by a finite number of singularities.

The starting point for the present work is the observation that a big part of simple singularities do not satisfy condition **C** above, see Subsection 1.2. Therefore singularities satisfying **C** will be called *fully simple*. A precise definition of fully simple singularities requires arcs – maps $F : [a, b] \to \mathbb{R}^n$.

DEFINITION. We will say that an arc $F : [a, b] \to \mathbb{R}^n$ represents a multigerm γ if the image of F contains $0 \in \mathbb{R}^n$, the points $F(a)$ and $F(b)$ are different from 0, and γ defines the singularity of F at $0 \in \mathbb{R}^n$. (The singularity of an arc $F : [a, b] \to \mathbb{R}^n$ at a point p of its image, different from $F(a)$ and $F(b)$, is the singularity at p of the curve $F|_{(a,b)}$.)

MAIN DEFINITION. Let γ be a multigerm of a parameterized curve in $\mathbb{R}^n$ and let $F : [a, b] \to \mathbb{R}^n$ be an arc representing γ. The multigerm γ (and the singularity defined by γ) is called fully simple if there exists $k < \infty$ such that the singularities of all arcs $\tilde{F} : [a, b] \to \mathbb{R}^n$ sufficiently C^k-close to the arc F at all points of their images sufficiently close to $0 \in \mathbb{R}^n$ are exhausted by a finite number of singularities.

One can easily prove that the definition is correct (the choice of an arc F representing γ is irrelevant) and that any fully simple multigerm is simple. The definition remains the same if we restrict ourselves to the singularities at $0 \in \mathbb{R}^n$ of those arcs representing γ, the image of which contains $0 \in \mathbb{R}^n$.

In the present work we determine and classify all fully simple singularities of plane and space curves. We show that the set of such singularities is much smaller than the set of simple singularities and their classification is much more natural. In particular, any fully simple singularity is quasi-homogeneous and for the fully simple singularities of plane curves there is a natural bijection with the classical A, D, E_6, E_7, E_8 simple singularities of functions.

The reason why a simple singularity might be not fully simple is the adjacency of a simple singularity described by a multigerm with d components to a singularity class consisting of non-simple multigerms with more than d components. By definition, a singularity class of multigerms is any set of multigerms which is closed with respect to the RL-equivalence, that is the union of a finite or infinite number of singularities. (A single singularity is also a singularity class.)

Definition. Let Q be a class of a multigerms of curves in $\mathbb{R}^n$ and let γ be a fixed multigerm represented by an arc $F : [a, b] \to \mathbb{R}^n$. The multigerm γ (and the singularity defined by γ) adjoins the class Q (notation $Q \leftarrow \gamma$) if for any $k < \infty$ then there exists a sequence of arcs $F_i : [a, b] \to \mathbb{R}^n$ tending to the arc F in the C^k-topology, with images that contain $0 \in \mathbb{R}^n$, and with singularities at $0 \in \mathbb{R}^n$ defined by multigerms of the class Q. A singularity class Q adjoins a singularity class $\tilde{Q}$ (notation: $\tilde{Q} \leftarrow Q$) if any singularity in Q adjoins $\tilde{Q}$.

It is easy to check that the definition is correct, that is the choice of the arc Γ representing γ is irrelevant. It is also easy to check that if a singularity adjoins a class containing no fully simple multigerms (in particular, no simple multigerms) then this singularity is not fully simple.

1.2. *Examples of simple but not fully simple singularities*

Let us give few obvious examples of simple singularities with $d \geqslant 1$ components which adjoin certain singularity classes consisting of multigerms with $d_1 > d$ components and containing no simple multigerms. Such simple singularities are not fully simple. We need the following definition.

Definition. The *multiplicity* of a curve germ $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0)$ is the minimal p such that $j^p\gamma \neq 0$ (the multiplicity is ∞ if γ has zero Taylor series). The multiplicity of a multigerm $(\gamma_1, \ldots, \gamma_d)$ is the sum of the multiplicities of its components.

For example, the multiplicity of the plane curve multigerm with two components $(t_1^2, t_1^{2k+1}), (t_2^4, t_2^3)$ is equal to $2 + 3 = 5$.

Proposition 1.1. *There are no fully simple multigerms of plane curves of multiplicity* 4 *or more. There are no fully simple multigerms of space curves of multiplicity* 5 *or more.*

On the other hand there is a big number of simple plane curve singularities of multiplicity 4 and simple space curve singularities of multiplicity 5, see [**3–6**]. Some examples: the plane curve singularities with one component $(t^4, t^5 \pm t^7), (t^4, t^5), (t^4, t^7 \pm t^9)$; the plane curve singularities with two components $\big((t_1, 0), (t_2^4, t_2^3)\big)$, $\big((t_1^2, t_1^{2k+1}), (t_2^{2s+1}, t_2^2)\big)$.

To prove Proposition 1.1 introduce the following notation.

- By $(I, I, \ldots, I)_{\mathbb{R}^n}$ (or simply $(I, I, \ldots, I)$) with I repeated r times we denote the class of multigerms of curves in $\mathbb{R}^n$ consisting of r *immersed* components.

Proposition 1.1 is a direct corollary of the following two claims.

Proposition 1.2. *Any multigerm of multiplicity at least r adjoins the class $(I, I, \ldots, I)$ with I repeated r times.*

Proposition 1.3. *The classes $(I, I, I, I))_{\mathbb{R}^2}$ and $(I, I, I, I, I))_{\mathbb{R}^3}$ contain no simple multigerms.*

Proof of Proposition 1.2. It is easy to prove that if a multigerm γ adjoins a class Q and a multigerm $\tilde{\gamma}$ adjoins a class $\tilde{Q}$ then the multigerm $(\gamma, \tilde{\gamma})$ with $d + \tilde{d}$ components adjoins the class $(Q, \tilde{Q}) = \{(\psi, \tilde{\psi}) : \psi \in Q, \tilde{\psi} \in \tilde{Q}\}$. This statement reduces Proposition 1.2 to the case of one component. It suffices to consider a curve germ of multiplicity r, that is, a germ $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0)$ of the form $\gamma(t) = t^r f(t)$, where $f : (\mathbb{R}, 0) \to R^n$ is a map germ such that $f(0) \neq 0$. Let $F(t)$ be a non-vanishing function defined on the interval $t \in [-1, 1]$ with the germ $f(t)$ at the point $t = 0$. Consider the arc $\Gamma_\epsilon = (t - \epsilon_1) \cdot \ldots \cdot (t - \epsilon_r)F(t), t \in [-1, 1]$. The arc Γ_0 represents the germ γ. If $\epsilon_1, \ldots, \epsilon_r \in (-1, 1)$ are distinct numbers then the singularity at $0 \in \mathbb{R}^n$ of the arc Γ_ϵ consists of r immersed curves. □

Proposition 1.3 is a well-known statement. In the RL-classification of multigerms of the class $(I, I, I, I)_{\mathbb{R}^2}$ (respectively $(I, I, I, I, I)_{\mathbb{R}^3}$) a modulus, that is, parameter distinguishing close non-equivalent multigerms occurs already in the classification of 1-jets – it is the cross-ratio invariant in the classification of tuples consisting of four (respectively five) 1-dimensional subspaces of $\mathbb{R}^2$ (respectively $\mathbb{R}^3$) with respect to the group of linear transformations.

Let us give one more example of simple but not fully simple germ $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$. Introduce the following class of multigerms.

- $(I \shortparallel I \cdot \ldots \cdot \shortparallel I)_{\mathbb{R}^n}$ is the subclass of $(I, I, \ldots, I)_{\mathbb{R}^n}$ consisting of multigerms such that the images of all components have the same tangent line at 0.

Proposition 1.4. *The class $(I \shortparallel I \shortparallel I)_{\mathbb{R}^2}$ contains no simple multigerms.*

This statement is also well known: there is a modulus in the RL-classification of the 2-jets of multigerms of this class. A generic 2-jet can be described by the normal form $(t_1, 0), (t_2, t_2^2), (t_3, at_3^2), a \notin \{0, 1\}$, and it is easy to prove that the parameter a is a modulus (it is a modulus in the classification of a tuple consisting of three parabolas which are tangent at 0 with respect to the group of 2-jets of local diffeomorphisms; the same modulus as in the classification of the singularity class J_{10} of function germs, see [**2**]).

Consider now a plane curve singularity defined by a germ of the form

$$\gamma : x(t) = t^3, \ y(t) = t^7 \cdot f(t). \tag{1.1}$$

Consider the deformation

$$\gamma_\epsilon : x = (t - \epsilon_1)(t - \epsilon_2)(t - \epsilon_3), \ y = \big((t - \epsilon_1)(t - \epsilon_2)(t - \epsilon_3)\big)^2 \cdot t \cdot f(t).$$

If $\epsilon_1, \epsilon_2, \epsilon_3$ are distinct numbers then the singularity at $0 \in \mathbb{R}^2$ of the curve γ_ϵ consists of three immersed curves tangent to the x-axes. Therefore the germ γ adjoins the class $(I \shortparallel I \shortparallel I)_{\mathbb{R}^2}$ and consequently it is not fully simple. On the other hand, if $f(0) \neq 0$ then γ is simple, see [**3**].

1.3. *Fully simple singularities with one component*

Our results for the case of one component are as follows: all fully simple plane curve singularities with one component are exhausted by the singularities

$$A_{2k}, \quad (3, 4), \quad (3, 5)$$

and all fully simple space curve singularities with one component are exhausted by the singularities

$$A_{2k}, \quad (3, 4, 5), \quad (3, 4, 0), \quad (3, 5, 7), \quad (3, 5, 0), \quad (3, 7, 8), \tag{1.2}$$

$$(4, 5, 6), \quad (4, 5, 7), \quad (4, 6, 7). \tag{1.3}$$

Here we use the following usual notation. A_{2k} denotes the singularity defined by the germ (t^2, t^{2k+1}) in the case of plane curves and $(t^2, t^{2k+1}, 0)$ in the case of space curves; (q, p) denotes the plane curve singularity defined by the germ (t^q, t^p); by (q, p, r) and $(q, p, 0)$ we denote the space curve singularity defined by the germ (t^q, t^p, t^r) and $(t^q, t^p, 0)$, respectively.

