Lecture 3. Constructing a mini-versal deformation. Infinitesimal method

1. Introduction

In this lecture I will define an *infinitesimal* (mini-) versal deformation and I will explain a simple way for constructing it. One of difficult and very powerful theorems of singularity theory is as follows:

Versality theorem. In the problem of classification of C^∞ or analytic map germs $\mathbb{R}^n \to \mathbb{R}^m$ with respect to local diffeomorphisms (changes of coordinates) either in the source space or in the target space, or in both, any infinitesimal (mini-) versal deformation is a (mini-) versal deformation.

The versality theorem (infinitesimal versal deformation = versal deformation) also holds for finite-dimensional problems of linear algebra.

I will consider a general case (sections 2,3,4), after that I will apply the construction to the problem of classifying matrices (section 5), functions (section 6), and parameterized curves (section 7) to get examples of mini-versal deformations including those mentioned in Lectures 1-2.

2. The general case I will consider

I will explain the notion of infinitesimal versal deformation in terms of math of 19th century, avoiding Lie algebras and Lie groups. It is possible in the following case (which holds for many, but not for all classification problems):

1. The space of objects is a vector space V of finite or infinite dimension. The equivalence of two objects is induced by the action of a group G on V of finite or infinite dimension (if $dimG = \infty$ it is more accurate to say that G is a pseudo-group). Two objects $a, b \in G$ are equivalent if they belong to one orbit of the action of G, i.e. there exists $g \in G$ such that $g.a = b$.

2. The group G belongs to a certain vector space W: $G \subset W$ such that for any fixed $w \in W$ one has $id + \epsilon w \in G$ for sufficiently small ϵ. Here id is the identity element of G.

Remark. Usually G is a finite dimensional or infinite dimensional manifold, i.e. G is a Lie group or Lie pseudo-group. In both cases the vector space W has a very important structure of Lie algebra and one can define the exponential map $W \to G$.

3. Linearization of the group action

Fix $a \in V$ and $w \in W$. Consider the family $(id + \epsilon w).a \in V$ defined for small ϵ. Write it in the form

$$(id + \epsilon w).a = a + \epsilon b + o(\epsilon), \tag{1}$$

where b is a certain element (point, vector) in V. It depends on $a \in V$ and on $w \in W$:

$$b = L_a(w).$$

The map $w \to L_a(w)$ is linear. We obtain a linear operator

$$L_a : \ W \to \ V, \quad L_a(w) = b, \quad b \text{ is defined by (1).}$$

Definition. The linear operator $L_a : W \to V$ is the linearization of the action of G at the point a. The image of this operator (which is a subspace of V) is the tangent space to the orbit of a at the point a.

4. Definition of an infinitesimal versal deformation

Consider the tangent space $Image(L_a) \subset V$. Let U be any subspace such that

$$V = Image(T_a) + U \tag{2}$$

Assume that $dimU < \infty$. Let $u_1, ..., u_m$ be a basis of U.

Definition. The family $a + \delta_1 u_1 + \cdots + \delta_m u_m$ is an infinitesimal versal deformation of a. It is an infinitesimal mini-versal deformation if the sum in (2) is direct, i.e. U is a complementary vector space to $Image(T_a)$ in V.

5. Example: versal deformation of matrices

Consider the problem of classifying $n \times n$ matrices with respect to similarity: A is equivalent to B if $T^{-1}AT = B$ for some non-singular matrix T. In this case V is the vector space of all $n \times n$ matrices, G is the group of all non-singular $n \times n$ matrices, and $W = V$. Requirement 2 (section 2) holds since for any fixed $n \times n$ matrix A and sufficiently small ϵ the matrix $I + \epsilon A$ is non-singular.

