Math 106A. Fall 2008. M. Zhitomirskii

LN2: The vector space of all solutions of the system $X' = AX$

Fix an $n \times n$ real matrix A. The set of all solutions of the system $X' = AX$ is a subspace of the vector space of all smooth vector-functions. This follows from the linearity properties: the sum of any two solutions is a solution, and multiplying any solution by a number we also get a solution. The vector space of all vector functions is infinite-dimensional. But its subspace consisting of all solutions of the system $X' = AX$, with a fixed matrix A, is finite-dimensional.

Theorem 1. Fix a real $n \times n$ real matrix A. The space of all solutions of the system $X' = AX$ is a finite dimensional vector space over $\mathbb{R}$. Its dimension is equal to n.

In the class I proved this theorem, using the existence and uniqueness theorems for solutions of ODEs.

From linear algebra you know that:

if V is a vector space of dimension n and $v_1, ..., v_n \in V$ are n linearly independent vectors then $\{v_1, ..., v_n\}$ is a basis of V.

This statement and Theorem 1 imply:

Theorem 2 (corollary of Theorem 1). Fix a real $n \times n$ real matrix A. Assume that the vector functions $X^{(1)}(t), ..., X^{(n)}(t)$ are linearly independent solutions of the system $X' = AX$. Then $\{X^{(1)}(t), ..., X^{(n)}(t)\}$ is a basis of the vector space of all solutions of this system.

By Theorem 2, to present a basis of the vector space of all solutions it suffices to find n linearly independent solutions. Is it possible to do this by explicit formulas? It is. For some A it is a difficult task, for some A it is much easier. The simplest, but already very important case, is the case that A has n distinct eigenvalues. We consider this case in LN3.