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LN5: The case of $< n$ complex eigenvalues (repeated eigenvalues)

In this case we need a few more statements and definitions from linear algebra.

At first recall the principal theorem of algebra. Its complete formulation is as follows.

Theorem 1. Any polynomial $P(x)$ of degree $n \geq 1$ has at least one complex root and not more than n complex roots. Let s be the number of complex roots and let $x_1, ..., x_s$ be these complex roots. Then $P(x)$ can be written in the form (factorization)

$$P(x) = a \cdot (x - x_1)^{\mu_1}(x - x_2)^{\mu_2} \cdot \cdots \cdot (x - x_s)^{\mu_s}$$

where $\mu_1, ..., \mu_s$ are integers ≥ 1 whose sum is equal to n and a is the coefficient of $P(x)$ at x^n.

Definition. The number μ_i is called the multiplicity of the root x_i.

Recall that a complex number is an eigenvalue of an $n \times n$ matrix A if it is a root of the characteristic polynomial $P(\lambda) = det(A - \lambda n)$. The characteristic polynomial has degree n. Its coefficient at λ^n is equal to 1. Therefore

$$P(\lambda) = (\lambda - \lambda_1)^{\mu_1}(\lambda - \lambda_2)^{\mu_2} \cdot \cdots \cdot (\lambda - \lambda_s)^{\mu_s} \quad (1)$$

where $\lambda_1, ..., \lambda_s$ are (distinct) eigenvalues of the matrix A.

Definition. The multiplicity of an eigenvalue λ_i, also called the algebraic multiplicity of λ_i, is the number μ_i in (1), i.e. the multiplicity of the root λ_i of the characteristic polynomial.

The sum of the multiplicities of the eigenvalues of an $n \times n$ matrix A is equal to n:

$$\mu_1 + \cdots + \mu_s = n$$

. It follows that if A has n distinct eigenvalues (i.e. $s = n$) then $\mu_1 = \cdots = \mu_s = 1$. If A has less than n eigenvalues then the multiplicity of at least one of the eigenvalues is ≥ 2.

Example. For a 2×2 matrix A one of the following holds:

a. A has two eigenvalues, each of algebraic multiplicity 1

b. A has one eigenvalue of algebraic multiplicity 2.

Example. For a 3×3 matrix A one of the following holds:

a. A has three eigenvalues, each of algebraic multiplicity 1

b. A has two eigenvalues, one of algebraic multiplicity 1, the other of algebraic multiplicity 2

c. A has one eigenvalue of algebraic multiplicity 3.

The terminology "algebraic multiplicity" suggests that there is another, "geometric multiplicity".

Definition. The geometric multiplicity of an eigenvalue λ_i of an $n \times n$ matrix A is the number

$$n - rank(A - \lambda_i I)$$

also called the corank of the matrix $A - \lambda_i I$.

Since $det(A - \lambda_i I) = 0$ then $rank(A - \lambda_i I) < n$ and consequently *the geometric multiplicity of any eigenvalue is an integer* ≥ 1. Since the minimal possible value for the rank of any $n \times n$ matrix A is 0 (the rank is equal to zero if and only if $A = 0$), *the geometric multiplicity of any eigenvalue of an* $n \times n$ *matrix does not exceed* n.

Is any integer from 1 to n realizable? More precisely, fix an integer $q \in \{1, ..., n\}$. Can we construct an $n \times n$ matrix such that one of its eigenvalue has geometric multiplicity q? The answer is yes. Nevertheless, if we fix the algebraic multiplicity the answer is no. Namely, the following theorem holds.

Theorem 2. The geometric multiplicity of any eigenvalue of any $n \times n$ matrix does not exceed the algebraic multiplicity of this eigenvalue.