1.4. *Fencing singularity classes*

A tuple of singularity classes will be called fencing (for fully simple singularities of curves in $\mathbb{R}^n$) if any of these classes contains no fully simple multigerms and any singularity which is not fully simple adjoins one of these classes, with a possible exception of infinitely-degenerate singularities (a certain class of singularities of infinite codimension). It turns out that for fully simple singularities of plane and space curves there is a tuple of fencing classes with multigerms that consist of immersed curves only. To present these classes we use the following notation.

• By $(I \amalg I \amalg I)^{\#}_{\mathbb{R}^3}$ we denote the subclass of the class $(I \amalg I \amalg I)_{\mathbb{R}^3}$ consisting of multigerms of space curves with *planar* 2-jet. Here we use the following definition.

Definition. A multigerm γ of a space curve is called planar if its image (that is, the union of the images of the components) is contained in a non-singular surface. The r-jet of γ is called planar if there exists a planar multigerm $\tilde{\gamma}$ such that $j^r\tilde{\gamma} = j^r\gamma$.

• By $(I, I, I, I)^{\#}_{\mathbb{R}^3}$ we denote the subclass of the class $(I, I, I, I)_{\mathbb{R}^3}$ consisting of multigerms of space curves such that the tangent lines at $0 \in \mathbb{R}^3$ to the images of the four components do not span $T_0\mathbb{R}^3$.

Proposition 1.5. *The classes $(I \amalg I \amalg I)^{\#}_{\mathbb{R}^3}$ and $(I, I, I, I)^{\#}_{\mathbb{R}^3}$ contain no simple multigerms.*

Proof. The definitions of these classes and Propositions 1.3 and 1.4 imply that there is a modulus in the RL-classification of the 1-jets of multigerms of the class $(I, I, I, I)^{\#}_{\mathbb{R}^3}$ and of the 2-jets of multigerms of the class $(I \amalg I \amalg I)^{\#}_{\mathbb{R}^3}$. □

Define also the following two classes of infinite codimension.

• By $(A_\infty, *)$ we denote the class of multigerms of curves in $\mathbb{R}^n$ with any number 1 or more of components such that the Taylor series of one of the components is RL-equivalent to $(t^2, 0, \ldots, 0)$;
• By $((I, I)_\infty, *)$ we denote the class of multigerms of curves in $\mathbb{R}^n$ with any number 2 or more of components such that two of them, say γ_1 and γ_2, are immersed curves with tangency of infinite order, that is, in suitable coordinates the image of γ_1 is the x_1-axes and $\gamma_2 : \ x_i = a_i(t)$ where the function germs $a_2(t), \ldots, a_n(t)$ have zero Taylor series.

Theorem A. *A singularity of a parameterized curve in $\mathbb{R}^n$, $n = 2, 3$ is fully simple if and only if it does not belong to the classes $(A_\infty, *)$, $((I, I)_\infty, *)$ and does not adjoin any of the following classes:*

$$n = 2 : \quad (I \amalg I \amalg I)_{\mathbb{R}^2}, \ (I, I, I, I)_{\mathbb{R}^2}; \tag{1.4}$$

$$n = 3 : \quad (I \amalg I \amalg I)^{\#}_{\mathbb{R}^3}, \ (I, I, I, I)^{\#}_{\mathbb{R}^3}, \ (I, I, I, I, I)_{\mathbb{R}^3}. \tag{1.5}$$

By Theorem A the following conjecture holds for $n = 2, 3$.

Conjecture A1. Denote by $\mathcal{F}_n$ the class of non-simple multigerms of curves in $\mathbb{R}^n$ with at most $(n + 2)$ immersed components. A singularity of a curve in $\mathbb{R}^n$ is fully simple if and only if it does not belong to the classes $(A_\infty, *)$, $((I, I)_\infty, *)$ and does not adjoin the class $\mathcal{F}_n$.

1.5. *Full simplicity and quasi-homogeneity*

The quasi-homogeneity of a singularity of a curve in $\mathbb{R}^n$ is the following property: a multigerm defining this singularity is RL-equivalent to a multigerm with components of the form $x_i = a_i t^{r\lambda_i}$, $i = 1, \dots, n$, where $\lambda_1, \dots, \lambda_n > 0$ are positive numbers (weights) which are the same for all components, whereas the coefficients $a_1, \dots, a_n \in \mathbb{R}$ might vary from a component to a component.

A number of simple singularities of plane and space curves, including singularities with one components (that is simple germs) are not quasi-homogeneous, see [**3**] and [**4**]. The simplest examples are the plane curve germs $(t^3, t^7 + t^8)$ and $(t^4, t^5 + t^7)$ and the space curve germs $(t^3, t^7 + t^8, t^{11})$ and $(t^4, t^5 + t^7, t^{11})$. On the other hand, the classification results of the present work imply the following.

THEOREM B. *Any fully simple singularity of a parameterized plane or space curve, with any number of components, is quasi-homogeneous.*

CONJECTURE B1. Theorem B holds for singularities of parameterized curves in $\mathbb{R}^n$ for any n.

1.6. *Plan for the paper*

In Section 2 we present results on determination and classification of fully simple singularities of plane curves (Theorem C) and establish the bijection between such singularities and the classical A, D, E_6, E_7, E_8 simple singularities of functions. We also compare the list of fully simple singularities of plane curves with a much more involved list of simple singularities. (In this purpose we explain in canonical terms some of the normal forms obtained in [**5**, **6**].) In Section 3 we determine and classify all fully simple space curves singularities (Theorem D). The proof of Theorems C and D and simultaneously the proof of Theorem A is contained in Sections 4–7. Theorem B is a direct corollary of the obtained classification results.

1.7. *Analytic curves* $(\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$

The definition of fully simple singularities can be extended, in a natural way, to such curves. All results remain the same up to the following obvious changes: $A^{\pm}_{2i+1} \hookrightarrow A_{2i+1}$, $D^{\pm}_{2i+4} \hookrightarrow D_{2i+4}$, $i \geqslant 0$, (Subsection 2.2), and consequently the second column of Table 1 contains all simple V-singularities (not only those which have the property of zeros); $\mathbf{c}^{\pm} \hookrightarrow \mathbf{c}$ and $\mathbf{b}^{\pm} \hookrightarrow \mathbf{b}$ (Tables 3 and 4, Subsection 3.3).

1.8. *Notations for singularity classes*

We will use the following notations for certain singularity classes, continuing the notations in Subsection 1.4.

• By $(I, I)^i$ we denote the class of multigerms consisting of two immersed components γ_1, γ_2 with tangency of finite order $i \geqslant 0$.

Here and in what follows the order of tangency between immersed components γ_1, γ_2 is the order of tangency between their images. It will be denoted $\mathrm{ord}(\gamma_1, \gamma_2)$. Fix a local coordinate system $x_1, \dots, x_n$ such that the image of γ_1 is the x_1-axes. Let $\gamma_2 : \ x_i = a_i(t)$. Then $\mathrm{ord}(\gamma_1, \gamma_2)$ is the minimal i such that at least one of the function germs $a_2(t), \dots, a_n(t)$ has non-zero $(i + 1)$-jet. The zero order of tangency means that the images of γ_1 and γ_2 are not tangent.

• By $(I, I, \dots, I)^{i,0}$ with I repeated $r \geqslant 3$ times we denote the class of multigerms consisting of r immersed components such that, up to numeration of the components, no two of the first $(r-1)$ components are tangent and one of them has tangency of order i with the last component. The case $i = 0$ means that no two of the r components are tangent.

• Given a class Q of singularities with one component we denote by (I, Q) the class of multigerms (μ, ψ) consisting of an immersed component μ and a singular component $\psi \in Q$. Similarly, by $((I, I)^i, Q)$ we denote the class of multigerms (μ_1, μ_2, ψ) consisting of two immersed components μ_1, μ_2 such that $(\mu_1, \mu_2) \in (I, I)^i$ and a singular component $\psi \in Q$.

• Given two classes Q_1, Q_2 of singularities with one component we denote by (Q_1, Q_2) the class of multigerms (ψ_1, ψ_2) such that $\psi_1 \in Q_1, \psi_2 \in Q_2$.

2. *Fully simple singularities of plane curves*

2.1. *Determination and classification of fully simple singularities*

Recall the definition of the tangent line to a curve germ $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0)$ with non-zero Taylor series. It is the limit as $t \to 0$ of the 1-dimensional subspaces $\mathrm{span}(\dot\gamma(t)) \subset T_{\gamma(t)}\mathbb{R}^n$. Let $j^{p-1}\gamma = 0$ and $j^p\gamma \neq 0$. Then $\gamma(t) = t^{p-1}\tilde\gamma(t)$, where $\tilde\gamma$ is an immersed curve germ, and the tangent line to γ is the tangent line at 0 to the image of $\tilde\gamma$ (if γ is immersed then $p = 1$ and $\tilde\gamma = \gamma$).

Notation. The tangent line to a curve germ $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0)$ with non-zero Taylor series will be denoted $\ell(\gamma)$. Given a class Q of plane curve singularities with one component, we denote by $I \pitchfork Q$ and $I \shortparallel Q$ the subclass of the class (I, Q) consisting of multigerms with an immersed component μ and a singular component $\psi \in Q$ such that $\ell(\mu) \neq \ell(\psi)$ and $\ell(\mu) = \ell(\psi)$, respectively.

Theorem C. *A plane curve singularity is fully simple if and only if it belongs to one of the classes in the first column of Table* 1. *Each of these classes is a singularity, that is all its multigerms are* RL-*equivalent* (*to the normal form given in the first column*).

