Lecture 7. Stability of equilibria of systems $X' = F(X)$

Consider an autonomous system of ODE's

$$X' = F(X), \quad X = X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \dots \\ x_n(t) \end{pmatrix}, \quad F(X) = \begin{pmatrix} f_1(x_1, ..., x_n) \\ f_2(x_1, ..., x_n) \\ \dots \\ f_n(x_1, ..., x_n) \end{pmatrix}. \tag{1}$$

We assume that $f_1, ..., f_n$ are C^∞ functions of n variables.

Example 1.

$$x_1' = sin(x_1 x_2), \quad x_2' = x_1 - x_2^2. \tag{2}$$

Definition. A point $X_0 \in \mathbb{R}^n$ is called a singular point of the system (1), or an equilibrium point of this system, if $F(X_0) = 0 \in \mathbb{R}^n$.

Like for the case $n = 1$ (studied in the beginning of the course) the singular points correspond to constant solutions: the constant vector-function $X(t) \equiv X_0$ is a solution of (1) if and only if X_0 is a singular point of this system.

Example 2. The system (2) has infinitely many singular points. They are:

$$\begin{pmatrix} (\pi k)^{2/3} \\ (\pi k)^{1/3} \end{pmatrix}, \quad k \in \mathbb{Z}.$$

Definition. A singular point $X_0 \in \mathbb{R}^n$ is called asymptotically stable if each of the following two conditions holds:

1. For any $\epsilon > 0$ there exists $\delta > 0$ such that if $v \in \mathbb{R}^n$ and $||v - X_0|| < \delta$ then (1) has a solution $X(t)$ satisfying the initial condition $X(0) = v$ defined for all $t \geq 0$ and such that $||X(t) - X_0|| < \epsilon$ for any $t \geq 0$.

2. There exists $\delta > 0$ such that if $v \in \mathbb{R}^n$ and $||v - X_0|| < \delta$ then (1) has a solution $X(t)$ satisfying the initial condition $X(0) = v$ defined for all $t \geq 0$ and such that $X(t) \to X_0$ as $t \to +\infty$.

Remarks. As I explained in the class, in general requirement 1. does not imply requirement 2. and requirement 2. does not imply requirement 1. If only requirement 1. holds then the equilibrium point X_0 is called stable by Lyapunov.

The stability of equilibria is very important for applications. How to determine if an equilibrium point X_0 is asympototically stable? In "most" (though not all) cases the answer can be given in terms of the linear approximation of (1) at X_0.

Any C^∞ vector-function $F(X)$ of n variables satisfies the following equation at any point $A \in \mathbb{R}^n$:

$$F(X) = F(A) + F'(A) \cdot (X - A) + o(||X - A||) \text{ as } X \to A.$$

Here $F'(A)$ is the Jacobi matrix:

$$F'(A) = \begin{pmatrix} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} & \dots & \frac{\partial f_1(x)}{\partial x_n} \\ \frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} & \dots & \frac{\partial f_2(x)}{\partial x_n} \\ \dots \\ \frac{\partial f_n(x)}{\partial x_1} & \frac{\partial f_n(x)}{\partial x_2} & \dots & \frac{\partial f_n(x)}{\partial x_n} \end{pmatrix} (A).$$

If $A = X_0$ is a singular point then $F(X_0) = 0$ and we have

$$F(X) = F'(X_0) \cdot (X - X_0) + o(||X - A||) \quad \text{as } X \to X_0.$$

Introducing

$$Y = Y(t) = X(t) - X_0$$

we obtain

$$Y' = F'(X_0)Y + o(||Y||) \quad \text{as } Y \to 0 \in \mathbb{R}^n.$$

Definition. The linear system $Y' = F'(X_0)Y$ is called the linearization of (1) at the singular (equilibrium) point $X_0 \in \mathbb{R}^n$.

