Lecture 6. Systems of linear ODEs with constant coefficients

$$X' = AX, \quad A \text{ constant matrix } n \times n \tag{1}$$

In (1) A is a constant $n \times n$ matrix and $X = X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \dots \\ x_n(t) \end{pmatrix}$.

1. **Existence of solutions defined for all t.** For any $X_0 \in \mathbb{R}^n$ there is a unique solution satisfying the initial condition $X(0) = X_0 \in \mathbb{R}^n$. This solution can be expressed by the formulae

$$X(t) = e^{At} \cdot X_0,$$

where

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots. \tag{2}$$

Here I is the identity matrix. Whatever is the matrix A, the series converges to a matrix whose entries are C^∞ functions.

2. **Shift of time.** Since (1) is autonomous system (i.e. the right hand side part does not depend directly on t), given any solution $X(t)$, the vector function $X(t+s)$ is also a solution, for any $s \in \mathbb{R}$. It follows that the solution of (1) satisfying the initial condition $X(t_0) = X_0 \in \mathbb{R}^n$ can be expressed by the formulae

$$X(t) = e^{A(t-t_0)} \cdot X_0. \tag{3}$$

Here $e^{A(t-t_0)}$ is the series (2) with t replaced by $t - t_0$.

3. **Using (2) or (3).** In applications these formulae can be used only in a small neighborhood of $t = 0$ (series (2)) or $t = t_0$ (series (3)).

Example. Consider the system

$$x_1' = 2x_1 - 4x_2, \quad x_2' = x_1 + 5x_2$$

and the initial condition

$$x_1(10) = 1, \quad x_2(10) = 0.$$

The solution has the form

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \left(I + \begin{pmatrix} 2 & -4 \\ 1 & 5 \end{pmatrix} (t-10) + \frac{\begin{pmatrix} 2 & -4 \\ 1 & 5 \end{pmatrix}^2}{2} (t-10)^2 \right) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + o(t-10)^2 \text{ as } t \to 10$$

and we obtain

$$x_1(t) = 1 + 2 \cdot (t-10) + o(t-10)^2 \text{ as } t \to 10$$
$$x_2(t) = (t-10) + 3.5(t-10)^2 + o(t-10)^2 \text{ as } t \to 10$$

4. **The case of complex A. The complex-valued solutions.** A complex-valued solution of (1) is a function $\mathbb{R} \to \mathbb{C}^n$ satisfying this equation. The complex-valued solutions are defined if A is either real or complex matrix. For any matrix A (either real or complex) the complex-valued solution of (1) satisfying the initial

condition $X(t_0) = X_0 \in \mathbb{C}^n$ can be expressed by the same formulae (3), where the exponent is defined by the same formula (2).

Example. Consider the system

$$x_1' = -x_2, \quad x_2' = x_1 \tag{4}$$

It is easy to check that the solution satisfying the initial condition

$$x_1(0) = 0, \quad x_2(0) = 0$$

is

$$x_1(t) = cos(t), \quad x_2(t) = sin(t).$$

On the other hand, introducing $z = x_1 + ix_2$ we can write (4) as one equation for complex-valued function $z(t)$:

$$z' = x_1' + ix_2' = -x_2 + ix_1 = i \cdot (x_1 + ix_2) = ib \cdot z.$$

The initial condition is $z(0) = 1 + 0i = 1$. The solution is $z(t) = e^{it} \cdot 1 = e^{it}$. Now the uniqueness theorem implies the famous

Euler formulae:

$$e^{it} = cos(t) + i \cdot sin(t)$$

5. **Theorem.** If A is a real $n \times n$ matrix then the set of all real-valued solutions of (1) (defined for all t) is a subspace of the vector space of C^∞ real-valued vector-functions, and this subspace has dimension n over the field $\mathbb{R}$. The set of complex-valued solutions of (1) is a subspace of the vector space of C^∞ complex-valued vector-functions, and this subspace has dimension n over the field $\mathbb{C}$. If $X(t)$ is a non-real solution of (1) then the complex-conjugate vector function $\bar{X}(t)$ is also a solution of (1).