2.2. *Fully simple plane curve singularities and simple singularities of functions*

Table 1 implies that there is a natural bijection between fully simple plane curve singularities and the classical simple V-singularities of functions of two variables having the property of zeros.

Recall from [**2**] that the V-equivalence of function germs $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ means that g can be obtained from f by a local diffeomorphism (change of coordinates) and multiplication by a non-vanishing function. The V-equivalence class of a function germ is called the V-singularity. A V-singularity defined by a function germ f is called simple if there exists $k < \infty$ such that the V-singularities defined by function germs with the k-jet sufficiently close to the k-jet of f are exhausted by a finite number of V-singularities. It is classically known (see [**2**]) that the simple V-singularities are exhausted by the series $A^{\pm}_{2k-1} : y^2 - x^{2k}, A_{2k} : y^2 - x^{2k+1}, D^{\pm}_{2k+2} : xy^2 \pm x^{2k+1}, D_{2k+3} : xy^2 - x^{2k+2}, (k \geqslant 1)$ and $E_6 : y^3 - x^4, E_7 : y^3 - x^3y, E_8 : y^3 - x^5$. We will say that a V-singularity defined by a function germ f has *the property of zeros* if any function germ vanishing on the set $\{f = 0\}$ belongs to the ideal generated by f. It is clear that the singularities A^+_{2k-1} and D^+_{2k+2} do not have the property of zeros, and it is easy to prove that all other simple V-singularities have this property. Thus the second column of Table 1 contains all simple V-singularities having the property of zeros.

The correspondence between the singularities in the first and the second column of Table 1 is as follows. Given a multigerm γ, we associate to it the ideal $\mathcal{I}_\gamma$ consisting of function germs vanishing on the image of each of the components of γ. If γ does not belong to a certain class of infinite codimension (containing no simple multigerms) then the ideal $\mathcal{I}_\gamma$ is 1-generated. Let f_γ be one of the generators. The V-singularity defined by f_γ is invariantly related to the singularity of a parameterized plane curve defined by the multigerm γ: a multigerm $\tilde\gamma$ is RL-equivalent to γ if and only if the function germ $f_{\tilde\gamma}$ is V-equivalent to f_γ. We will say that the V-singularity defined by f_γ is the *functional realization* of the plane curve singularity defined by γ.

It is easy to check that every singularity in the second column of Table 1 is the functional realization of the singularity in the first column and the same row. Therefore Theorem C implies the following corollary.

TABLE 1. *Fully simple plane curve singularities and simple V-singularities of functions. Indexes: $k \geqslant 1, i \geqslant 0$.*

Fully simple singularities of plane curves	Simple V-singularities of functions
$A_{2k} : (t^2, t^{2k+1})$	$A_{2k} : y^2 - x^{2k+1}$
$(3,4) : (t^3, t^4)$	$E_6 : y^3 - x^4$
$(3,5) : (t^3, t^5)$	$E_8 : y^3 - x^5$
$(I,I)^i : (t_1, 0), (t_2, t_2^{i+1})$	$A^-_{2i+1} : y^2 - x^{2i+2}$
$I \pitchfork A_{2k} : (t_1^2, t_1^{2k+1}), (0, t_2)$	$D_{2k+3} : xy^2 - x^{2k+2}$
$I \shortparallel A_2 : (t_1^2, t_1^3), (t_2, 0)$	$E_7 : y^3 - x^3 y$
$(I,I,I)^{0,i} : (t_1, 0), (0, t_2), (t_3^{i+1}, t_3)$	$D^-_{2i+4} : xy^2 - x^{2i+3}$

THEOREM C1. *A singularity of a parameterized plane curve is fully simple if and only if its functional realization is simple.*

Is it possible to prove Theorem C1 without using Theorem C? It is not hard to prove that the adjacency of two plane curve singularities implies the adjacency of their functional realizations, that is, the part 'if' in Theorem C1. On the other hand, the 'only if' part of Theorem C1 is non-trivial and maybe surprising because the adjacency of the functional realizations of two plane curve singularities does not imply the adjacency of these singularities. Trivial examples can be found already within Table 1:

$$\begin{array}{ccc} A_{2k} & \nleftarrow & (I,I)^k \\ \updownarrow & & \updownarrow \\ A_{2k} & \leftarrow & A^-_{2k+1} \end{array} \qquad\qquad \begin{array}{ccc} (3,4) & \nleftarrow & I \shortparallel A_2 \\ \updownarrow & & \updownarrow \\ E_6 & \leftarrow & E_7 \end{array}$$

2.3. *Fully simple versus simple plane curve singularities*

According to results in [**3**] for the case of one component and [**5**, **6**] for the case of 2 or more components components, a plane curve singularity is simple if and only if it belongs to one of the classes in Table 2 below.

TABLE 2. *Simple and fully simple plane curve singularities. Indexes: $k, s \geqslant 1$, $i \geqslant 0$*

Class of simple singularities	Fully simple?	Class of simple singularities	Fully simple?
A_{2k}	Yes	$I \shortparallel A_{2k}$, $k \geqslant 1$	Only if $k = 1$
E_{6k}, E_{6k+2}	Only if $k = 1$	$I \pitchfork E_{6k}$, $I \pitchfork E_{6k+2}$	No
$(t^4, t^6 + t^{7+2i})$	No	$I \shortparallel (3,4)$, $I \shortparallel (3,5)$	No
$(t^4, t^5 \pm t^7)$, (t^4, t^5), $(t^4, t^7 \pm t^9)$	No	$A_{2k} \pitchfork A_{2s}$	No
$(I,I)^i$	Yes	$(I,I,I)^{i,0}$	Yes
$I \pitchfork A_{2k}$	Yes	$((I,I)^0, A_{2k})$	No

In this table the usual notation E_{6k} and E_{6k+2} are used for the class of germs which are RL-equivalent to germs of the form $(t^3, t^{3k+1} + \text{higher order terms})$ and $(t^3, t^{3k+2} + \text{higher order terms})$, respectively. Note that $E_6 = (3,4)$ and $E_8 = (3,5)$. The class $A_{2k} \pitchfork A_{2s}$ consists of multigerms with two singular components $\psi_1 \in A_{2k}, \psi_2 \in A_{2s}$ such that $\ell(\psi_1) \neq \ell(\psi_2)$. The table shows which of the simple singularities are fully simple. Each of the classes in Table 2 consists of a finite number of singularities. For example, the class $I \shortparallel A_{2k}$ consists of $(2k-1)$ singularities $(t_1^2, t_1^{2k-1}), (t_2, t_2^{j+1}), j = 1, \ldots, 2k-1$.

3. *Fully simple singularities of space curves*

To formulate the results on determination and classification of such singularities we need, except the tangent line $\ell(\gamma)$ to an immersed or singular germ $\gamma : (\mathbb{R}, 0) \to (R^3, 0)$, (see the beginning of Section 2) the definition of the tangent plane to a multigerm of a space curve (Subsection 3.1) and the definition of the order of tangency between a non-singular curve germ $\mu : (\mathbb{R}, 0) \to (\mathbb{R}^3, 0)$ and a space cusp $\psi \in A_{2k}$ (Subsection 3.2). A theorem on determination and classification of fully simple space curve singularities is formulated in Subsection 4.3. Though the list of such singularities is much bigger than that of fully simple plane curve singularities, it is far from being as involved as the classification of simple space curve singularities obtained in [**5**, **6**].

3.1. *Tangent plane to a space curve multigerm*

One of the equivalent definitions is as follows. At first we define the order of tangency between any space curve multigerm γ with d components $\gamma_i : x = a_i(t), y = b_i(t), z = c_i(t)$ and a germ of a non-singular surface $S \subset \mathbb{R}^3$. Let $S = \{H(x, y, z) = 0\}$. Consider the functions $R_i(t) = H(a_i(t), b_i(t), c_i(t))$.

Definition and Notation. The *order of tangency* between γ and S is the minimal integer r such that at least one of the functions $R_i(t), i = 1, \ldots, d$ has non-zero $(r+1)$-jet. The order of tangency will be denoted $\operatorname{ord}(\gamma, S)$.

If the image of γ is contained in S then $\operatorname{ord}(\gamma, S) = \infty$. If γ consists of one immersed component transversal to S then $\operatorname{ord}(\gamma, S) = 0$.

Definition and Notation. The tangent plane $L(\gamma)$ to a multigerm γ of a space curve is well defined if all non-singular surfaces $S \subset \mathbb{R}^3$ for which $\operatorname{ord}(\gamma, S)$ takes maximal possible value have the same tangent plane at 0. In this case $L(\gamma) = T_0 S$.

Examples. We need two examples when the tangent plane is well defined.

(1) It is well known that if γ consists of one singular component then the tangent plane $L(\gamma)$ is well defined, provided that the Taylor series of γ is not RL-equivalent to a series of the form $(at^r, 0, 0)$, where $r \geqslant 2, a \in \mathbb{R}$. Under this assumption, excluding a class of infinite codimension, γ is RL-equivalent to a germ of the form $(x, y, z) = (t^q, t^p, 0) + o(t^p)$, where $2 \leqslant q < p$ and $p \neq 0 \bmod q$. In the coordinates of this normal form one has $L(\gamma) = \operatorname{span}(\partial/\partial x, \partial/\partial y)$. (In the same coordinates the tangent line $\ell(\gamma)$, which is always a subspace of the tangent plane, is spanned by the vector $\partial/\partial x$). If γ is *planar* (for example, $\gamma \in A_{2k} \cup (3,4,0) \cup (3,5,0)$) then $L(\gamma)$ is the tangent plane at 0 to some (and then any) non-singular surface containing the image of γ.

