# Welcome to ODEs!

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Lecture notes for 104285. Spring Semester 2017

1. Introduction
   - What is an ODE and systems of ODEs?
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   - What does it mean to solve ODE?
   - Syllabus
   - Books
   - Organization of the course, grades, tests

2. Autonomous ODEs $x'(t) = f(x(t))$
   - Equation $x'(t) = kx(t) + a$ and its alive interpretations

# 1. Introduction

At first I will explain what is an ODE (ordinary differential equation) and a system of ODEs.

After that I will give you an idea about the syllabus (topics) we will be doing and the books (on top of the present computer-presentation) which can help.

And we will fix everything on the organization of the course (homeworks, grades, midterm (bohan), etc.) and we will start the first topic.

An ODE (ordinary differential equation) of order $k$ is an equation of the form

$$x^{(k)}(t) = F\left(t, x(t), x'(t), x''(t), ..., x^{(k-1)}(t)\right)$$

where $x(t)$ is the unknown function and $F$ is a given function of $k+1$ variables. The $x^{(i)}(t)$ denotes the derivative of $x(t)$ of order $i$. The word ordinary corresponds to the following: the unknown function is a function one variable.

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A system of $n$ ODEs of order $k$ has the same form but $x$ and $F$ vector-function: $x = (x_1, ...x_n)$, $F = (F_1, ..., F_n)$ where $x_i$ are functions of one variable $t$ and $F_i$ are functions of $1 + nk$ variables.

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A solution is a function $x(t)$ satisfying the equation or the system and defined on an open interval $(t_-, t_+)$. The case $t_- = -\infty$ and/or $t_+ = \infty$ are not excluded.

# 1.4. What does it mean to solve ODE? Important FAQs

Question 1. Is it possible to give a formula for the solution of an ODE which involves elementary functions only ($sin, cos, exp, \ldots$)?

Answer. Yes for certain class of equations (including some with alive applications) and no for most of equations (including some with alive applications).

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Question 3. Do we really need a formula for a solution, with or without integrals?

Answer. Depends on applications. In many cases we need a qualitative analysis only - points of local max and local min, $\lim_{t\to\infty} x(t)$, graph of solution, ... and in many cases the qualitative analysis can be done without any formula for a solution.

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Question 6. Assume we have a formula for solution, like $x(t) = \int_1^t e^{-s} s^2 ds$. The integral can be "taken", should we do it?

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Question 7. And if my (current) purpose is just not to lose points in tests?

Answer. For each problem of each test where the final answer is a formula, it will be written if integrals in the final answer are OK or not.

Topic 1. ODEs of the form $x'(t) = f(x(t))$. Such equations are called autonomous first order ODEs. We will start with simplest examples, including alive examples, and will end up with a complete theory. We will also discuss the application on optimization by a rigid plan versus optimization by feedback.

# 1.5. Syllabus

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Topic 4. $k$-th order ODEs of the form

$$x^{(k)}(t) = a_0 x(t) + a_1 x'(t) + \cdots + a_{k-1} x^{(k-1)}(t) + f(t)$$

where $a_i$ are constant numbers. They are called linear $k$-th order ODEs with constant coefficients. Such equations also have many applications, especially for $k = 1, 2$.

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Understanding each topic requires understanding all previous topics.

Understanding the course requires good understanding of

• main notions and theorems of infi or hedva including convergence/divergence of improper integrals

• main notions and theorems of linear algebra including eigenvalues, eigenvectors, and linearization of linear operators

# Books

M. Zhitomirskii, the present computer-exposition
go to Math Faculty - People- Faculty- Zhitomirskii -
Personal Page- Teaching- ODE 104285 Year 2017

It is beamer. Press Enter or PageDown when a pause.
Do the same to go to the next frame.

This computer exposition will be updated on the day of each lecture and will contain all lectures given by that day

Do not ask me to make it in Hebrew

M. Zhitomirskii, the computer exposition of the same course in Spring 2016
go to Math Faculty - People- Faculty- Zhitomirskii -
Personal Page- Teaching- ODE 104285 Year 2016

The order of topics in 2016 was different. In 2017 there will be few new topics; some topics will be shorter. You can use the computer exposition of 2017 to see the whole course, the bohan and the final tests with solutions.

