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LN4: The case of n eigenvalues, not all are real, PART 1

Consider a system $X' = AX$ where A is an $n \times n$ real matrix such that some (or all) its eigenvalues are not real. How to find a basis of the space of all solutions in this case?

In this case one should work over complex numbers $\mathbb{C}$ rather than on real numbers $\mathbb{R}$, even though the final answer contains no complex numbers.

Definition. A complex-valued function of a real variable t is a function of the form $z(t) = A(t) + iB(t)$. Here $i = \sqrt{-1}$. The function $A(t)$ is the real part of $z(t) : A(t) = Re(z(t))$. The function $B(t)$ is the imaginary part of $z(t)$: $B(t) = Im(z(t))$. The derivative of $z(t)$ is the complex-valued function $A'(t) + iB'(t)$.

In a natural way we can define the sum, difference, ratio, and multiplication of complex valued functions. We can multiply a complex-valued function by a complex number.

A complex-valued vector function $X(t)$ is a solution of the system $X' = AX$ if it satisfies this equation. The set of all complex-valued solutions, when A is fixed, is closed with respect to the sum and with respect to the multiplication by complex numbers. Therefore the set of all solutions is a vector space over $\mathbb{C}$.

The idea of solving the system $X' = AX$ in the case that some of the eigenvalues of the real $n \times n$ matrix A are not real is as follows: at first we find n linearly independent complex valued solutions, then we transfer them to n linearly independent real-valued solutions. As we know, any n linearly independent real-valued solutions form a basis of the vector space of all real-valued solutions.

Definition Given a complex number $\lambda = a + bi \in \mathbb{C}$ introduce the following (EXTREMELY IMPORTANT) complex-valued function

$$(1) \qquad e^{\lambda t} = e^{at}\big(cos(bt) + i \cdot sin(bt)\big)$$

In what follows in this course we will give another definition, then (1) will be a beautiful theorem. Till that (1) will be the definition of $e^{\lambda t}$.

Theorem 1. For any complex numbers $\lambda, \lambda_1, \lambda_2$ one has

$$e^{\lambda_1 t} \cdot e^{\lambda_2 t} = e^{(\lambda_1+\lambda_2)t}$$

$$(e^{\lambda t})' = \lambda e^{\lambda t}$$

We checked these equations in the class.

The second equation in Theorem 1 implies the following statement:

Theorem 2 (simple corollary of Theorem 1). If λ is complex eigenvalue of A and $v \in \mathbb{C}^n$ is a corresponding eigenvalue then the complex-valued vector function $X(t) = e^{\lambda t} \cdot v$ is a solution of the system $X' = AX$.

In the same way as in the case that all eigenvalues are real we prove:

Theorem 3. If a complex (in particular real) $n \times n$ matrix A has n distinct complex eigenvalues $\lambda_1, ..., \lambda_n \in \mathbb{C}$ and $v_1, ..., v_n \in \mathbb{C}^n$ are corresponding eigenvectors then the complex valued functions

$$e^{\lambda_1 t} \cdot v_1, \ \dots \ , e^{\lambda_n t} \cdot v_n$$

are linearly independent.

Important remark. Saying that A has n complex eigenvalues does not exclude the case that some or even all eigenvalues are real because $\mathbb{R}$ is a part of $\mathbb{C}$.

Recall from linear algebra that if λ is a non-real eigenvalue of a real matrix A then $\bar{\lambda}$ is also an eigenvalue of A. Furthemore, if $v \in \mathbb{C}^n$ is an eigenvector corresponding to λ then $\bar{v}$ is an eigenvector corresponding to $\bar{\lambda}$. Therefore the eigenvalues and the corresponding eigenvectors can be arranges as follows:

eigenvalues	corresponding eigenvectors
$\lambda_1, ..., \lambda_s \in \mathbb{R}$	$v_1, ..., v_s \in \mathbb{R}^n$
$\mu_1, \bar{\mu}_1, ..., \mu_q, \bar{\mu}_q \in \mathbb{C} - \mathbb{R}$	$w_1, \bar{w}_1, ..., w_q, \bar{w}_q \in \mathbb{C}^n - \mathbb{R}^n$

Here s is the number of real eigenvalues and $2q$ is the number of non-real eigenvalues, therefore $s + 2q = n$. The case $s = 0$ means that there are no real eigenvalues. The case $q = 0$ means that all the eigenvalues are real.

By Theorems 2 and 3 the complex-valued vector functions

$$
\begin{aligned}
(1) \qquad & e^{\lambda_1 t}v_1, ..., e^{\lambda_s t}v_s, \\
& e^{\mu_1 t}w_1, e^{\bar{\mu}_1 t}\bar{w}_1, ..., e^{\mu_q t}w_q, e^{\bar{\mu}_q t}\bar{w}_q
\end{aligned}
$$

are n linearly independent complex-valued solutions of the system $X' = AX$.