6. **The method for finding the basis of the space of all solutions.** The method is as follows. We introduce a new vector-function $Y = Y(t)$ related to $X = X(t)$ via a (transition) invertible $n \times n$ constant matrix T:

$$X = TY.$$

Then, substituting to (1) we obtain

$$X' = TY' = AX = ATY$$

and we obtain the following equation for Y:

$$Y' = \big(T^{-1}AT\big)Y$$

We can take any invertible T and of course one should take T so that the matrix $T^{-1}AT$ has the simplest possible form.

7. **The case that A is diagonalizable over $\mathbb{R}$.** This case holds if the eigenvalues of A are all real and each of them has the same algebraic and geometric multiplicity. Within this case the most important one is the case when A has n real distinct eigenvalues $\lambda_1, ..., \lambda_n$ (then the algebraic and the geometric multiplicity of each of the eigenvalues is 1). Consider this case. Denote

$$T_i = \text{ eigenvector corresponding to } \quad \lambda_i$$

and consider the matrix

$$T = \text{ matrix with columns } T_1, ..., T_n.$$

This matrix is invertible and

$$T^{-1}AT = diag(\lambda_1, ..., \lambda_n),$$

where $diag(\lambda_1, ..., \lambda_n)$ denotes the diagonal matrix with $\lambda_1, ..., \lambda_n$ on the diagonal. Introducing Y such that $X = TY$ (section 6) we obtain

$$Y' = diag(\lambda_1, ..., \lambda_n) \cdot Y$$

For this system an example of the basis of all real-valued solutions can be easily calculated:

$$Y^{(1)}(t) = e^{\lambda_1 t} \cdot \begin{pmatrix} 1 \\ 0 \\ ... \\ 0 \\ 0 \end{pmatrix}, \quad \cdots, \quad Y^{(n)}(t) = e^{\lambda_n t} \cdot \begin{pmatrix} 0 \\ 0 \\ ... \\ 0 \\ 1 \end{pmatrix}$$

Now we should transfer this basis from Y to X. Since $X = TY$ we simply multiply the vector functions above by T. We obtain an example of the basis

$$X^{(1)}(t) = e^{\lambda_1 t} T_1, \quad \cdots, \quad X^{(n)}(t) = e^{\lambda_n t} T_n.$$

Example. Let us find the solution of the system

$$x_1' = x_1 + x_2, \quad x_2' = -2x_1 + 4x_2$$

satisfying the initial condition

$$x_1(0) = 1, \ x_2(0) = -1.$$

We calculate the eigenvalues of the matrix $\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$. They are $\lambda_1 = 2, \lambda_2 = 3$, therefore the matrix is diagonalizable over $\mathbb{R}$. Calculate the corresponding eigenvectors $T_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $T_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Now we know an example of the basis of the space of all real-valued solutions:

$$X^{(1)}(t) = e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad X^{(2)}(t) = e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Any solution has the form

$$X(t) = C_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

where $C_1, C_2 \in \mathbb{R}$. The coefficients C_1, C_2 depend on the initial condition. The initial condition in our example is $X(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Substituting $t = 0$ we obtain

$$C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and solving this system we obtain $C_1 = 3, C_2 = -2$.

8. **The case that the real $n \times n$ matrix A is not diagonalizable over $\mathbb{R}$ but diagonalizable over $\mathbb{C}$**

Again, within this case I will consider the most important subcase that A has n distinct eigenvalues, but some of them are not real. Since A is real, the set of all eigenvalues is as follows:

$$\lambda_1, ..., \lambda_r, \quad a_1 \pm b_1 i, \quad a_2 \pm b_2 i, \quad ..., \quad a_s \pm b_s i,$$

where $b_1, ..., b_s \neq 0$ and $r + 2s = n$. We use the same method (section 6) and we find an example of a basis of the space of all solutions in the same way as in section 7, but now we work over the field $\mathbb{C}$, i.e. we find an example of a basis of the space of complex-valued solutions. It is