Uri Eliash, Introduction to ODEs. Math Faculty, Technion, 2009 (Hebrew)

W.E. Boyce, R.C. DiPrima, Elementary differential equations...

These two books are mainly for students of ODE courses of lower level. They do not contain a good part of qualitative theory and examples-applications of the present course.

V.I. Arnol'd, Ordinary differential equations
masterpiece, qualitative theory only, but very difficult

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- Certainly the main purpose of homeworks is experience which you need to get a good grade on the bohan and the final test. I strongly recommend to do as many as possible exercises from the present computer presentation, preferably all.
- The midterm (bohan) will be on April 30 (Sunday) during the lecture time. Mogen 20 percent.
- The final tests: July 10 and September 27

- All tests: you can use any notes, any books, and any computers. You do not need computer for computations and/or programs on ODEs available on the web, but if you wish you can use computer for that (I doubt that it will help, but maybe). Mistake of a computer is your mistake.

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- Math questions - you can e-mail to me (mzhi@tx) provided no formulas and in English. Otherwise - on my office hours.
- Organization questions, special circumstances - you can e-mail to me provided in English. Otherwise - on my office hours.
- My office hours: Sunday and Thursday 13:30 - 14:30.

# 2. Autonomous ODEs $x'(t) = f(x(t))$

The universal physical interpretation: a body moves along a straight line (the $x$-axes) and its velocity depends on its position only: for example if the body is at the point $x = 2$ km its velocity is 4 km/hour and it it so if the body is at that point yesterday, today, or tomorrow.

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We will start (section 2.1. ) with very simple but very important equation $x'(t) = kx(t) + a$ where $k, a$ are given numbers. The solution $x(t)$ can be expressed by a simple formula. We will discuss alive interpretations of this equation.

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In sections 2.2-2.4, for the general case $x'(t) = f(x(t))$, we will formulate very important questions on existence and uniqueness and prolongation of solution satisfying a fixed initial condition. We will continue with examples that there is no uniqueness or no prolongation. We will formulate and discuss (with a help of alive examples) the existence and uniqueness theorem and a theorem on prolongation of solutions. The latter two theorems will be proved. The existence theorem will be proved at the end of the course, if time allows.

In section 2.5 we will develop a complete theory of equations $x'(t) = f((x)t))$ with any function $f(x)$ satisfying the assumption of the existence and uniqueness theorem.

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We will end this part of the course with a nice and very important application in section 2.6 showing the advantage of optimization (of any aspect of life) by the feedback versus optimization by rigid plan.

# 2.1. Equation $x'(t) = kx(t) + a$ and its alive interpretations

Let us start with the equation $x'(t) = kx(t)$. It is the equation, for example, for:

• Increase or decrease of population under assumption that each year (or each month, each week each hour) the increase/decrease is proportional to the number of people (or cats, ...) with constant coefficient. For example, if the population increases by 2 percent each year we have $x'(t) = kx(t)$ with $k = ?$

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$k = 0.02$ (provided that the time $t$ is measured in years).

• The bank gives interest for your money continuously. Assume that the interest rate (ribit) is 3 percent and you have 1000 shekels. If you get in a year 3 percent you will have 1030 shekels. If you get in 1/2 year 1.5 percent you will have in 1/2 year 1015 shekels and in 1/2 year more you will have $1015 + 0.015 \cdot 1015 = 1030 + 0.015 \cdot 15 = 1032.5 > 1030$ shekels. If they give interest every 1/4 year you will get more.

And if the bank gives interest continuously your money changes by the equation $x'(t) = 0.03x(t)$ where $t$ is the time in years. If $t_0$ is the time today (in years), in one year you will have $x(t_0 + 1)$ shekels where $x(t)$ is the solution of this equation.

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Therefore the interest rate R percent given continuously gives in one year

$$Y = \frac{x(t_0+1) - x(t_0)}{x(t_0)} \cdot 100 \quad \text{percent}$$

where $x(t)$ is the solution of the equation $x'(t) = \frac{R}{100}x(t)$.