To compute the linearization of the action of G note that for any fixed $n \times n$ matrix B and small ϵ one has

$$(I + \epsilon B)^{-1} A (I + \epsilon B) = (I - \epsilon B + o(\epsilon))(A + \epsilon AB) =$$

$$= A + \epsilon(AB - BA) + o(\epsilon).$$

Therefore the linearization of the action of G at the point $A \in Mat(n, n)$ is the linear operator

$$L_A : \ Mat(n,n) \to Mat(n,n), \quad L_A(B) = AB - BA$$

and the tangent space to the orbit of A at the point A is the subspace

$$\big\{AB - BA, \ B \in Mat(n,n)\big\} \subset Mat(n,n). \tag{3}$$

Example. Let us find an infinitesimal mini-versal (and consequently mini-versal) deformation of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easy to compute that the vector space (3) consists of matrices of the form

$$Image(L_A) = \begin{pmatrix} a & b & c \\ 0 & -a & 0 \\ 0 & d & 0 \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}.$$

Fix a complementary vector space, for example

$$U = \begin{pmatrix} 0 & 0 & 0 \\ r_1 & r_2 & r_3 \\ r_4 & 0 & r_5 \end{pmatrix}, \quad r_1, ..., r_5 \in \mathbb{R}.$$

We obtain an example of an infinitesimal mini-versal (and consequently mini-versal) deformation of A:

$$\begin{pmatrix} 1 & 1 & 0 \\ \delta_1 & 1+\delta_2 & \delta_3 \\ \delta_4 & 0 & 1+\delta_5 \end{pmatrix}$$

with 5 parameters $\delta_1, ..., \delta_5$.

Exercise. Find an example of mini-versal deformation of the matrices

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and explain the number of parameters in the mini-versal deformation (see Lect. 1).

6. Classification of function germs

Consider the infinite dimensional vector space

$$V = \text{function germs } f : \mathbb{R}^n \to \mathbb{R} \text{ at } 0 \in \mathbb{R}^n.$$

Consider the problem of classifying V with respect to local C^∞ diffeomorphisms of $\mathbb{R}^n$ (changes of coordinates in the source space). In this case $G = Diff(n)$ is the infinite dimensional group of local C^∞ diffeomorphisms $\Psi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$. In local coordinates any $\Psi \in Diff(n)$ is a vector-function germ

$$\Psi = \begin{pmatrix} \Psi_1 \\ \cdots \\ \Psi_n \end{pmatrix} \tag{4}$$

and $\Psi \in Diff(n)$ is and only if the Jacobian does not vanish at 0 (and consequently near 0):

$$det \begin{pmatrix} \frac{\partial \Psi_1}{\partial x_1} & \cdots & \frac{\partial \Psi_1}{\partial x_n} \\ \cdots & & \\ \frac{\partial \Psi_n}{\partial x_1} & \cdots & \frac{\partial \Psi_n}{\partial x_n} \end{pmatrix} (0) \neq 0.$$

Consider the following vector space W covering the group $Diff(n)$:

$$W = V^n = \text{ all vector function germs (4)}.$$

The requirement 2 (section 2) holds because for any fixed function germs $\psi_1(x), ..., \psi_n(x)$ the map $x_i \to x_i + \epsilon\psi_i(x)$ is a local diffeomorphism for sufficiently small ϵ:

$$det\Bigg(I + \epsilon \begin{pmatrix} \frac{\partial \psi_1}{\partial x_1} & \cdots & \frac{\partial \psi_1}{\partial x_n} \\ \cdots & & \\ \frac{\partial \psi_n}{\partial x_1} & \cdots & \frac{\partial \psi_n}{\partial x_n} \end{pmatrix} (0) \Bigg) \neq 0.$$

Remark. Why we do not restrict W to vector function germs vanishing at $0 \in \mathbb{R}^n$? Because working with families we work in extended space $\mathbb{R}^{n+1} = \mathbb{R}^n(x) \times \mathbb{R}(\epsilon)$ and a map $x_i \to x_i + \epsilon\psi_i(x)$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ should be understood as a map

$$x_i \to x_i + \epsilon\psi_i(x), \quad \epsilon \to \epsilon \quad \text{from } \mathbb{R}^{n+1} \text{ to } \mathbb{R}^{n+1}.$$

This map is a local diffeomorphism at $0 = \{x = 0, \epsilon = 0\} \in \mathbb{R}^{n+1}$ preserving $0 \in \mathbb{R}^{n+1}$ for ANY $\psi_1(x), ..., \psi_n(x)$.