$$e^{\lambda_1 t} T_1, \ \ldots, e^{\lambda_r t} T_r, \tag{6}$$

$$e^{(a_1+b_1 i)t} U_1, \ e^{(a_1-b_1 i)t} \bar{U}_1, \ \cdots, \ e^{(a_s+b_s i)t} U_s, \ e^{(a_s-b_s i)t} \bar{U}_s, \tag{7}$$

where $T_1, ..., T_r$ are real eigenvectors corresponding to the real eigenvalues $\lambda_1, ..., \lambda_r$ and $U_1, ..., U_s$ are complex eigenvectors (in $\mathbb{C}^n$) corresponding to the eigenvalues $a_1 + b_1 i, ..., a_s + b_s i$ (Then the complexly-conjugate vectors $\bar{U}_1, ..., \bar{U}_s$ are the eigenvectors corresponding to the eigenvalues $a_1 - b_1 i, ..., a_s - b_s i$).

Now we have to transfer this basis for the space of complex-valued solutions to a basis for the space of real-valued solutions. Replacing in (7) each of the couples

$$e^{(a_k+b_k i)t} U_1, \ e^{(a_k-b_k i)t} \bar{U}_1, \quad k = 1, ..., s$$

by the couple

$$\left(e^{(a_1+b_1 i)t} U_1 + e^{(a_1-b_1 i)t} \bar{U}_1\right)/2, \quad \left(e^{(a_1+b_1 i)t} U_1 - e^{(a_1-b_1 i)t} \bar{U}_1\right)/(2i),$$

we obtain another example of a basis for the space of complex-valued solutions:

$$e^{\lambda_1 t} T_1, \ \ldots, e^{\lambda_r t} T_r, \tag{8}$$

(9)

$$Re\Big(e^{(a_1+b_1 i)t} U_1\Big), \ Im\Big(e^{(a_1+b_1 i)t} U_1\Big), \ \cdots, \ Re\Big(e^{(a_s+b_s i)t} U_s\Big), \ Im\Big(e^{(a_s+b_s i)t} U_s\Big),$$

where Re and Im denote the real and the imaginary parts. Since each of the vector functions in (8)-(9) is real-valued then (8)-(9) is a basis simultaneously for complex-valued and real-valued solutions. (This means that vector-functions in (8)-(9) are linearly independent over $\mathbb{C}$ and consequently linearly independent over $\mathbb{R}$, and that any real-valued solution is a linear combination of these vector-functions with real coefficients, and any complex-valued solution is a linear combination of these vector-functions with complex coefficients).

Example Let us find the general solution of the system

$$x_1' = x_1 - 10x_2, \quad x_2' = x_1 + 3x_2.$$

The eigenvalues of the matrix $\begin{pmatrix} 1 & -10 \\ 1 & 3 \end{pmatrix}$ are $\lambda_{1,2} = 2 \pm 3i$, therefore this matrix is diagonalizable over $\mathbb{C}$, but not over $\mathbb{R}$. Find the eigenvector corresponding to $\lambda_1 = 2 + 3i$, one of example is $\begin{pmatrix} -10 \\ 1 + 3i \end{pmatrix}$. Therefore an example of a basis for the space of all complex-valued solutions is

$$e^{(2+3i)t} \begin{pmatrix} -10 \\ 1 + 3i \end{pmatrix}, \quad e^{(2-3i)t} \begin{pmatrix} -10 \\ 1 - 3i \end{pmatrix} \tag{10}$$

and for real-valued solutions:

$$Re\Big(e^{(2+3i)t}\begin{pmatrix}-10\\1+3i\end{pmatrix}\Big),\quad Im\Big(e^{(2+3i)t}\begin{pmatrix}-10\\1+3i\end{pmatrix}\Big). \tag{11}$$