(2) If $\gamma = (\mu_1, \mu_2)$ is a multigerm of the class $(I, I)^i$ then $L(\gamma)$ is the tangent plane at 0 to some (and then any) non-singular surface containing the images of the immersed curves μ_1 and

μ_2. Any multigerm of the class $(I, I)^i$ is RL-equivalent to the multigerm $(t_1, 0, 0), (t_2, t_2^{i+1}, 0)$; in these coordinates $L(\gamma) = \mathrm{span}(\partial/\partial x, \partial/\partial y)$, where x and y are the first two coordinates.

3.2. *The order of tangency between an immersed curve and a cusp in* $\mathbb{R}^3$

Let μ be an immersed space curve germ and let $\psi \in A_{2k}$.

Definition and Notation. The order of tangency between μ and ψ is the number $\mathrm{ord}(\mu, \psi) = \min \mathrm{ord}(\mu, S)$, where the minimum is taken over all non-singular surfaces S containing the image of ψ.

Lemma 3.1. *Let μ be an immersed space curve germ and let $\psi \in A_{2k}$. If $\ell(\mu) \not\subset L(\psi)$ then $\mathrm{ord}(\mu, \psi) = 0$. If $\ell(\mu) \subset L(\psi)$, but $\ell(\mu) \neq \ell(\psi)$ then $\mathrm{ord}(\mu, \psi) = 1$. If $\ell(\mu) = \ell(\psi)$ then $1 \leqslant \mathrm{ord}(\mu, \psi) \leqslant 2k$ and for generic couple (μ, ψ) within this case one has $\mathrm{ord}(\mu, \psi) = 1$.*

Proof. Take local coordinates in which

$$\psi : x = t^2, y = t^{2k+1}, z = 0, \quad \mu : x = a(t), y = b(t), z = c(t).$$

Any non-singular surface S containing the image of ψ is described by equation $z - (y^2 - x^{2k+1})g(x, y) = 0$, with an arbitrary function $g(x, y)$. Let

$$R(t) = c(t) - (b^2(t) - a^{2k+1}(t)) \cdot g(a(t), b(t)).$$

The lemma follows from the following observations.

(1) The condition $\ell(\mu) \not\subset L(\psi)$ means that $c'(0) \neq 0$. In this case $R'(0) \neq 0$.

(2) The condition $\ell(\mu) \subset L(\psi), \ell(\mu) \neq \ell(\psi)$ means that $c'(0) = 0$, $b'(0) \neq 0$. In this case $R'(0) = 0$ and if $g(0, 0)$ is a generic number then $R''(0) \neq 0$.

(3) The condition $\ell(\mu) = \ell(\psi)$ means that $c'(0) = b'(0) = 0$. If $c''(0) \neq 0$ then $R''(0) \neq 0$. Since μ is immersed then $a'(0) \neq 0$ and it is easy to see that if $g(x, y) \equiv r$ with a generic $r \in \mathbb{R}$ then $j^{2k+1}R \neq 0$. □

3.3. *Determination and classification of fully simple singularities*

In Tables 3 and 4 below we denote immersed components by $\mu, \mu_1, \mu_2, \ldots$, and singular components by $\psi, \psi_1, \psi_2, \ldots$.

Theorem D. *Any fully simple space curve singularity has not more than four components. A space curve singularity with one component is fully simple if and only if it is one of the singularities* (1.2), (1.3). *A space curve singularity with two components* (*respectively three or four components*) *is fully simple if and only if it belongs to one of the classes given in the first column of Table* 3 (*respectively Table* 4) *and satisfies the restrictions given in the second column. Any such singularity is* RL-equivalent *to one and only one of the normal forms given in the third column. The normal forms are distinguished in coordinate-free terms in the part 'Cases' of the second column of Tables* 3 *and* 4.

Here by coordinate-free terms we mean the mutual position of the tangent lines and the tangent planes to the components and/or the orders of tangency between the components. It is easy to see that the second column of Tables 3 and 4 contains all possible cases within the given restrictions. For example, within the restriction $L(\psi_1) \neq L(\psi_2)$ for multigerms of the class (A_2, A_2), see Table 3, the case $\ell(\psi_1) \subset L(\psi_2), \ell(\psi_2) \subset L(\psi_1), \ell(\psi_1) \neq \ell(\psi_2)$ is impossible.

4. *Proof of Theorems* **C, D** *and* **A** *(outline)*

Throughout the proof we will use the following notation.

NOTATION. By W we denote the union of the classes in Table 1. We will use the same notation W for the union of space curve singularities (1.2) and (1.3), and the subclasses of the classes in Tables 3 and 4 consisting of multigerms satisfying the restrictions given in the second column of these tables.

TABLE 3. *Fully simple space curve singularities with two components.*

Class	A singularity is fully simple if and only if Cases	Singularities
$(I,I)^i, i \geqslant 0$	No restrictions	$((t_1,0,0),(t_2,t_2^{i+1},0)$
(I,A_2)	No restrictions	
	a : $\ell(\mu) \not\subset L(\psi)$	**a** : $(t_1^2,t_1^3,0),(0,0,t_2)$
	b : $\ell(\mu) \subset L(\psi)$; $\ell(\mu) \neq \ell(\psi)$	**b** : $(t_1^2,t_1^3,0),(0,t_2,0)$
	c : $\ell(\mu) = \ell(\psi)$; $\operatorname{ord}(\mu,\psi)=1$	**c** : $(t_1^2,t_1^3,0),(t_2,0,t_2^2)$
	d : $\ell(\mu) = \ell(\psi)$; $\operatorname{ord}(\mu,\psi)=2$	**d** : $(t_1^2,t_1^3,0),(t_2,0,0)$
(I,A_4)	if $\ell(\mu)=\ell(\psi)$ then $\operatorname{ord}(\mu,\psi)=1$	
	a : $\ell(\mu) \not\subset L(\psi)$	**a** : $(t_1^2,t_1^5,0),(0,0,t_2)$
	b : $\ell(\mu) \subset L(\psi)$; $\ell(\mu) \neq \ell(\psi)$	**b** : $(t_1^2,t_1^5,0),(0,t_2,0)$
	c : $\ell(\mu) = \ell(\psi)$; $\operatorname{ord}(\mu,\psi)=1$	**c** : $(t_1^2,t_1^5,0),(t_2,0,t_2^2)$
$(I,A_{2k}), k \geqslant 3$	$\ell(\mu) \neq \ell(\psi)$	
	a : $\ell(\mu) \not\subset L(\psi)$	**a** : $(t_1^2,t_1^{2k+1},0),(0,0,t_2)$
	b : $\ell(\mu) \subset L(\psi)$	**b** : $(t_1^2,t_1^{2k+1},0),(0,t_2,0)$
$(I,(3,4,5))$	No restrictions	
	a : $\ell(\mu) \not\subset L(\psi)$	**a** : $(t_1^3,t_1^4,t_1^5),(0,0,t_2)$
	b : $\ell(\mu) \subset L(\psi)$; $\ell(\mu) \neq \ell(\psi)$	**b** : $(t_1^3,t_1^4,t_1^5),(0,t_2,0)$
	c : $\ell(\mu) = \ell(\psi)$;	**c** : $(t_1^3,t_1^4,t_1^5),(t_2,0,0)$
$(I,(3,4,0))$ $(I,(3,5,7))$ $(I,(3,5,0))$	$\ell(\mu) \not\subset L(\psi)$	$(t_1^3,t_1^4,0),(0,0,t_2)$ $(t_1^3,t_1^5,t_1^7),(0,0,t_2)$ $(t_1^3,t_1^5,0),(0,0,t_2)$
(A_2,A_2)	$L(\psi_1) \neq L(\psi_2)$	
	a : $\ell(\psi_1) \not\subset L(\psi_2), \ell(\psi_2) \not\subset L(\psi_1)$	**a** : $(t_1^2,t_1^3,0),(0,t_2^3,t_2^2)$
	b : $\ell(\psi_1) \subset L(\psi_2), \ell(\psi_2) \not\subset L(\psi_1)$ (up to numeration of ψ_1,ψ_2)	**b** : $(t_1^2,t_1^3,0),(t_2^3,0,t_2^2)$
	c : $\ell(\psi_1) = \ell(\psi_2)$	$\mathbf{c}^{\pm}$: $(t_1^2,t_1^3,0)$, $(\pm t_2^2,0,t_2^3)$
$(A_2,A_{2k}), k \geqslant 2$	$\ell(\psi_2) \not\subset L(\psi_1)$ (assuming $\psi_1 \in A_2, \psi_2 \in A_{2k}$)	
	a : $\ell(\psi_1) \not\subset L(\psi_2)$	**a** : $(t_1^2,t_1^3,0),(0,t_2^{2k+1},t_2^2)$
	b : $\ell(\psi_1) \subset L(\psi_2)$	**b** : $(t_1^2,t_1^3,0),(t_2^{2k+1},0,t_2^2)$

Using this notation, Theorems C and D can be joined as follows.

THEOREM 4.1.
(i) *A plane or space curve singularity is fully simple if and only if it belongs to the class* W.
(ii) *Any multigerm with at least two components of a plane or space curve of the class* W *is* RL*-equivalent to the normal form given in Tables* 1, 3 *and* 4.

