Math 106A. Fall 2008. M. Zhitomirskii

LN3: The case of n real eigenvalues:
solving the system $X' = AX$;
the phase portraits in the two-dimensional case.

Assume that an $n \times n$ matrix A has n distinct real eigenvalues $\lambda_1, ...\lambda_n$. Then we know (see LN1) n solutions of the system $X' = AX$: the vector functions

$$X^{(1)}(t) = e^{\lambda_1 t} v_1, \ ..., \ X^{(n)}(t) = e^{\lambda_n t} v_n, \tag{1}$$

where v_i is an eigenvector corresponding to λ_i.

Theorem 1. If $\lambda_1, ..., \lambda_n$ are real distinct eigenvalues then the vector functions (1) are linearly independent over $\mathbb{R}$.

Proof. Assume that the linear combinations of the vector functions (1) with real coefficients $C_1, ..., C_n$ is the zero vector in the space of all functions, i.e.

$$C_1 X^{(1)}(t) + \cdots + C_n X^{(n)}(t) \equiv 0.$$

We have to show that $C_1 = \cdots = C_n = 0$. Substitute $t = 0$. We obtain

$$C_1 v_1 + \cdots + C_n v_n = 0. \tag{2}$$

Now we use the following fact from linear algebra:

If an $n \times n$ matrix A has n distinct eigenvalues $\lambda_1, ..., \lambda_n$ then the corresponding eigenvectors $v_1, ..., v_n$ are linearly independent.

This fact and (2) imply $C_1 = \cdots = C_n = 0$ and we are done.

Theorem 1 and Theorem 2 of LN-2 imply:

Theorem 2. (corollary of Th. 1 and Th. 2 of LN-2) Assume that an $n \times n$ matrix A has n real distinct eigenvalues $\lambda_1, ..., \lambda_n$. Let $v_1, ..., v_n$ be corresponding eigenvectors. Then

$$\left\{ e^{\lambda_1 t} v_1, \ ..., \ e^{\lambda_n t} v_n \right\} \tag{1}$$

is a basis of the space of all solutions of the system $X' = AX$.

Theorem 2 gives a simple way to find the solution of the system $X' = AX$ satisfying any initial condition, provided that we are within the case of n real eigenvalues. In fact, by Theorem 2 any solution has the form

$$X(t) = C_1 e^{\lambda_1 t} v_1 + ... + C_n e^{\lambda_n t} v_n,$$

where $C_1, ..., C_n$ are certain numbers. These numbers depend on the initial conditions and are uniquely determined by the initial conditions. To see this, fix the initial condition $X(0) = X_0$, where X_0 is a certain vector in $\mathbb{R}^n$. Substituting $t = 0$ we obtain $C_1 v_1 + \cdots + C_n v_n = X_0$. It is a system of n linear algebraic equations with respect to n unknowns $C_1, ..., C_n$. This system has a unique solution because the eigenvectors $v_1, ..., v_n$ are linearly independent and consequently the coefficient matrix of this system is non-singular (its determinant is not 0). **Examples: in the class, in the homeworks**.

Theorem 2 in combination with our analysis of the straight line phase curves (LN-1) and with additional arguments explained in the class allows to draw the oriented phase curve for any system $X' = AX$ where A is a 2×2 matrix with two real eigenvalues $\lambda_1 \neq \lambda_2$. If one of the eigenvalues is positive and the other is negative the phase portrait is called **saddle**. If each of the eigenvalues is negative the phase portrait is called sink, and If each of the eigenvalues is positive the phase portrait is called source. Principally different phase portrait holds if one of the eigenvalues is 0.

All phase portraits are shown and explained in fig. 1-3.