Using the Euler formulae (section 4) we can write down the basis (11) without complex numbers:

$$Re\Big(e^{(2+3i)t}\begin{pmatrix}-10\\1+3i\end{pmatrix}\Big) = e^{2t}\cdot Re\Big(cos(3t)+i\cdot sin(3t))\cdot\begin{pmatrix}-10\\1+3i\end{pmatrix}\Big) =$$

$$= e^{2t}\cdot\begin{pmatrix}-10cos(3t)\\cos(3t)-3sin(3t)\end{pmatrix} \tag{12}$$

$$Im\Big(e^{(2+3i)t}\begin{pmatrix}-10\\1+3i\end{pmatrix}\Big) = e^{2t}\cdot Im\Big(cos(3t)+i\cdot sin(3t))\cdot\begin{pmatrix}-10\\1+3i\end{pmatrix}\Big) =$$

$$= e^{2t}\cdot\begin{pmatrix}-10sin(3t)\\3cos(3t)+3sin(3t)\end{pmatrix} \tag{13}$$

The general real-valued solution (i.e. the set of all real-valued solutions) is the linear combination of the vector functions in (12)-(13) with real coefficients C_1, C_2. To find the solution corresponding to the given initial condition we substitute $t = t_0$ and obtain a system of linear equations for C_1, C_2.

Another way to find a solution corresponding to given initial conditions is to work over $\mathbb{C}$ and use basis (10). We know that any complex-valued solution (and in particular any real-valued solution!) is a linear combination of complex-valued functions (10) with complex coefficients C_1, C_2. Substituting $t = t_0$ we obtain a linear system for C_1, C_2. The matrix of this system has complexly-conjugate columns and the right hand side is a vector in $\mathbb{R}^2$ (if of course the initial conditions are real, as in applications). Therefore solving this system for C_1, C_2 we obtain $C_2 = \bar{C}_1$. The solution takes the form

$$C_1e^{(2+3i)t}\begin{pmatrix}-10\\1+3i\end{pmatrix} + \bar{C}_1e^{(2-3i)t}\begin{pmatrix}-10\\1-3i\end{pmatrix} = 2\cdot Re\Big(C_1e^{(2+3i)t}\begin{pmatrix}-10\\1+3i\end{pmatrix}\Big),$$

where C_1 is a certain complex number.

9. **The case that the $n \times n$ matrix A is not diagonalizable over $\mathbb{C}$.** This requires the whole theory of Jordan normal forms. In this Lecture Notes I will explain the construction of a basis in the simplest case when:

the $n \times n$ matrix A has $(n-1)$ distinct eigenvalues $\lambda_1, ..., \lambda_{n-1}$, each of the eigenvalues $\lambda_2, ..., \lambda_{n-1}$ has algebraic multiplicity 1 (and consequently geometric multiplicity 1) and the eigenvalue λ_1 has algebraic multiplicity 2 and geometric multiplicity 1.

Denote by $T_1, ..., T_{n-1}$ eigenvectors corresponding to $\lambda_1, ..., \lambda_{n-1}$.

Lemma. In the case under consideration there exists a vector $\widehat{T_1}$, called associate vector to T_1, such that

$$(A - \lambda_1 I)\widehat{T_1} = \lambda_1 T_1. \tag{14}$$

Construct $n \times n$ matrix

$$T = \; n \times n \text{ matrix with columns } T_1, \widehat{T}_1, T_2, ..., T_{n-1}.$$

Lemma. The vectors $T_1, \widehat{T}_1, T_2, ..., T_{n-1}$ are linearly independent and consequently the matrix T is invertible.

The equations $AT_i = \lambda_i T_i,\ i = 1, ..., n-1$ and the equation (14) imply

$$AT = TJ, \quad J = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 & ... & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & ... & ... & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & \lambda_3 & 0 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 0 & ... & 0 & \lambda_{n-1} \end{pmatrix}$$

and consequently

$$T^{-1}AT = J$$

The matrix J is one of Jordan normal forms. Now we use the method in section 6: we introduce Y such that $X = TY$, for Y we obtain the system

$$Y' = JY,$$

this system can be easily solved using the method of variation of constant (see example below), after that we transfer Y to X.