To linearize the action of $Diff(n)$ note that

$$f(x_1 + \epsilon\psi_1(x), \cdots, x_n + \epsilon\psi_n(x)) = f(x) + \epsilon\Big(\frac{\partial f}{\partial x_1}\psi_1 + \cdots + \frac{\partial f}{\partial x_n}\psi_n\Big) + o(\epsilon).$$

Therefore the linearization of the action of $Diff(n)$ at the function germ $f = f(x) \in V$ is the linear operator

$$L_f: \quad V^n \to V, \quad L_f(\psi_1, ..., \psi_n) = \frac{\partial f}{\partial x_1}\psi_1 + \cdots + \frac{\partial f}{\partial x_n}\psi_n.$$

The tangent space to the orbit of f at $f \in V$ is the vector space

$$\Big\{\frac{\partial f}{\partial x_1}\psi_1 + \cdots + \frac{\partial f}{\partial x_n}\psi_n, \quad \psi_i \in V\Big\}. \tag{5}$$

Example. Let $n = 1$, $f(x) = x^n$. The vector space (5) consists of function germs of the form $nx^{n-1}\phi(x)$ where $\phi(x)$ is an arbitrary function germ. A complementary vector space can be chosen to be $span(1, x, ..., x^{n-2})$. We obtain an infinitesimal mini-versal (and consequently mini-versal)) deformation of x^n:

$$x^n + \delta_0 + \delta_1 x + \cdots + \delta_{n-2}x^{n-2}$$

which is the classical result mentioned in Lecture 2.

Example. Let $n = 2$, $f(x_1, x_2) = x_1^3 + x_2^3$. The vector space (5) consists of function germs of the form $3x_1^2\phi_1(x) + 3x_2^2\phi_2(x)$ where $\phi_1(x), \phi_2(x)$ are arbitrary function germs. A complementary vector space can be chosen to be $span(1, x_1, x_2, x_1x_2)$. We obtain an infinitesimal mini-versal (and consequently mini-versal)) deformation of $x_1^3 + x_2^3$:

$$x_1^3 + x_2^3 + \delta_1 + \delta_2 x_1 + \delta_3 x_2 + \delta_4 x_1 x_2.$$

Exercise. Find mini-versal deformations of the function germs

$$x_1^n + x_2^m, \; x_1^2 x_2 + x_2^6$$

(you can work on the level of formal power series, the same results hold in C^∞ and analytic categories).

7. Classification of parameterized curves (one branch)

Consider the vector space of germs of parameterized curves $\gamma : \mathbb{R} \to \mathbb{R}^n$ at a fixed point $t = 0 \in \mathbb{R}$. In the notations of the previous section it is the vector space V^n. Consider the equivalence induced by the changes of coordinates in both the source space $\mathbb{R}$ (reparameterization) and the target space $\mathbb{R}^n$ (see lecture 2). In this case $G = Diff(1) \times Diff(n)$ where $Diff(1)$ is the group of local diffeomorphisms of $\mathbb{R}$ and $Diff(n)$ is the group of local diffeomorphisms of $\mathbb{R}^n$. The requirement 2 (section 2) holds for $W = V \times V^n$. (Explanation: the same as in the previous section. Constructing W we do not take away function germs that do not vanish at 0 - see the remark in the previous section).