TABLE 4. *Fully simple space curve singularities with 3 and 4 components.*

Class	A singularity is fully simple if and only if: Cases	Singularities
$(I,I,I)^{i,0}$	No restrictions	$\mu_1=(t_1,0,0)$ $\mu_2=(t_2,t_2^{i+1},0)$
	$\mathbf{a}:\ell(\mu_3)\not\subset L(\mu_1,\mu_2)$ $\mathbf{b}:\ell(\mu_3)\subset L(\mu_1,\mu_2)$ (assuming $\mathrm{ord}(\mu_1,\mu_2)=i$)	$\mathbf{a}:\mu_3=(0,0,t_3)$ $\mathbf{b}:\mu_3=(0,t_3,0)$
$I \amalg I \amalg I$	the 2-jet of the multigerm is not planar	$\mu_1=(t_1,0,0)$ $\mu_2=(t_2,t_2^2,0)$ $\mu_3=(t_3,0,t_3^2)$
$((I,I)^0,A_2)$	$\mathrm{span}(\ell(\mu_1),\ell(\mu_2))\neq L(\psi)$	$\psi=(t_1^2,t_1^3,0)$ $\mu_1=(0,0,t_2)$,
	$\mathbf{a}:\ell(\mu_1),\ell(\mu_2)\not\subset L(\psi)$ $\mathbf{b}:\ell(\mu_2)\subset L(\psi),\ell(\mu_2)\neq\ell(\psi)$ $\mathbf{c}_1:\ell(\mu_2)=\ell(\psi),\mathrm{ord}(\mu_2,\psi)=1$ $\mathbf{c}_2:\ell(\mu_2)=\ell(\psi),\mathrm{ord}(\mu_2,\psi)=2$ (up to numeration of μ_1,μ_2)	$\mathbf{a}:\mu_2=(0,t_3,t_3)$ $\mathbf{b}:\mu_2=(0,t_3,0)$ $\mathbf{c}_1:\mu_2=(t_3,0,t_3^2)$ $\mathbf{c}_2:\mu_2=(t_3,0,0)$
$((I,I)^0,A_{2k}),k\geqslant 2$	$\ell(\mu_1),\ell(\mu_2),\ell(\psi)$ span $T_0\mathbb{R}^3$	$\psi=(t_1^2,t_1^{2k+1},0)$ $\mu_1=(0,0,t_2)$
	$\mathbf{a}:\ell(\mu_1),\ell(\mu_2)\not\subset L(\psi)$ $\mathbf{b}:\ell(\mu_1)\not\subset L(\psi),\ell(\mu_2)\subset L(\psi)$ (up to numeration of μ_1,μ_2)	$\mathbf{a}:\mu_2=(0,t_3,t_3)$ $\mathbf{b}:\mu_2=(0,t_3,0)$
$((I,I)^i,A_2),i\geqslant 1$	$\ell(\mu_1)=\ell(\mu_2)\not\subset L(\psi)$	$\psi=(t_1^2,t_1^3,0)$, $\mu_1=(0,0,t_2)$,
	$\mathbf{a}:\ell(\psi)\not\subset L(\mu_1,\mu_2)$ $\mathbf{b}:\ell(\psi)\subset L(\mu_1,\mu_2)$	$\mathbf{a}:\mu_2=(0,t_3^{i+1},t_3)$ $\mathbf{b}^{\pm}:\mu_2=(\pm t_3^{i+1},0,t_3)$ ($\pm\hookrightarrow +$ iff i is even)
$(I,I,I,I)^{i,0},i\geqslant 0$	$\ell(\mu_1),\ell(\mu_2),\ell(\mu_3),\ell(\mu_4)$ span $T_0\mathbb{R}^3$	$\mu_1=(t_1,0,0)$, $\mu_2=(0,t_2,0)$, $\mu_3=(0,0,t_3)$,
	$\mathbf{a}:\ell(\mu_2),\ell(\mu_3)\not\subset L(\mu_1,\mu_4)$ $\mathbf{b}:\ell(\mu_2)\subset L(\mu_1,\mu_4)$ or $\ell(\mu_3)\subset L(\mu_1,\mu_4)$ (assuming that $\ell(\mu_1),\ell(\mu_2),\ell(\mu_3)$ span $T_0\mathbb{R}^3$ and $\mathrm{ord}(\mu_1,\mu_4)=i$)	$\mathbf{a}:\mu_4=(t_4,t_4^{i+1},t_4^{i+1})$ $\mathbf{b}:\mu_4=(t_4,t_4^{i+1},0)$

The second statement of this theorem can be proved by the standard normalization techniques. It also follows from a straightforward analysis of the list of normal forms obtained in [**5**, **6**].

Let us show that Theorem 4.1, (i) follows from Propositions 4.2 and 4.3 below. In what follows by *the fencing classes* we mean the fencing singularity classes defined in Subsection 1.4: the classes (1.4) in the case of plane curves and the classes (1.5) in the case of space curves.

PROPOSITION 4.2. *Any plane or space curve singularity beyond the class W and the classes $(A_\infty, *)$ and $((I,I)_\infty, *)$ adjoins one of the fencing classes.*

PROPOSITION 4.3. *Any plane or space curve singularity of the class W does not adjoin any of the fencing classes.*

Theorem 4.1(i) *from Propositions* 4.2 *and* 4.3. Since, as we showed in Subsection 1.4, the fencing classes contain no simple multigerms then Proposition 4.2 implies the 'only if' part of Theorem 4.1(i). The proof of the 'if' part is as follows. Assume, to obtain a contradiction, that a multigerm $\gamma \in W$ is not fully simple. Represent γ by an arc Γ defined on the segment $[-1,1]$. Then for any k there exists a sequence of arcs $\Gamma_1, \Gamma_2, \dots$ defined on $[-1,1]$ and tending to Γ in the C^k topology such that the singularities at the origin of the arcs Γ_{i_1} and Γ_{i_2} are different for any $i_1 \neq i_2$. The class W consists of a countable number of singularities, see Tables 1, 3 and 4. It is easy to check (using the notion of the codimension of singularities) that any *fixed* singularity of the class W adjoins not more than a finite number of singularities in this class. Therefore the sequence Γ_i can be chosen in such a way that the singularities at the origin of Γ_i do not belong to the tuple W. By Proposition 4.2 these singularities adjoin one of the fencing classes. This means that for any k and any fixed i there exists a sequence of arcs $\Gamma_{i,1}, \Gamma_{i,2}, \dots$ defined on $[-1,1]$, tending to Γ_i in the C^k-topology, and such that the singularities at the origin of $\Gamma_{i,1}, \Gamma_{i,2}, \dots$ belong to one of the fencing classes. Consider the sequence of arcs $\Gamma_{1,1}, \Gamma_{2,2}, \dots$. It tends to the arc Γ in the C^k topology and the singularity at the origin of $\Gamma_{i,i}$ belongs to one of the fencing classes. Therefore γ adjoins one of the fencing classes which contradicts to Proposition 4.3.

Theorem A is a direct corollary of Theorem 4.1(i) and Propositions 4.2 and 4.3. Therefore to prove Theorems C, D and A it suffices to prove Propositions 4.2 and 4.3. Proposition 4.2 is proved in Section 5. Proposition 4.3 is proved in Sections 6 and 7.

5. *Proof of Proposition 4.2*

Throughout the proof we use the following deformation of t^r:

$$P_{r,\epsilon}(t) = (t - \epsilon_1) \cdot (t - \epsilon_2) \cdot \ldots \cdot (t - \epsilon_r).$$

5.1. *The case of one component*

The classification results in [**3**] imply that any plane curve germ $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ beyond the singularities $A_{2k}, (3,4), (3,5)$ and the class A_∞ either has multiplicity 4 or more, or is RL-equivalent to a germs of form (1.1). As we showed in Subsection 1.2, in the first case γ adjoins the fencing class $(I,I,I,I)_{\mathbb{R}^2}$ and in the second case it adjoins the fencing class $(I \shortparallel I \shortparallel I)_{\mathbb{R}^2}$.

The classification results in [**4**] imply that any space curve germ $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^3, 0)$ beyond the singularities (1.2), (1.3) and the class A_∞ either has multiplicity 5 or more, (and then it adjoins the fencing class $(I,I,I,I,I)_{\mathbb{R}^3}$, see Proposition 1.2) or is RL-equivalent to a germ of one of the following forms:

$$x(t) = t^3, \quad y(t) = t^7 \cdot b(t), \quad z(t) = t^{10} \cdot c(t); \tag{5.1}$$

$$x(t) = t^4, \quad y(t) = t^5 \cdot b(t), \quad z(t) = t^9 \cdot c(t). \tag{5.2}$$

Let us show that germ (5.1) adjoins the fencing class $(I \amalg I \amalg I)^{\#}_{\mathbb{R}^3}$ and germ (5.2) adjoins the fencing class $(I, I, I, I)^{\#}_{\mathbb{R}^3}$. To prove the adjacency $(I \amalg I \amalg I)^{\#}_{\mathbb{R}^3} \leftarrow$ (5.1) it suffices to consider the deformation

$$x = P_{3,\epsilon}(t), \quad y = P^2_{3,\epsilon}(t) \cdot t \cdot b(t), \quad z = P^3_{3,\epsilon}(t) \cdot t \cdot c(t). \tag{5.3}$$

In fact, if $\epsilon_1, \epsilon_2, \epsilon_3$ are distinct then the singularity at $0 \in \mathbb{R}^3$ of the curve (5.3) consists of three immersed curves tangent to the x-axes and having tangency of order 2 or more with the plane $z = 0$.

The adjacency $(I, I, I, I)^{\#}_{\mathbb{R}^3} \leftarrow$ (5.2) can be realized by the deformation

$$x = P_{4,\epsilon}(t), \quad y = P_{4,\epsilon}(t) \cdot t \cdot b(t), \quad z = P^2_{4,\epsilon}(t) \cdot t \cdot c(t). \tag{5.4}$$

In fact, the singularity at $0 \in \mathbb{R}^3$ of the curve (5.4) consists of four immersed curves tangent to the plane $z = 0$.

5.2. *Plane curve singularities with at least two components*

It is easy to check that any plane curve singularity with at least two components beyond the singularities in Table 1 and the classes $(A_\infty, *)$ and $((I, I)_\infty, *)$ either has multiplicity 4 or more ((and then it adjoins the fencing class $(I, I, I, I)_{\mathbb{R}^2}$ by Proposition 1.2), or belongs to the fencing class $(I \amalg I \amalg I)_{\mathbb{R}^2}$, or belongs to the class $I \amalg A_{2k \geqslant 4}$. The latter class adjoins the fencing class $(I \amalg I \amalg I)_{\mathbb{R}^2}$ by the following lemma.

