106803. Homework 4. Deadline: May 25.

1. Find the form of the vector field $V = \frac{\partial}{\partial x_1} + x_1^2 \frac{\partial}{\partial x_2} + x_1 x_2 \frac{\partial}{\partial x_3}$ in the new coordinates y_1, y_2, y_3 (near $0 \in \mathbb{R}^3$) such that

a) $x_1 = y_1 + y_2^2,\ \ x_2 = y_2 + y_3^2,\ \ x_3 = 2y_3$ b) $y_1 = x_1 + x_2^2,\ \ y_2 = x_2 + x_3^2,\ \ y_3 = 2x_3$.

2. Find a coordinate system y_1, y_2, y_3 near $0 \in \mathbb{R}^3$ such that the following vector fields take the form $\frac{\partial}{\partial y_1}$:

a) $V = \frac{\partial}{\partial x_1} + x_1^2 \frac{\partial}{\partial x_2} + x_1 x_2 \frac{\partial}{\partial x_3}$ b) $V = (1 + x_1) \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$

c) $V = \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} + x_1 x_2 \frac{\partial}{\partial x_3}$

The answer can be written in the form
$x_1 = \Phi_1(y_1, y_2, y_3), x_2 = \Phi_2(y_1, y_2, y_3), x_3 = \Phi_3(y_1, y_2, y_3)$.

3. Find a global coordinate system y_1, y_2, y_3 on $\mathbb{R}^3$ such that the following vector fields have the form $\lambda_1 \frac{\partial}{\partial y_1} + \lambda_2 \frac{\partial}{\partial y_2} + \lambda_3 \frac{\partial}{\partial y_3}$ for some $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$:

a) $V = \big(2x_1 + 3x_2 + x_3\big) \frac{\partial}{\partial x_1} + \big(x_2 + 5x_3\big) \frac{\partial}{\partial x_2} + 3x_3 \frac{\partial}{\partial x_3}$

b) $V = \big(a_{11}x_1 + a_{12}x_2 + a_{13}x_3\big) \frac{\partial}{\partial x_1} + \big(a_{21}x_1 + a_{22}x_2 + a_{23}x_3\big) \frac{\partial}{\partial x_2} +$
$+ \big(a_{31}x_1 + a_{32}x_2 + a_{33}x_3\big) \frac{\partial}{\partial x_3}$, where the 3×3 matrix (a_{ij}) has eigenvectors $\big(1\ \ 2\ \ 4\big), \big(0\ \ 1\ \ 1\big), \big(1\ \ 1\ \ 1\big)$.

4. Recall that a (k, n) distribution $D = span\,(V_1, ..., V_k)$ is called integrable if $[V_i, V_j] \in span\,(V_1, ..., V_k)$ for any $i, j = 1, ..., k$ and at any point of $\mathbb{R}^n$. Prove the following theorem (begin with $k = 2, n = 3$, after that generalize to any k, n):

Theorem (Frobenius) *Any C^∞ integrable (k, n) distribution is locally (near any point of $\mathbb{R}^n$) C^∞-diffeomorphic to the distribution span* $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_k}\right)$.

5. Prove the following theorem.

Theorem (J. Martinet, 1970). *Any C^∞ $(2, 3)$ distribution on $\mathbb{R}^3$ with the growth vector $(2, 2, 3)$ at the point $0 \in \mathbb{R}^3$ is locally (near $0 \in \mathbb{R}^3$) C^∞-diffeomorphic to the distribution span* $\big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} + x_2^2 \frac{\partial}{\partial x_3}\big)$.

Proving this theorem you should use the following lemma from singularity theory.

Lemma. *A C^∞ function $f(x_1, x_2, x_3)$ such that*

$$f(0,0,0) = \frac{\partial f}{\partial x_1}(0,0,0) = 0, \quad \frac{\partial^2 f}{\partial x_1^2}(0,0,0) \neq 0$$

can be reduced by a local (near $0 \in \mathbb{R}^3$) change of coordinates of the form

$$x_1 = \phi(y_1, y_2, y_3),\ x_2 = y_2,\ x_3 = y_3$$

to the form $\pm y_1^2 + g(y_2, y_3)$ with some function $g(y_1, y_2)$.