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**Definition 2.1**

*The number $Y$ is called (in USA where most banks give the interest continuously, i.e. according to the equation above) interest yield (ribit-be-ribit).*

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Why we have infinitely many solutions defined for all $t$? In practical language it is clear: if we speak about increase of money in a bank the equation $x'(t) = kx(t)$ does not know how money we have today. If we speak about the change of population according to the same equation, it does not know how many people (cats, etc. ) are there today.

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And if we know this information, will the solution (defined for all $t$) be unique? In math terms such information is called the initial condition.

## Definition 2.2

*The initial condition for the first order ODE (in general case: $x'(t) = F(t, x(t))$ is the condition*

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Given an initial condition, is there a solution satisfying this condition? Is it defined for all $t \in \mathbb{R}$? If yes, is it unique? We will study these (very important) questions later.

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For the equation $x'(t) = kx(t)$ all answer are yes.

# Solution of the equation $x'(t) = kx(t)$

What is very easy is to see that this equation has a solution defined for all $t$ and satisfying any initial condition $x(t_0) = x_0$.

In fact, we know that $x(t) = Ce^{kt}$ is a solution for any $C \in \mathbb{R}$. Substituting $t_0$ we obtain $x_0 = Ce^{kt_0}$ which can be solved with respect to $C$. We obtain:

**Claim 2.1**

*The equation $x'(t) = kx(t)$ has the following solution defined for all $t \in \mathbb{R}$ and satisfying the initial condition $x(t_0) = x_0$:*

$$x(t) = e^{k(t-t_0)}x_0.$$

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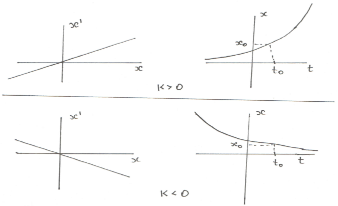
What is non-trivial and what follows from the general uniqueness theorem that we will study is:

**Claim 2.2**

*This solution is unique (within solutions defined for all $t$).*

The qualitative analysis of this solution is very simple:



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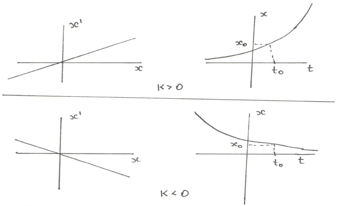


## Exercise 2.1

*The interest rate in a bank is 5 percent a year and the interest is given continuously. Compute the one year interest yield (see Definition 2.1). Does it depend on the initial deposit (how much money you have today)? Compute how much money you will have in 100 months if today you have 1000 shek.*

# Equation $x'(t) = kx(t) + a$ and its alive interpretations

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Some of alive interpretations:

• There are 10 million people in a country. If the country does not accept people from other countries its population decreases by 2 percent a year. The country accepts 10,000 people a year. We have the equation $x'(t) = -0.02x(t) + 0.01$ and the initial condition $x(t_0) = 10$ where the time $t$ is measured in years and the number of people $x(t)$ is measured in millions.

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• You have 10,000 shekels in a bank the bank gives continuous interest with the interest rate 5 percent a year, and you spent each year all what you earn plus 700 shekels. We have the equation $x'(t) = 0.05x(t) - 700$ and the initial condition $x(t_0) = 10,000$.

# Solution of the equation $x'(t) = kx(t) + a$ with the initial condition $x(t_0) = x_0$

This equation is linear with respect to $x$ and due to it it can be solved by the following method (which later will be applied in more general cases).

# Solution of the equation $x'(t) = kx(t) + a$ with the initial condition $x(t_0) = x_0$

This equation is linear with respect to $x$ and due to it it can be solved by the following method (which later will be applied in more general cases).

At first forget about the initial condition. Assume that we have a single solution $x(t) = x^*(t)$ of the equation $x'(t) = kx(t) + a$. Let $x(t) = U(t)$ be any fixed solution of the equation $x'(t) = kx(t)$ (the same equation but without $a$). Is it true that $x(t) = x^*(t) + U(t)$ is a solution of the equation $x'(t) = kx(t) + a$?

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It is very easy to see that yes. Certainly it is not so for any ODEs but for those which are linear with respect to $x$ it is so, like in linear algebra.