LEMMA 5.1. *If $k \geqslant 2$ then any plane curve germ $\psi \in A_{2k}$ adjoins the class of multigerms with two immersed components μ_1, μ_2 tangent to the line $\ell(\psi)$ and such that the order of tangency between μ_1, μ_2 is equal to $(k-1)$.*

Proof. Take local coordinates in which ψ has the form $x = t_1^2, y = t_1^{2k+1}$. Then $\ell(\psi) = \mathrm{span}(\partial/\partial x)$. The required adjacency is realized by the deformation $\gamma_\epsilon :\ x = P_{2,\epsilon}(t), y = P^k_{2,\epsilon}(t) \cdot t$. In fact, if $\epsilon_1 \neq \epsilon_2$ then the singularity at $0 \in \mathbb{R}^2$ of the curve γ_ϵ consists of two immersed curves tangent to the x-axes; it is easy to see that the order of tangency is equal to $(k-1)$. □

5.3. *Space curve singularities with at least two components*

The proof is based on the following lemmas.

LEMMA 5.2. *Any space curve germ $\psi \in A_{2k}$ adjoins the class of multigerms with two immersed components μ_1, μ_2 such that $\ell(\mu_1), \ell(\mu_2) \subset L(\psi)$. If $k \geqslant 2$ then ψ adjoins the subclass of this class consisting of multigerms (μ_1, μ_2) such that μ_1 and μ_2 are tangent to the line $\ell(\psi)$ and* $\mathrm{ord}(\mu_1, \mu_2) = (k-1)$.

Proof. Follows from Lemma 5.1 and the fact that any space curve germ $\psi \in A_{2k}$ is planar: its image is contained in a non-singular surface tangent to the plane $L(\psi)$ which contains the line $\ell(\psi)$. □

Lemma 5.3. *Any germ* $\psi : (\mathbb{R}, 0) \to (\mathbb{R}^3, 0)$ *satisfying the condition*

$$j^2\psi = 0, \quad j^3\psi \neq 0, \quad \psi \notin (3,4,5) \cup (3,4,0) \cup (3,5,7) \cup (3,5,0) \tag{5.5}$$

adjoins the class of multigerms consisting of three components whose images have the same tangent line at $0 \in \mathbb{R}^3$.

Proof. In suitable coordinates ψ has the form $x = t^3,\ y = t^7 b(t),\ z = t^7 c(t)$, see [**4**]. The required adjacency is realized by the deformation $x = P_{3,\epsilon}(t),\ y = P^2_{3,\epsilon}(t) \cdot t \cdot b(t),\ z = P^2_{3,\epsilon}(t) \cdot t \cdot c(t)$. In fact, if $\epsilon_1, \epsilon_2, \epsilon_3$ are distinct then the singularity at $0 \in \mathbb{R}^3$ of this curve consists of three immersed components tangent to the x-axes. □

Lemma 5.4. *Any germ* $\psi \in (3,4,0) \cup (3,5,7) \cup (3,5,0)$ *adjoins the class of multigerms with three immersed components tangent to the plane* $L(\psi)$.

Proof. In suitable coordinates ψ has the form $x = t^3,\ y = t^4 b(t),\ z = t^7 c(t)$ and $L(\psi) = \operatorname{span}(\partial/\partial x, \partial/\partial y)$. The required adjacency is realized by the deformation $x = P_{3,\epsilon}(t), y = P_{3,\epsilon}(t) \cdot t \cdot b(t), z = P^2_{3,\epsilon}(t) \cdot t \cdot c(t)$. In fact, if $\epsilon_1, \epsilon_2, \epsilon_3$ are distinct then the singularity at $0 \in \mathbb{R}^3$ of this curve consists of three immersed components tangent to the plane $z = 0$. □

Let us show that these lemmas imply that any space curve singularity with at least two components beyond the class W adjoins one of the fencing classes. We start with space curve singularities with at least two components which do not belong to any of the classes given in the first column of Tables 3 and 4, neither to the classes $(A_\infty, *)$ and $((I, I)_\infty, *)$. It is easy to check that any such singularity γ either belongs to one of the the fencing classes $(I \amalg I \amalg I)^{\#}_{\mathbb{R}^3}$, $(I, I, I, I)^{\#}_{\mathbb{R}^3}$ or satisfies one of the following conditions:

(a) γ has multiplicity 5 or more;
(b) $\gamma \in (A_{2k}, A_{2s})$, where $k, s \geqslant 2$ or $\gamma \in ((I, I)^i, A_{2k})$ where $i \geqslant 1$ and $k \geqslant 2$;
(c) $\gamma = (\mu, \psi)$, where the singular component ψ satisfies (5.5).

In case (a) the singularity adjoins the fencing class $(I, I, I, I, I)_{\mathbb{R}^3}$ by Proposition 1.2. In case (b) Lemma 5.2 implies that γ adjoins the fencing class $(I, I, I, I)^{\#}_{\mathbb{R}^3}$. The same holds in case (c) by Lemma 5.3.

Now we analyze space curve singularities γ with at least two components which belong to one of the classes in the first column of Tables 3 and 4, but do not satisfy the restriction given in the second column of these tables. In the case of classes $I \amalg I \amalg I$ and $(I, I, I, I)^{i,0}$ such singularities belong to one of the fencing classes $(I \amalg I \amalg I)^{\#}_{\mathbb{R}^3}, (I, I, I, I)^{\#}_{\mathbb{R}^3}$. In the case of other classes in Tables 3 and 4, such that the restrictions in the second column are not empty, γ satisfies one of the following conditions:

(d) $\gamma = (\mu, \psi) \in (I, A_4), \quad \ell(\mu) = \ell(\psi), \quad \operatorname{ord}(\mu, \psi) \geqslant 2$;
(e) $\gamma = (\mu, \psi) \in (I, A_{2k \geqslant 6}), \quad \ell(\mu) = \ell(\psi)$;
(f) $\gamma = (\mu, \psi), \quad \psi \in (3,4,0) \cup (3,5,7) \cup (3,5,0), \quad \ell(\mu) \subset L(\psi)$;
(g) $\gamma = (\psi_1, \psi_2) \in (A_2, A_2), \quad L(\psi_1) = L(\psi_2)$;
(h) $\gamma = (\psi_1, \psi_2) \in (A_2, A_{2k \geqslant 4}), \quad \ell(\psi_2) \subset L(\psi_1)$;
(i) $\gamma = (\mu_1, \mu_2, \psi) \in ((I, I)^i, A_2), \ i \geqslant 0, \quad \ell(\mu_1), \ell(\mu_2) \subset L(\psi)$;
(j) $\gamma = (\mu_1, \mu_2, \psi) \in ((I, I)^i, A_{2k}), \ i \geqslant 0, \quad k \geqslant 2,$ the lines $\ell(\mu_1), \ell(\mu_2), \ell(\psi)$ do not span $T_0\mathbb{R}^3$.

Lemma 5.2 implies that in the cases (g), (h), (i), (j) the singularity γ adjoins the fencing class $(I, I, I, I)^{\#}_{\mathbb{R}^3}$ and in the cases (d) and (e) it adjoins the fencing class $(I \amalg I \amalg I)^{\#}_{\mathbb{R}^3}$. In the remaining case (f) the singularity adjoins the fencing class $(I, I, I, I)^{\#}_{\mathbb{R}^3}$ by Lemma 5.4.

6. *Proof of Proposition* 4.3

For singularities with immersed components only Proposition 4.3 is trivial. In what follows we consider singularities with at least one singular component. For such singularities, in this section we prove Proposition 4.3 modulo several lemmas which are proved in the next section.

LEMMA 6.1. *A singularity of a curve in $\mathbb{R}^n$ of multiplicity less than r does not adjoin the class consisting of multigerms with r immersed components.*

It is easy to check that this lemma implies the following corollary.

COROLLARY 6.2. *Plane and space curve singularities of the class W do not adjoin the fencing class $(I, I, I, I)_{\mathbb{R}^2}$ and $(I, I, I, I, I)_{\mathbb{R}^3}$, respectively. A plane or space curve singularity of the class A_{2k} does not adjoin any of the fencing classes.*

The next lemma excludes certain adjacencies of singularities consisting of immersed curve(s) and a cusp.

LEMMA 6.3. *Let $n = 2$ or $n = 3$. Consider the arc*

$$F : x = t^2,\ \ y = t^{2k+1}\ \ (n = 2), \quad F :\ x = t^2,\ \ y = t^{2k+1},\ \ z = 0\ \ (n = 3)$$

defined on $[-1, 1]$. Assume that a sequence of arcs $F_i : [-1, 1] \to \mathbb{R}^n$ tends to the arc F in the C^3-topology and the singularity of F_i at $0 \in \mathbb{R}^n$ consists of two immersed components $\mu_i^{(1)}, \mu_i^{(2)}$. Then the following holds:

1. *the sequences of tangent lines $\ell(\mu_i^{(1)})$, $\ell(\mu_i^{(2)})$ tend to the line* $\mathrm{span}(\partial/\partial x)$*;*
2. *if $k = 1$ then $\ell(\mu_i^{(1)}) \neq \ell(\mu_i^{(2)})$ for sufficiently large i;*
3. *if $k = 1$ and $n = 3$ then the sequence of planes spanned by $\ell(\mu_i^{(1)})$ and $\ell(\mu_i^{(2)})$ tends to the plane* $\mathrm{span}(\partial/\partial x, \partial/\partial y)$*.*

The first two statements of Lemma 6.3 imply the absence of the following adjacencies.

COROLLARY 6.4. *The plane curve singularities $I \pitchfork A_{2k}$ and $I \amalg A_2$ do not adjoin the class $(I \amalg I \amalg I)_{\mathbb{R}^2}$.*

Corollaries 6.2 and 6.4 and the obvious adjacency $(3, 4) \leftarrow (3, 5)$ reduce Proposition 4.3 for plane curve singularities to the following lemma.

