Math 106A. Fall 2008. M. Zhitomirskii

Straight line phase curves and equilibrium points for linear autonomous systems of first order ODEs with constant coefficients

A linear autonomous systems of first order ODEs with constant coefficients is a system of the form $X' = AX$, where $X = \begin{pmatrix} X_1(t) \\ ... \\ X_n(t) \end{pmatrix}$ is the unknown vector-function, and A is a constant $n \times n$ matrix.

In what follows we work with **real** matrix A.

Definition. A straight line phase curve (= solution curve) is a phase curve in $\mathbb{R}^n$ which is a PART OF a straight line passing through $0 \in \mathbb{R}^n$, but not a single point.

Recall that if a phase curve is a single point then this point is an equilibrium point.

Remark. This definition excludes phase curves contained in a straight line that does not pass through the origin $0 \in \mathbb{R}^n$. In fact, in "most" cases such phase curves do not exist. Later on we will return to this question and will explain what is "most", at least for $n = 2$.

Definition. A straight line solution is a solution $\begin{pmatrix} X_1(t) \\ ... \\ X_n(t) \end{pmatrix}$ such that the corresponding phase curve is a straight line phase curve.

Definition. The the ray generated by a vector $v \in \mathbb{R}^n$ is the set of points $\{rv, r > 0\} \subset R^n$. The line generated (or spanned) by v is the set of points $\{rv, r \in \mathbb{R}\} \subset R^n$. See fig. 1.

Example 1. In the class I explained that for the system

$$X' = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} X$$

there are exactly four straight line phase curves:

1. The ray generated by the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Corresponding straight line solution: $\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}$

2. The ray generated by the vector $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$. Corresponding straight line solution: $\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} -e^{2t} \\ 0 \end{pmatrix}$

3. The ray generated by the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Corresponding straight line solution: $\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ e^{-3t} \end{pmatrix}$

4. The ray generated by the vector $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$. Corresponding straight line solution: $\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ -e^{-3t} \end{pmatrix}$.

The following theorem gives a complete description of all straight line phase curves for any system $X' = AX$. It also gives the orientation of the straight line phase curves.

Theorem 1.

1. Let $v \in \mathbb{R}^n$ be a real eigenvector of an $n \times n$ matrix A corresponding to a **nonzero** real eigenvalue λ. Then the ray generated by v is a straight line phase curve for the system $X' = AX$. This ray is oriented as follows: to $0 \in \mathbb{R}^n$ if $\lambda < 0$ and from $0 \in \mathbb{R}^n$ if $\lambda > 0$. One of corresponding straight line solutions is the vector function $X(t) = e^{\lambda t} \cdot v$.

2. Any straight line solution for the system $X' = AX$, where A is an $n \times n$ matrix, is the ray generated by a real eigenvector of A corresponding to a nonzero real eigenvalue.

In the class I proved the first statement and left the second statement as a homework.

Remarks. Two proportional eigenvectors generate the same line, but not necessarily the same ray. They generate the same ray if and only if one of them can be obtained from the other by multiplication by a **positive** number. Therefore as soon as we know an eigenvector v we know TWO straight line solutions: the rays generated by v and by $-v$.

Example 2. For the matrix A in Example 1 we have two real eigenvectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ corresponding to the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -3$. Any other eigenvector is proportional to one of these two eigenvectors. Therefore we have exactly four oriented straight line phase curves: the rays at fig.2.

Example 3. In the class we discussed the system of ODEs describing a fight between two armies:

$$X' = AX, \quad A = \begin{pmatrix} 0 & -kw_2 \\ -kw_1 & 0 \end{pmatrix}$$

Here $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, $x_1(t)$ is the number of soldiers in the first army, $x_2(t)$ is the number of soldiers in the second army, and w_1 and w_2 in the number of weapon in the first and the second army which is assumed to be constant during the fight (this is of course not realistic, at least in now-days). Computing the eigenvalues and the eigenvectors of the matrix A we obtain:

eigenvalue $\lambda_1 = k\sqrt{w_1 w_2}$, corresponding eigenvector $\begin{pmatrix} \sqrt{w_2} \\ -\sqrt{w_1} \end{pmatrix}$

eigenvalue $\lambda_2 = -k\sqrt{w_1 w_2}$, corresponding eigenvector $\begin{pmatrix} \sqrt{w_2} \\ \sqrt{w_1} \end{pmatrix}$

Therefore the straight line phase curves are the four rays showed at fig.3. The most important is the ray in the first quarter (the other three have no sense for our problem since the number of soldiers cannot be negative). The fact that $x_1(t)$ and $x_2(t)$ are decreasing functions imply the phase portrait within the first quarter showed at fig. 4. According to this phase portrait, we see that the second army wins if and only the point $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$ (corresponding to the beginning of the fight, $t = 0$) lies above the line $x_2 = \sqrt{\frac{w_1}{w_2}} x_1$. This means that $\frac{x_2(0)}{x_1(0)} > \sqrt{\frac{w_1}{w_2}}$. For example, if the first army has nine times more weapon than the second army, then the second army wins if and only if at the beginning of the fight its number of soldiers is more than three times the number of soldiers in the first army. We see that within our model of a fight the number of soldiers is much more important than the number of weapon.

Equilibrium points. Any system $X' = AX$ has the equilibrium point $0 \in \mathbb{R}^n$. Is this equilibrium point unique? This question can be put in the following equivalent form: is it true that the system of algebraic linear equations $AX = 0$ with respect to the vector $X \in \mathbb{R}^n$ has solution $v = 0$ ONLY? You know the answer from liner algebra course: this is so if and only if $detA \neq 0$. The latter condition is equivalent to the condition that $\lambda = 0$ is not an eigenvalue of A. Another equivalent condition: $rankA < n$ (assuming that A is an $n \times n$ matrix). Thus:

Proposition 1. Let A be an $n \times n$ matrix. The following conditions are all equivalent:

(i) the origin $0 \in \mathbb{R}^n$ is unique equilibrium point

(ii) $detA \neq 0$

(iii) $rankA < n$

(iv) $\lambda = 0$ is not an eigenvalue of A.

Assume that $detA = 0$. Then there are equilibrium points except $0 \in \mathbb{R}^n$. The corresponding vectors in $\mathbb{R}^n$ are solutions of the equation $AX = 0$. Equivalently, they are eigenvectors corresponding to the zero eigenvalue. We obtain:

Theorem 2. If $detA \neq 0$ then $0 \in \mathbb{R}^n$ is the only equilibrium point of the system $X' = AX$. If $detA = 0$ then any eigenvector of A corresponding to the zero eigenvalues generates the line of equilibrium points.

Example 4. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. One has $detA = 0$, therefore there is at least one line of equilibrium points. In fact there is exactly one line of equilibrium points: all eigenvectors corresponding to the zero eigenvalue are proportional to $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and the line of equilibrium points is generated by this vector. See fig. 5. Except the zero eigenvalue, A also has eigenvalue $\lambda = 2$. The corresponding eigenvector is $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Therefore there are two straight line solutions: the rays generated by v and $-v$. See fig. 5.