Example. Let us find the solution of the system

$$x_1' = 2x_1 + 3x_2, \quad x_2' = -3x_1 + 8x_2$$

satisfying the initial condition

$$x_1(0) = 1,\ x_2(0) = 3.$$

The matrix $A = \begin{pmatrix} 2 & 3 \\ -3 & 8 \end{pmatrix}$ has only one eigenvalue $\lambda_1 = 5$ with algebraic multiplicity 2 and geometric multiplicity 1. Calculate one of eigenvectors $T_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. By the given above lemma there exists an associate vector $\widehat{T}$ satisfying the equation

$$(A - 5I) \cdot \widehat{T}_1 = T_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

This associate vector can be easily found: for example $\widehat{T}_1 = \begin{pmatrix} 0 \\ 1/3 \end{pmatrix}$ (the solution for $\widehat{T}_1$ is not unique). Now we construct the matrix

$$T = (T_1, \widehat{T}_1) = \begin{pmatrix} 1 & 0 \\ 1 & \frac{1}{3} \end{pmatrix}$$

and introduce Y such that $X = TY$. For Y we obtain the system

$$Y' = \begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix} Y$$

with the initial condition

$$Y(0) = \begin{pmatrix} 1 & 0 \\ 1 & 1/3 \end{pmatrix}^{-1} \cdot X(0) = \begin{pmatrix} 1 & 0 \\ -3 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}.$$

Let $Y = (y_1(t), y_2(t))$. Then

$$y_1' = 5y_1 + y_2, \quad y_2' = 5y_2, \quad y_1(0) = 1, \; y_2(0) = 6.$$

The equation $y_2' = 5y_2$ and the initial condition $y_2(0) = 6$ implies

$$y_2(t) = 6e^{5t}$$

and then $y_1' = 5y_1 + 6e^{5t}, \quad y_1(0) = 1$. This is a linear equation which can be solved by the method of variation of constant:

$$y_1(t) = C(t)e^{5t}, \quad C'(t)e^{5t} = 6e^{5t}, \; C'(t) = 6, \; C(t) = 6t + D,$$

so

$$y_1(t) = (6t + D)e^{5t}$$

and the initial condition $y_1(0) = 1$ gives $D = 1$. We obtain

$$y_1(t) = (6t + 1)e^{5t}, \quad y_2(t) = 6e^{5t}.$$

Now we return to $X(t)$:

$$X(t) = \begin{pmatrix} 1 & 0 \\ 1 & \frac{1}{3} \end{pmatrix} \cdot Y(t) = \begin{pmatrix} 1 & 0 \\ 1 & \frac{1}{3} \end{pmatrix} \cdot \begin{pmatrix} (6t+1)e^{5t} \\ 6e^{5t} \end{pmatrix} =$$

$$= e^{5t} \cdot \begin{pmatrix} 6t + 1 \\ 6t + 3 \end{pmatrix}.$$

Answer:

$$x_1(t) = e^{5t} \cdot (6t + 1), \quad x_2(t) = e^{5t} \cdot (6t + 3).$$

9. **Invariant subspaces.** A subspace $L \subset \mathbb{C}^n$ (in particular $L \subset \mathbb{R}^n$) is called invariant with respect to the system $X' = AX$ if the following holds:

if $X(t)$ is a solution such that $X(0) \in L$ then $X(t) \in L$ for any $t \in \mathbb{R}$.

Consider the case that the $n \times n$ real matrix A has n distinct complex eigenvalues (and consequently diagonalizable over $\mathbb{C}$). Divide the eigenvalues onto the following three groups:

• the eigenvalues $\lambda_1, ..., \lambda_r$ whose real part is negative (i.e. $Re(\lambda_i) < 0$). This group contains real negative eigenvalues and the couples $a \pm bi$ where $a < 0, b > 0$.

• the eigenvalues $\mu_1, ..., \mu_s$ whose real part is positive (i.e. such that $Re(\mu_i) > 0$. This group contains real positive eigenvalues and the couples $a \pm bi$ where $a > 0, b > 0$.

• the eigenvalues $\theta_1, ..., \theta_k$ whose real part is equal to 0. This group contains the zero eigenvalue and the couples $\pm bi$ where $b > 0$.

Here $r + s + k = n$.