LEMMA 6.5. *The singularity $(3, 5)$ does not adjoin the class $(I \amalg I \amalg I)_{\mathbb{R}^2}$.*

Now we consider space curve singularities. Lemmas 6.1 and 6.3 imply the following statement.

COROLLARY 6.6. *Let γ be a space curve multigerm of one of the classes*

$$(I, A_{2k}),\ \ k \neq 2,\ \ (A_2, A_{2k}),\ \ ((I, I)^0, A_{2k}),\ \ ((I, I)^i, A_2).$$

If γ satisfies the restrictions in the second column of Tables 3 and 4 then it does not adjoin any of the classes $(I, I, I, I)^{\#}_{\mathbb{R}^3}$, $I \amalg I \amalg I$.

Corollaries 6.2 and 6.6 and the obvious adjacencies $(3,4,5) \leftarrow (3,4,0) \leftarrow (3,5,7) \leftarrow (3,5,0) \leftarrow (3,7,8)$; $(4,5,6) \leftarrow (4,5,7) \leftarrow (4,6,7)$ reduce Proposition 4.3 for space curve singularities to the following statement:

(*) none of the singularities $(3,7,8), (4,6,7)$ and none of the singularities of the classes $W \cap (I, A_4), W \cap (I,(3,4,5)), W \cap (I,(3,5,0))$ adjoins any of the fencing classes $(I \amalg I \amalg I)^{\#}_{\mathbb{R}^3}, (I,I,I,I)^{\#}_{\mathbb{R}^3}$.

Remind that the intersection of W with a class in Tables 3 and 4 is the subclass consisting of multigerms satisfying the restrictions in the second column. By Table 3, each of the classes $W \cap (I, A_4)$, $W \cap (I,(3,4,5))$ consists of three singularities $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and their coordinate free definition in the second column implies the adjacencies $\mathbf{a} \leftarrow \mathbf{b} \leftarrow \mathbf{c}$. The class $W \cap (I,(3,5,0))$ consists of a single singularity. Therefore **(*)** reduces to the following statement.

(**) none of the following space curve singularities adjoins any of the fencing classes $(I \amalg I \amalg I)^{\#}_{\mathbb{R}^3}, (I,I,I,I)^{\#}_{\mathbb{R}^3}$:

$$(3,7,8); \quad (4,6,7); \quad \mathbf{c} \in (I, A_4); \quad \mathbf{c} \in (I,(3,4,5)); \quad W \cap (I,(3,5,0)). \tag{6.1}$$

The singularity $W \cap (I,(3,5,0))$ consists of multigerms $(\mu, \psi) \in (I,(3,5,0))$ such that $\ell(\mu) \not\subset L(\psi)$. Such multigerms do not adjoin the class $(I,I,I,I)^{\#}_{\mathbb{R}^3}$ by Lemma 6.5. They do not adjoin the class $I \amalg I \amalg I$ and consequently the class $(I \amalg I \amalg I)^{\#}_{\mathbb{R}^3}$ by the following lemma.

Lemma 6.7. *Let $F_i : [-1,1] \to \mathbb{R}^3$ be a sequence of arcs tending to the arc $x = t^3, y = t^5, z = 0$ in the C^r-topology with sufficiently large r and such that the singularity of F_i at $0 \in \mathbb{R}^3$ consists of two immersed components $\mu_i^{(1)}, \mu_i^{(2)}$ tangent to the same line ℓ_i. Then $\ell_i \to \ell(t^3, t^5, 0) = \mathrm{span}(\partial/\partial x)$.*

Now, let us prove the adjacencies

$$(I, A_4) \ni \mathbf{c} \longleftarrow (3,7,8) \longleftarrow (4,6,7); \quad (I,(3,4,5)) \ni \mathbf{c} \longleftarrow (4,6,7). \tag{6.2}$$

The adjacency $(3,7,8) \leftarrow (4,6,7)$ is realized by the deformation $(\epsilon t^3 + t^4, t^6, t^7)$ (this germ belongs to the class $(3,7,8)$ if $\epsilon \neq 0$). The other two adjacencies are realized by the following deformations of the curves (t^3, t^7, t^8) and (t^4, t^6, t^7):

$$x = (t-\epsilon_1)^2(t-\epsilon_2), \quad y = (t-\epsilon_1)^5(t-\epsilon_2)^2, \quad z = (t-\epsilon_1)^6(t-\epsilon_2)^2; \tag{6.3}$$
$$x = (t-\epsilon_1)^3(t-\epsilon_2), \quad y = (t-\epsilon_1)^4(t-\epsilon_2)^2, \quad z = (t-\epsilon_1)^5(t-\epsilon_2)^2. \tag{6.4}$$

In fact, the singularity at $0 \in \mathbb{R}^3$ of the curve (6.3) (respectively (6.4)) consists of an immersed component μ and a singular component $\psi \in A_4$ (respectively $\psi \in (3,4,5)$) such that $\ell(\mu) = \ell(\psi)$ is the x-axes.

Adjacencies (6.2) and Lemma 6.7 reduce the claim (**) to the following statement.

Lemma 6.8. *The singularity $(4,6,7)$ does not adjoin any of the fencing classes $(I \amalg I \amalg I)^{\#}_{\mathbb{R}^3}, (I,I,I,I)^{\#}_{\mathbb{R}^3}$.*

7. *Proof of the lemmas in Section 6*

7.1. *Auxiliary statements*

Throughout the proof the following statements will be used.

Proposition 7.1. *Let $P(t) = a_0 + a_1 t + \ldots + a_k t^k$, where $k \geqslant 0$ and $a_k \neq 0$. Let $f_i(t), t \in [-1,1]$ be a sequence of functions tending to $P(t)$ in the C^k-topology. Then for sufficiently large i the set $f^{-1}(0)$ consists of at most k points.*

Proposition 7.2. *Let $a_i^{(1)}, \ldots, a_i^{(p)}$ be sequences of points in $[-1,1]$. Let $q, r \geqslant p$. If the sequence of functions*

$$F_i(t) = (t - a_i^{(1)}) \cdot \ldots \cdot (t - a_i^{(p)}) \cdot G_i(t), \quad t \in [-1,1]$$

tends to the function t^q in the C^r-topology then the sequences $a_i^{(1)}, \ldots, a_i^{(p)}$ tend to 0 and the sequence $G_i(t)$ tends to the function t^{q-p} in the C^{r-p} topology.

To prove Proposition 7.1 note that the sequence of kth derivatives $f_i^{(k)}(t)$ tends to a non-zero constant. Therefore for sufficiently large i one has $f_i^{(k)}(t) \neq 0, t \in [-1,1]$. This implies Proposition 7.1 by the classical Rolle theorem.

The first statement of Proposition 7.2 is also obvious: if one of the sequence $a_i^{(1)}, \ldots, a_i^{(p)}$ does not converge to 0 then it has a limit point $a^* \neq 0$ which contradicts to the condition $F_i(t) \to t^q$. To prove the second statement it suffices to consider the case $p = 1$. Thus we have to prove the following assertion.

(*) if the sequence of points a_i tends to 0 and the sequence of functions of the form $f_i(t) = (t - a_i) \cdot \phi_i(t), t \in [-1,1]$ tends to 0 in the $C^{r \geqslant 1}$-topology then the sequence of functions $\phi_i(t), t \in [-1,1]$, tends to 0 in the C^{r-1}-topology.

Fix $\delta \in (0,1)$ and a number m such that $|a_i| < 1 - \delta$ for any $i > m$. Consider the functions $\tilde{\phi}_i(t) = \phi_{i+m}(t + a_i)$ defined for $t \in [-\delta, \delta]$. Then the sequence of functions $t\tilde{\phi}_i(t), t \in [-\delta, \delta]$ tends to 0 in the C^r-topology and it follows that the sequence $\tilde{\phi}_i(t)$ tends to 0 in the C^{r-1}-topology (the latter implication is a known fact; we leave the proof to a reader). Therefore the sequence $\phi_i(t)$, $t \in [-\delta/2, \delta/2]$ tends to 0 in the C^{r-1}-topology and (*) follows.

7.2. *Proof of Lemma* 6.1

This lemma is almost straightforward corollary of Proposition 7.1.

7.3. *Proof of Lemma* 6.3

We will consider the case $n = 3$ (the proof for $n = 2$ is the same). The arcs F_i have the form

$$F_i : x(t) = R_i(t)A_i(t), \quad y(t) = R_i(t)B_i(t), \quad z(t) = R_i(t)C_i(t), \tag{7.1}$$

$$R_i(t) = \big(t - a_i^{(1)}\big) \cdot \big(t - a_i^{(2)}\big), \quad a_i^{(1)} \neq a_i^{(2)}. \tag{7.2}$$

The lines $\ell\big(\mu_i^{(1)}\big)$, $\ell\big(\mu_i^{(2)}\big)$ are spanned by the vectors

$$v_i^{(j)} = A(a_i^{(j)})\frac{\partial}{\partial x} + B(a_i^{(j)})\frac{\partial}{\partial y} + C(a_i^{(j)})\frac{\partial}{\partial z}, \quad j = 1, 2. \tag{7.3}$$

By Proposition 7.2 the sequences $a_i^{(1)}, a_i^{(2)}$ tend to 0 and the sequences of functions $A_i(t), B_i(t), C_i(t)$ tend to $1, t^{2k-1}, 0$, respectively, in the C^1-topology. Therefore the sequences of vectors $v_i^{(1)}$ and $v_i^{(2)}$ tend to the vector $\partial/\partial x$. This implies the first statement of the lemma.