## Solution of the equation $x'(t) = kx(t) + a$ with the initial condition $x(t_0) = x_0$

Furthermore, like in linear algebra, it is easy to see that any solution $x(t)$ of the equation $x'(t) = kx(t) + a$ is the sum of any fixed solution $x^*(t)$ of this equation and a solution $x(t) = U(t)$ of the equation $x'(t) = kx(t)$.

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We know how to solve the equation $x'(t) = kx(t)$ - it has solutions $Ce^{kt}, C \in \mathbb{R}$. It does not have other solutions (we have not proved it yet). Therefore all the solutions of the equation $x'(t) = kx(t) + a$ have the form $x(t) = x^*(t) + Ce^{kt}, C \in \mathbb{R}$ where $x^*(t)$ is any fixed solution.

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How to find a single solution $x^*(t)$? In general it might be not immediate (later we will learn a certain method for that, for linear with respect to $x$ equations), but for our equation it is very simple:

the equation $x'(t) = kx(t) + a$ with $k \neq 0$ has a constant solution $x(t) \equiv -\frac{a}{k}$.

# Solution of the equation $x'(t) = kx(t) + a$ with the initial condition $x(t_0) = x_0$

We obtain:

**Claim 2.3**

*The general solution of the equation $x'(t) = kx(t) + a$ (i.e. the set of all solutions) defined for all $t \in \mathbb{R}$ is*

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Now we can find $C$ to satisfy the initial condition $x(t_0) = x_0$. Substituting $t_0$ to the general solution we obtain

$$C = (x_0 + \frac{a}{k})e^{-kt_0}$$

It follows:

## Claim 2.4

*The equation $x'(t) = kx(t) + a$ with $k \neq 0$ has a solution defined for all $t \in \mathbb{R}$ and satisfying the initial condition $x(t_0) = x_0$:*

$$x(t) = -\frac{a}{k} + \left(x_0 + \frac{a}{k}\right) e^{k(t-t_0)}.$$

*This solution is unique.*

Like for any ODE the qualitative analysis is necessary!

Differentiate the solution $x(t)$. We have

$$x'(t) = k(x_0 + \frac{a}{k})e^{k(t-t_0)}.$$

We see that if $k(x_0 + \frac{a}{k}) > 0$ the solution is an increasing function and if $k(x_0 + \frac{a}{k}) < 0$ the solution is a decreasing function. The limits of $x(t)$ as $t \to \pm\infty$ can be easily found.

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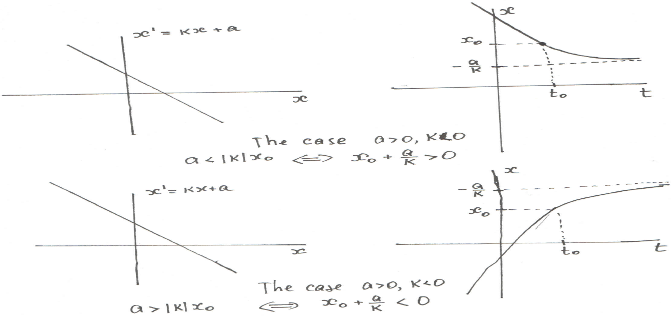
For example for $a > 0, k > 0$ we obtain one of the following graphs depending on the sign of the number $x_0 + \frac{a}{k}$:

The solution of the equation $x'(t) = kx(t) + a$ with $k < 0, a > 0$ with the initial condition $x(t_0) = x_0$ increases if $a > |k|x_0$ and decreased if $a < |k|x_0$:



For example, if a country has $x_0 = 10$ million people and without accepting people from other countries its population decreases by 2 percent a year, on order that the population increases the country should accept $a$ people a year with $a$ such that $10^7 + \frac{a}{-0.02} < 0$, i.e. $a > 200,000$. Note that the bigger is $x_0$ the more people should be accepted.

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## Exercise 2.2

*1. Draw the graph of the solution of the equation $x'(t) = kx(t) + a$ with $k > 0$ and $a < 0$.*

*2. A bank gives continuous interest with the interest rate 5 percent a year. You spent each year all what you earn plus 5000 shekels. How much money should you have in the bank today on order that you are able to live in this way all your life?*