To linearize the action of $Diff(1) \times Diff(n)$ write

$$\gamma(t) = \begin{pmatrix} \gamma_1(t) \\ ... \\ \gamma_n(t) \end{pmatrix}$$

and note that a reparameterization $t \to t + \epsilon\phi(t)$ brings $\gamma(t)$ to a curve of the form

$$\gamma(t) + \epsilon \begin{pmatrix} \gamma_1'(t)\phi(t) \\ \cdots \\ \gamma_n'(t)\phi(t) \end{pmatrix} + o(\epsilon) \tag{6}$$

and a change of coordinates of the form $x_i \to x_i + \epsilon\psi_i(x)$ brings (6) to a curve of the form

$$\gamma(t) + \epsilon \begin{pmatrix} \psi_1(\gamma_1(t), ..., \gamma_n(t)) + \gamma_1'(t)\phi(t) \\ \cdots \\ \psi_n(\gamma_1(t), ..., \gamma_n(t)) + \gamma_n'(t)\phi(t) \end{pmatrix} + o(\epsilon).$$

Therefore the linearization of the action of $Diff(1) \times Diff(n)$ at the point $\gamma(t)$ is the linear operator

$$L_\gamma : \ V \times V^n \ \to V^n, \qquad L_\gamma(\phi, \psi_1, ..., \psi_n) = \begin{pmatrix} \psi_1(\gamma_1(t), ..., \gamma_n(t)) + \gamma_1'(t)\phi(t) \\ \cdots, \\ \psi_n(\gamma_1(t), ..., \gamma_n(t)) + \gamma_n'(t)\phi(t) \end{pmatrix}.$$

Example. Let $n = 2$ and consider the cusp singularity $\gamma(t) = (t^2, t^3)$. Then L_γ maps a triple $\phi(t), \psi_1(x_1, x_2), \psi_2(x_1, x_2)$ to the vector function germ

$$\begin{pmatrix} \psi_1(t^2, t^3) + 2t\phi(t) \\ \psi_2(t^2, t^3) + 3t^2\phi(t) \end{pmatrix} \tag{7}$$

and the tangent space to the orbit of $\gamma = \gamma(t)$ at the point γ consists of vector function germs of this form, with arbitrary $\phi(t), \psi_1(x_1, x_2), \psi_2(x_1, x_2)$. Since any non-negative integer m except $m = 1$ can be expressed in the form $m = 2d_1 + 3d_2$ with non-negative integers d_1, d_2, by a choice of $\psi_1(x_1, x_2)$ and $\psi_2(x_1, x_2)$ we can get, in the first and in the second component, any function whose $f(t)$ such that $f'(0) = 0$ (it is clear on the level of formal power series; the same holds in C^∞ and analytic categories). Using it, it is easy to see that a complementary space to the vector space of functions of form (7) is one-dimensional and can be chosen to be *span* $\begin{pmatrix} 0 \\ t \end{pmatrix}$. We obtain an infinitesimal mini-versal (and consequently mini-versal) deformation of the cusp (t^2, t^3):

$$(t^2, \ t^3 + \epsilon t)$$

used in Lecture 2.

Example. Consider the curve germ (t^3, t^4). To construct its mini-versal deformation we have to find a complementary space to the vector space consisting of vector function germs of the form

$$\begin{pmatrix} \psi_1(t^3, t^4) + 3t^2\phi(t) \\ \psi_2(t^3, t^4) + 4t^3\phi(t) \end{pmatrix}$$

where $\phi(t), \psi_1(x_1, x_2), \psi_2(x_1, x_2)$ are arbitrary function germs. Using the fact that any non-negative integer m except $m = 1, 2, 5$ can be expressed in the form $m = 3d_+ 4d_2$ with non-negative integers d_1, d_2, it is easy to prove that on the level of formal power series a complementary vector space can be chosen to be

$$\left\{ \begin{pmatrix} r_1 t \\ r_2 t + r_3 t^2 \end{pmatrix}, \quad r_1, r_2, r_3 \in \mathbb{R} \right\}.$$

Therefore one of the mini-versal deformations is as follows:

$$(t^3 + \delta_1 t, \ t^4 + \delta_2 t + \delta_3 t^2).$$

Exercise. Find mini-versal deformations of the curve germs $(t^3, t^5), (t^4, t^5), (t^3, t^7)$.

Exercise. Understand constructing of mini-versal deformation of multi-germs of curves (see Lecture 2) and construct a mini-versal deformation of the multi-germ $(t_1^2, t_1^3) \ \cup \ (t_2, t_2)$.