Express now the vector $v_i^{(2)}$ in the form

$$v_i^{(2)} = v_i^{(1)} + \triangle v_i \cdot (a_i^{(2)} - a_i^{(1)}), \tag{7.4}$$

$$\triangle v_i = A'(t_{i,1})\frac{\partial}{\partial x} + B'(t_{i,2})\frac{\partial}{\partial y} + C'(t_{i,3})\frac{\partial}{\partial z}, \tag{7.5}$$

where $t_{i,1}, t_{i,2}, t_{i,3} \in [a_i^{(1)}, a_i^{(2)}]$. In the case $k = 1$ the sequences $A_i'(t)$ and $C_i'(t)$ tend to 0 and the sequence $B_i'(t)$ tends to 1 in the C^0-topology. Therefore $\triangle v_i \to \partial/\partial y$. This implies the second and the third statements of the lemma.

7.4. *Proof of Lemma* 6.5

Let $F_i : [-1, 1] \to \mathbb{R}^2$ be a sequence of arcs tending to the arc $x = t^3, y = t^5$ in the C^r-topology with sufficiently large r and such that the singularity of F_i at $0 \in \mathbb{R}^2$ consists of three immersed components $\mu_i^{(1)}, \mu_i^{(2)}, \mu_i^{(3)}$. We have to prove that for sufficiently large i either $\ell\big(\mu_i^{(1)}\big) \neq \ell\big(\mu_i^{(2)}\big)$ or $\ell\big(\mu_i^{(1)}\big) \neq \ell\big(\mu_i^{(3)}\big)$.

We will prove this statement with $r = 5$. Assume, to get contradiction, that $\ell(\mu_i^{(1)}) = \ell(\mu_i^{(2)}) = \ell(\mu_i^{(3)}) = \ell_i$ for sufficiently large i. Then we may assume that this holds for all i. Any sequence of 1-dimensional subspaces of $T_0\mathbb{R}^2$ has a convergent subsequence. Therefore there is no loss of generality to assume that $\ell_i = \ell^*$ is a fixed 1-dimensional subspace of $T_0\mathbb{R}^2$. Let

$$\ell^* = \operatorname{span}(a \cdot \partial/\partial x + b \cdot \partial/\partial y), \quad (a, b) \neq (0, 0).$$

The arcs F_i have the form

$$F_i : x = (t - a_i^{(1)})(t - a_i^{(2)})(t - a_i^{(3)})A_i(t), \quad y = (t - a_i^{(1)})(t - a_i^{(2)})(t - a_i^{(3)})B_i(t),$$

where $a_i^{(1)}, a_i^{(2)}, a_i^{(3)}$ are distinct points in $(-1, 1)$. Introduce the sequence

$$G_i(t) = b \cdot A_i(t) - a \cdot B_i(t).$$

The lines $\ell(\mu_i^{(j)})$ are spanned by the vectors $A_i(a_i^{(j)})\partial/\partial x + B_i(a_i^{(j)})\partial/\partial y$, $j = 1, 2, 3$, therefore one has

$$G_i(a_i^{(1)}) = G_i(a_i^{(2)}) = G_i(a_i^{(3)}) = 0. \tag{7.6}$$

The assumption that $F_i \to (t^3, t^5)$ in the C^5-topology implies by Proposition 7.2 that $A_i \to 1$, $B_i \to t^2$ in the C^2-topology. Consequently the sequence G_i tends to $b - at^2$ in the C^2-topology. Proposition 7.1 and (7.6) with $a_i^{(1)}, a_i^{(2)}, a_i^{(3)}$ imply that $a = b = 0$ and we get contradiction.

7.5. *Proof of Lemma* 6.7

We will prove this lemma with $r = 3$. The arcs F_i have the form (7.1), (7.2) and the lines $\ell(\mu_i^{(1)})$ and $\ell(\mu_i^{(2)})$ are spanned by the vectors (7.3). Express the vector $v_i^{(2)}$ in the form (7.4), (7.5). By Proposition 7.2 the assumption that $F_i \to (t^3, t^5, 0)$ in the C^3-topology implies that the sequences of points $a_i^{(1)}, a_i^{(2)}$ tend to 0 and the sequences of functions A_i, B_i, C_i tend to $t, t^3, 0$, respectively, in the C^1-topology. It follows that $v_i^{(1)}, v_i^{(2)} \to 0$ and $\triangle v_i \to \partial/\partial x$. Since the vectors $v_i^{(1)}$ and $v_i^{(2)}$ are proportional this implies the lemma.

7.6. *Proof of Lemma* 6.8

We have to prove the following statements for a sequence of arcs $F_i : [-1, 1] \to \mathbb{R}^3$ tending to the arc $x = t^4, y = t^6, z = t^7$ in the C^r-topology with sufficiently large r:

- (i) if the singularity of F_i at $0 \in \mathbb{R}^3$ consists of four immersed components $\mu_i^{(1)}, \ldots, \mu_i^{(4)}$ then for sufficiently large i the lines $\ell(\mu_i^{(1)}), \ldots, \ell(\mu_i^{(4)})$ span $T_0\mathbb{R}^3$;
- (ii) if the singularity of F_i at $0 \in \mathbb{R}^3$ consists of three immersed components $\mu_i^{(1)}, \mu_i^{(2)}, \mu_i^{(3)}$ tangent to the same line ℓ_i then for sufficiently large i the 2-jet of the multigerm $(\mu_i^{(1)}, \mu_i^{(2)}, \mu_i^{(3)})$ is not planar.

Proof of (i). We will prove (i) with $r = 7$. Assume, to obtain a contradiction, that the lines $\ell(\mu_i^{(1)}), \ldots, \ell(\mu_i^{(4)})$ do not span $T_0\mathbb{R}^3$ for arbitrarily large i. Then these lines are contained in some 2-dimensional subspace $L_i \subset T_0\mathbb{R}^3$. The sequence L_i has a convergent subsequence, therefore there is no loss of generality to assume that the lines $\ell(\mu_i^{(1)}), \ldots, \ell(\mu_i^{(4)})$ are contained, for all i, in a fixed 2-dimensional subspace

$$L^* \subset T_0\mathbb{R}^3, \quad L^* = \ker(a\,dx + b\,dy + c\,dz), \ (a, b, c) \neq (0, 0, 0).$$

The arcs F_i have the form

$$x = R_i(t)A_i(t), \quad y = R_i(t)B_i(t), \quad z = R_i(t)C_i(t),$$
$$R_i(t) = (t - a_i^{(1)})(t - a_i^{(2)})(t - a_i^{(3)})(t - a_i^{(4)}),$$

where $a_i^{(1)}, \ldots, a_i^{(4)}$ are distinct points in $(-1, 1)$. Let

$$G_i(t) = a \cdot A_i(t) + b \cdot B_i(t) + c \cdot C_i(t).$$

The assumption $\ell(\mu_i^{(j)}) \subset L^*$ means the relations

$$G_i(a_i^{(1)}) = G_i(a_i^{(2)}) = G_i(a_i^{(3)}) = G_i(a_i^{(4)}) = 0. \tag{7.7}$$

By Proposition 7.2 the condition that $F_i \to (t^4, t^6, t^7)$ in the C^7-topology implies that the sequences of functions A_i, B_i, C_i tend to $1, t^2, t^3$, respectively, in the C^3-topology. Consequently the sequence $G_i(t)$ tends to $a + bt^2 + ct^3$ in the C^3-topology. Therefore by Proposition 7.1 relation (7.7) with distinct points $a_i^{(j)}, j = 1, 2, 3, 4$ implies that $a = b = c = 0$ and we obtain the contradiction. □

Proof of (ii). We will prove (ii) with $r = 8$. Arguing like in the proof of of Lemma 6.7 it is easy to prove that $\ell_i \to \mathrm{span}(\partial/\partial x)$ and then there is no loss of generality to assume that $\ell_i = \mathrm{span}(\partial/\partial x)$ for all i. Assume now, to get contradiction, that there is a sequence S_i of non-singular surfaces such that $\mathrm{ord}(\mu_i^{(j)}, S_i) \geqslant 2$, $j = 1, 2, 3$. Then there is no loss of generality to assume also that the tangent planes T_0S_i are the same for all i. Since $\ell_i = \mathrm{span}(\partial/\partial x) \subset L_i$ then there exist fixed numbers $(b^*, c^*) \neq (0, 0)$ such that

$$S_i = \{(x, y, z) : b^*y + c^*z + f_i(x, y, z) = 0\}, \quad j^1 f_i = 0.$$

The arcs F_i have the form

$$F_i : x = R_i(t)A_i(t), \quad y = R_i^2(t)B_i(t), \quad z = R_i^2(t)C_i(t), \tag{7.8}$$
$$R_i(t) = (t - a_i^{(1)})(t - a_i^{(2)})(t - a_i^{(3)}),$$

where $a_i^{(1)}, a_i^{(2)}, a_i^{(3)}$ are distinct points in $(-1, 1)$. It is easy to calculate that the assumption $\mathrm{ord}(\mu_i^{(j)}, S_i) \geqslant 2$ means that the functions

$$G_i(t) = r_iA_i^2(t) + b^*B_i(t) + c^*C_i(t), \quad r_i = (1/2) \cdot (\partial^2 f_i/\partial x^2)(0)$$

satisfy the relations

$$G_i(a_i^{(1)}) = G_i(a_i^{(2)}) = G_i(a_i^{(3)}) = 0. \tag{7.9}$$

Now we use the condition that $F_i \to (t^4, t^6, t^7)$ in the C^8-topology. Proposition 7.2 implies that $A_i \to t, B_i \to 1, C_i \to t$ in the C^2-topology. If the sequence r_i is not bounded then then the sequence $G_i(t)/r_i$ has a subsequence tending to t^2 in the C^2-topology which contradicts to (7.9) by Proposition 7.1. If the sequence r_i is bounded then the sequence $G_i(t)$ has a subsequence tending to $r^*t^2 + b^* + c^*t$ in the C^2-topology, for some finite r^*. By Proposition 7.1 and (7.9) this contradicts to the condition $(b^*, c^*) \neq (0, 0)$. □

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