Since the set of all complex-valued solutions is a vector space over $\mathbb{C}$, the n vector functions

$$
\begin{aligned}
(2) \qquad & e^{\lambda_1 t}v_1, ..., e^{\lambda_s t}v_s, \\
& \frac{e^{\mu_1 t}w_1 + e^{\bar{\mu}_1 t}\bar{w}_1}{2}, \ \frac{e^{\mu_1 t}w_1 - e^{\bar{\mu}_1 t}\bar{w}_1}{2i}, \\
& ..., \\
& \frac{e^{\mu_q t}w_q + e^{\bar{\mu}_q t}\bar{w}_q}{2}, \ \frac{e^{\mu_q t}w_q - e^{\bar{\mu}_q t}\bar{w}_q}{2i}
\end{aligned}
$$

are also solutions of the system $X' = AX$. Now we use the following very simple statement from linear algebra.

Lemma. Let V be a vector space over $\mathbb{C}$ and let

$$a_1, ..., a_p, b_1, b_2 \in V$$

be linearly independent vectors. Then the vectors

$$a_1, ..., a_p, (b_1 + b_2)/2, (b_1 - b_2)/2i$$

are also linearly independent.

By this lemma the linear independence over $\mathbb{C}$ of solutions (1) implies the linear independence over $\mathbb{C}$ of solutions (2). Note now that (2) are real-valued solutions. In fact, they have the form

$$
\begin{aligned}
(3) \qquad & e^{\lambda_1 t}v_1, ..., e^{\lambda_s t}v_s, \\
& Re\Big(e^{\mu_1 t}w_1\Big), \ Im\Big(e^{\mu_1 t}w_1\Big), \ ... \ , Re\Big(e^{\mu_q t}w_q\Big), \ Im\Big(e^{\mu_q t}w_q\Big)
\end{aligned}
$$

The linear independence over $\mathbb{C}$ implies the linear independence over $\mathbb{R}$. Thus (3) are linearly independent over $\mathbb{R}$ real-valued solutions. There are n solutions in (3). Since the space of all real-valued solutions is n-dimensional, (3) is a basis of this space. Knowing a basis we can solve the system, i.e. to find the set of all solutions and to find the solution satisfying any given initial condition.

Example 1. Let A be a real 6×6 matrix with the eigenvalues

$$\lambda_1 = 1,\ \lambda_2 = -2, \lambda_3 = -3 + 4i,\ \lambda_4 = 5i$$

and the corresponding eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 4 \\ 7 \\ 0 \end{pmatrix},\quad v_2 = \begin{pmatrix} 3 \\ 5 \\ 0 \\ 0 \\ -4 \\ 1 \end{pmatrix},\quad v_3 = \begin{pmatrix} 8+2i \\ 7-3i \\ 0 \\ 5i \\ 1 \\ 6+i \end{pmatrix},\quad v_4 = \begin{pmatrix} 1 \\ 4+2i \\ 1+i \\ 0 \\ 3+5i \\ i \end{pmatrix}$$

Let us find the solution of the system $X' = AX$ satisfying the initial condition

$$X(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We know that there are also eigenvalues $\bar{\lambda}_3 = -3 - 4i$ and $\bar{\lambda}_4 = -5i$. Therefore there are $n = 6$ distinct eigenvalues and we can apply the construction of a basis of the space of all solutions which is given above. An example of a basis of the space of all real valued solutions is given by the vector functions

$$e^{\lambda_1 t v_1},\ e^{\lambda_2 t} v_2,\ Re(e^{\lambda_3 t} v_3),\ Im(e^{\lambda_3 t} v_3),\ Re(e^{\lambda_4 t} v_4),\ Im(e^{\lambda_4 t} v_4)$$

Compute

$$Re(e^{\lambda_3 t} v_3) = e^{-3t} Re\left(cos(4t) + i \cdot sin(4t)) \begin{pmatrix} 8+2i \\ 7-3i \\ 0 \\ 5i \\ 1 \\ 6+i \end{pmatrix} \right) =$$

$$e^{-3t} \begin{pmatrix} 8cos(4t) - 2sin(4t) \\ 7cos(4t) + 3sin(4t) \\ 0 \\ -5sin(4t) \\ cos(4t) \\ 6cos(4t) - sin(4t) \end{pmatrix} \tag{4}$$

$$Im(e^{\lambda_3 t}v_3) = e^{-3t} Im\left(cos(4t) + i \cdot sin(4t)) \begin{pmatrix} 8+2i \\ 7-3i \\ 0 \\ 5i \\ 1 \\ 6+i \end{pmatrix} \right) =$$

$$e^{-3t} \begin{pmatrix} 2cos(4t) + 8sin(4t) \\ -3cos(4t) + 7sin(4t) \\ 0 \\ 5cos(4t) \\ sin(4t) \\ cos(4t) + 6sin(4t) \end{pmatrix} \tag{5}$$

$$Re(e^{\lambda_4 t}v_4) = Re\left(cos(5t) + i \cdot sin(5t)) \begin{pmatrix} 1 \\ 4+2i \\ 1+i \\ 0 \\ 3+5i \\ i \end{pmatrix} \right) =$$

$$= \begin{pmatrix} cos(5t) \\ 4cos(5t) - 2sin(5t) \\ cos(5t) - sin(5t) \\ 0 \\ 3cos(5t) - 5sin(5t) \\ -sin(5t) \end{pmatrix} \tag{6}$$

$$Im(e^{\lambda_4 t}v_4) = Im\left(cos(5t) + i \cdot sin(5t)) \begin{pmatrix} 1 \\ 4+2i \\ 1+i \\ 0 \\ 3+5i \\ i \end{pmatrix} \right) =$$

$$= \begin{pmatrix} sin(5t) \\ 2cos(5t) + 4sin(5t) \\ cos(5t) + sin(5t) \\ 0 \\ 5cos(5t) + 3sin(5t) \\ cos(5t) \end{pmatrix} \tag{7}$$