Denote now by

$T_{\lambda_1}, T_{\lambda_2}, ..., T_{\lambda_r}$ eigenvectors corresponding to $\lambda_1, ..., \lambda_r$,

$T_{\mu_1}, T_{\mu_2}, ..., T_{\mu_s}$ eigenvector corresponding to $\mu_1, \mu_2, ..., \mu_s$,

$T_{\theta_1}, T_{\theta_2}, ..., T_{\theta_k}$ eigenvectors corresponding to $\theta_1, \theta_2, ..., \theta_k$.

Theorem. Let A be an $n \times n$ matrix with n distinct complex eigenvalues.

1. Each of the subspaces

$$L^{stable} = span\{T_{\lambda_1}, T_{\lambda_2}, ..., T_{\lambda_r}\}$$
$$L^{unstable} = span\{T_{\mu_1}, T_{\mu_2}, ..., T_{\mu_s}\}$$
$$L^{center} = span\{T_{\theta_1}, T_{\theta_2}, ..., T_{\theta_k}\}$$

is invariant with respect to the system $X' = AX$.

2. One has $\mathbb{C}^n = L^{stable} \oplus L^{unstable} \oplus L^{center}$.

3. A solution $X(t)$ of the system $X' = AX$ tends to $0 \in \mathbb{C}^n$ as $t \to +\infty$ if and only if $X(0) \in L^{stable}$.

4. A solution $X(t)$ of the system $X' = AX$ tends to $0 \in \mathbb{C}^n$ as $t \to -\infty$ if and only if $X(0) \in L^{unstable}$.

4. If $dimL^{center} \leq 3$ then a solution $X(t)$ of the system $X' = AX$ is periodic if and only if $X(0) \in L^{center}$.

Remarks.

1. The subspaces $L^{stable}, L^{unstable}, L^{center}$ are called invariant stable, unstable, center subspaces respectively.

2. In the last statement of the theorem a constant function is assumed to be periodic. The condition $dimL^{center} \leq 3$ means, in the case of n distinct complex eigenvalues, that there are no TWO couples of eigenvalues $\pm\omega_1 i, \pm\omega_2 i$. This condition holds if there is a zero eigenvalue and/or one couple of non-real complexly-conjugate eigenvalues on the imaginary axes. If $dimL^{center} \geq 4$ then the last statement of the theorem holds with "periodic" replaced by "almost periodic" (an example of an almost periodic function is $f(t) = sin(t) + sin(\sqrt{2}t)$).

3. If there are no eigenvalues to the left of the imaginary axes then $L^{stable} = \{0\}$. If there are no eigenvalues to the right of the imaginary axes then $L^{unstable} = \{0\}$. And if there are no eigenvalues on the imaginary axes (i.e. with zero real part) then $L^{center} = \{0\}$.

4. Invariant stable, unstable, and center subspaces can also be defined (with the same properties) if A is not diagonalizable over $\mathbb{C}$, but in this case the definition is more involved.

Example. Let A be a real 7×7 matrix with eigenvalue -3 and corresponding eigenvector $T_1 \in \mathbb{R}^7$, eigenvalue $-1 + 6i$ and corresponding eigenvector $T_2 \in \mathbb{C}^7$, eigenvalue $9i$ and corresponding eigenvector $T_3 \in \mathbb{C}^7$, and eigenvalue $2 + 3i$. Let $X(t)$ be the solution of the system $X' = AX$ satisfying the initial condition $X(0) = v \in \mathbb{R}^7$. Under which condition on v the solution $X(t)$ tends to $0 \in \mathbb{R}^7$ as $t \to +\infty$? Under which condition on v the solution $X(t)$ is periodic?

Solution. Since A is a real matrix, the stable invariant subspace of $\mathbb{C}^n$ is spanned by the vectors $T_1, T_2, \bar{T}_2$ and the center invariant subspace is spanned by the vectors $T_3, \bar{T}_3$. Therefore:

$$X(t) \to 0 \in \mathbb{R}^7 \iff v \in span\Big\{T_1, T_2, \bar{T}_2\Big\} = span\big\{T_1, Re(T_2), Im(T_2)\big\},$$

$$X(t) \text{ is periodic } \iff v \in span\Big\{T_3, \bar{T}_3\Big\} = span\Big\{Re(T_3), Im(T_3)\Big\}.$$