For the linear system $X' = AX$ the question on asymptotic stability of the singular point $0 \in \mathbb{R}^n$ is simple enough: the singular point $0 \in \mathbb{R}^n$ is asymptotically stable if and only if the real part of each of the eigenvalues of A is smaller than 0 (i.e. each of the eigenvalues is to the left of the imaginary axes). See Lecture Gn where this statement follows from the theorem on stable, unstable, and center invariant spaces for the case that A is diagonalizable over $\mathbb{C}$. This statement remains true for non-diagonalizable A. Is the stability of the equilibrium of the linearized system "responsible" for the stability of the non-linear system? In "most" cases yes, but not in all cases.

Theorem (Lyapunov). Let $X_0 \in \mathbb{R}^n$ be an equilibrium point of the system (1). Consider the Jacobi matrix $F'(X_0)$.

A. If $Re(\lambda) < 0$ for ANY eigenvalue λ of the matrix $F'(X_0)$ then X_0 is asymptotically stable equilibrium point of (1).

B. If there exists AT LEAST ONE eigenvalue λ of the matrix $F'(X_0)$ such that $Re(\lambda) > 0$ then X_0 is NOT asymptotically stable point of (1).

What remains beyond cases A. and B.? The case:

C. There are no eigenvalues with positive real part AND there are eigenvalue(s) with zero real part (i.e. on the imaginary axes).

In case C. it is impossible to determine if X_0 is asymptotically stable or not. Any variant is possible and this depends on the non-linear part of the Taylor expansion of the vector-function $F(X)$ at the point X_0. How depends? This is rather involved part of the qualitative theory of dynamical systems.

Calculating the eigenvalues of an $n \times n$ matrix is not an easy task. It can be avoided if $n = 2$. A simple analysis show the following.

Proposition. Let A be a real 2×2 matrix. The case A. holds if and only if $detA > 0$ AND $traceA < 0$. The case B. holds if and only if either $detA < 0$ OR $traceA > 0$. The case C. holds if either $detA = 0$ and $traceA < 0$ or $detA > 0$ and $traceA = 0$.

Example 3 Let us determine which of the equilibria of the system (1) are asymptotically stable. The equilibria are

$$E_k = \begin{pmatrix} (\pi k)^{2/3} \\ (\pi k)^{1/3} \end{pmatrix}$$

The Jacobi (linearization) matrix at E_k are:

$$F'(E_k) = \begin{pmatrix} x_2 \cdot cos(x_1x_2) & x_1 \cdot cos(x_1x_2) \\ 1 & -2x_2 \end{pmatrix} (E_k) = \begin{pmatrix} (-1)^k \cdot (\pi k)^{1/3} & (-1)^k \cdot (\pi k)^{2/3} \\ 1 & -2(\pi k)^{1/3} \end{pmatrix}$$

Calculate

$$det(F'(E_k)) = (-1)^{k+1} \cdot 3 \cdot (\pi k)^{2/3},$$

$$trace(F'(E_k)) = (\pi k)^{1/3} \cdot ((-1)^k - 2).$$

We see that:

• if k is even non-zero number then $det(F'(E_k)) < 0$ and consequently E_k is not asymptotically stable;

• if k is odd positive number then $det(F'(E_k)) > 0,\ \ trace(F'(F_k)) < 0$ and consequently E_k is asymptotically stable;

• if k is odd negative number then $trace(F'(F_k)) > 0$ and consequently E_k is not asymptotically stable;

• if $k = 0$ then $det(F'(E_k)) = 0,\ trace(F'(F_k)) = 0$ and in this case the Lyapunov theorem does not allow to determine if $E_k = E_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is asymptotically stable; this depends on the non-linear part of the Taylor series at the point E_0.

Conclusion: the equilibrium point E_k is asymptotically stable if k is an odd positive number. It is not asymptotically stable if k is an even non-zero number or if k is an odd negative number. In the remaining case that $k = 0$ we do not know whether or not $E_k = E_0$ is asymptotically stable.