Thus we obtained a basis

$$e^t v_1,\ e^{-2t}v_2,\ (4),\ (5),\ (6),\ (7)$$

Any solution has the form

(8) $X(t) = C_1 e^t v_1 + C_1 e^{-2t} v_2 + C_3 \cdot (4) + C_4 \cdot (5) + C_5 \cdot (6) + C_6 \cdot (7)$

To find solution satisfying the given initial condition, substitute $t = 0$. We obtain a system of linear algebraic equation for $C_1, .., C_6$:

$$C_1 \begin{pmatrix} 1 \\ 0 \\ 2 \\ 4 \\ 7 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 3 \\ 5 \\ 0 \\ 0 \\ -4 \\ 1 \end{pmatrix} + C_3 \begin{pmatrix} 8 \\ 7 \\ 0 \\ 0 \\ 1 \\ 6 \end{pmatrix} + C_4 \begin{pmatrix} 2 \\ -3 \\ 0 \\ 5 \\ 0 \\ 1 \end{pmatrix} + C_5 \begin{pmatrix} 1 \\ 4 \\ 1 \\ 0 \\ 3 \\ 0 \end{pmatrix} + C_6 \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving this system we obtain unique solution $C_1, ..., C_6$.

Example 2. Let A be a 6×6 matrix from the previous example. Let us find condition on the vector $X_0 \in \mathbb{R}^n$ under which the solution $X(t)$ of the system $X' = AX$ satisfying the initial condition $X(0) = X_0$

(a) tends to 0 if $t \to \infty$

(b) tends to 0 if $t \to -\infty$

(c) is periodic

We know that any solution has form (8). If $t \to \infty$ then $e^{-t} \to 0$, $(4) \to 0$, $(5) \to 0$ and the other functions in (8) do not tend to 0. Therefore the solution tends to 0 as $t \to \infty$ if $C_2 = C_5 = C_6 = 0$. This means that

$$X_0 \in span\Big\{v_1, Re(v_3), Im(v_3)\Big\} = span\{v_1, v_3, \bar{v}_3\}$$

.

If $t \to -\infty$ then $e^{2t} \to 0$ and the other functions in (8) do not tend to 0. Therefore the solution tends to 0 as $t \to \infty$ if $C_1 = C_3 = C_4 = C_5 = C_6 = 0$. This means that $X_0 \in span\{v_2\}$.

The functions (6) and (7) are periodic with the same period. The other functions in (8) are not periodic. Therefore the solution is periodic if $C_1 = C_2 = C_3 = C_4 = 0$. This means that

$$X_0 \in span\Big\{Re(v_4), Im(v_4)\Big\} = span\{v_4, \bar{v}_4\}$$

One can prove that the obtained conditions on X_0 are not only necessary, but also sufficient.

The following theorem generalizes this example.

Theorem 4. Assume that an $n \times n$ matrix A has n distinct complex eigenvalues. Let $v_1, ..., v_s$ be eigenvectors corresponding to the eigenvalues located in the left part of the complex plane (i.e. real negative eigenvalues and non-real eigenvalues with negative real part). Let $u_1, ..., u_p$ be eigenvectors corresponding to the eigenvalues located in the right part of the complex plane (i.e. real positive eigenvalues and non-real eigenvalues with positive real part). Let $w_1, ..., w_q$ be the eigenvectors corresponding to to the eigenvalues located on the imaginary axes of the complex plane (i.e. pure imaginary eigenvalues and the zero eigenvalue). Let $X(t)$ be the solution of the system $X' = AX$ satisfying the initial condition $X(0) = X_0$. The following statement hold:

1. $X(t) \to 0$ as $t \to \infty$ if and only if $X_0 \in span\{v_1, ..., v_s\}$
2. $X(t) \to 0$ as $t \to -\infty$ if and only if $X_0 \in span\{u_1, ..., u_p\}$
3. If $X(t)$ is periodic then $X_0 \in span\{w_1, .., w_q\}$. For $q = 2$ or $q = 3$ then if can be replaced by if and only if.

Remark. The case $q = 2$ means that there is a couple of pure imaginary eigenvalues $\pm\beta i, \beta \neq 0$ and there is no zero eigenvalue. The case $q = 3$ means that there is a couple of pure imaginary eigenvalues and the zero eigenvalue. If $q \geq 4$ and $X_0 \in span\{w_1, ..., w_q\}$ then the solution might be not periodic, but it is always "almost periodic" (explained in the class).