Example. Let A be a 2×2 matrix with only one eigenvalue λ_1. Then the algebraic multiplicity of λ_1 is equal to 2. The geometric multiplicity of λ_1 is either 1 or 2. It is equal to 2 if and only if
(2)

$$rank(A - \lambda_1 I) = 0 \;\Leftrightarrow\; A - \lambda_1 I = 0 \;\Leftrightarrow A = \lambda_1 I \;\Leftrightarrow A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}.$$

How to construct a 2×2 matrix with the only eigenvalue of geometric multiplicity 1? It is easy. One should use the following theorem.

Theorem 3. If $\lambda_1, ..., \lambda_s$ are all the eigenvalues of an $n \times n$ matrix A, with algebraic multiplicities $\mu_1, ..., \mu_s$ then

$$traceA = \mu_1 \lambda_1 + \cdots + \mu_s \lambda_s \;\; \text{(the “sum with multiplicities”)}$$

$$detA = \lambda_1^{\mu_1} \cdot \cdots \cdot \lambda_s^{\mu_s} \;\; \text{(the “product with multiplicities”)}$$

Recall that the couple (trace, det) defines uniquely the eigenvalues of any 2×2 matrix (but not a $n \times n$ matrix with $n \geq 3$). Therefore one has the following

Corollary. A 2×2 matrix A has only one eigenvalue λ_1 if and only if $traceA = 2\lambda_1$ and $detA = \lambda_1^2$. If this is so, the geometric multiplicity of λ_1 is equal to 1 provided that A is not diagonal (i.e. not of form (2)). If A has form (2) then the geometric multiplicity of λ_1 is equal to 2.

Example. To construct a 2×2 matrix with the only eigenvalue 3 of geometric multiplicity 1 take any non-diagonal matrix with trace 6 and determinant 9, for example $\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$.

Example Let us find the algebraic and geometric multiplicities of the matrix

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & 7 \end{pmatrix}$$

The characteristic polynomial is $P(\lambda) = (\lambda - a)(\lambda - d)(\lambda - 7)$. There are the following cases:

Case 1. $a \neq d$, $a \neq 7$, $d \neq 7$. In these case we have 3 eigenvalues $a, d, 7$ and we know (without computing) that each of them has algebraic multiplicity 1 and geometric multiplicity 1.

Case 2. $a \neq 7$, $d = 7$, i.e. the matrix has the form $\begin{pmatrix} a & b & c \\ 0 & 7 & e \\ 0 & 0 & 7 \end{pmatrix}$. In this case we have two eigenvalues: a and 7. The eigenvalue a has algebraic multiplicity 1 and consequently the geometric multiplicity 1. The eigenvalue 7 has algebraic multiplicity 2 and consequently its geometric multiplicity is either 1 or 2. To see when it is 1 and when it is 2 one has to compute the rank of the matrix

$$\begin{pmatrix} a & b & c \\ 0 & 7 & e \\ 0 & 0 & 7 \end{pmatrix} - 7I = \begin{pmatrix} a-7 & b & c \\ 0 & 0 & e \\ 0 & 0 & 0 \end{pmatrix}$$

Since $a \neq 7$, the rank is 2 if $e \neq 0$ and 1 if $e = 0$. Therefore the geometric multiplicity of the eigenvalue 7 is equal to 1 if $e \neq 0$ and to 2 if $e = 0$.

Case 3. $a = 7, d \neq 7$ i.e. the matrix has the form $\begin{pmatrix} 7 & b & c \\ 0 & d & e \\ 0 & 0 & 7 \end{pmatrix}$.
In this case we have two eigenvalues: d and 7. The eigenvalue d has algebraic multiplicity 1 and consequently the geometric multiplicity 1. The eigenvalue 7 has algebraic multiplicity 2 and consequently its geometric multiplicity is either 1 or 2. To see when it is 1 and when it is 2 one has to compute the rank of the matrix

$$\begin{pmatrix} 7 & b & c \\ 0 & d & e \\ 0 & 0 & 7 \end{pmatrix} - 7I = \begin{pmatrix} 0 & b & c \\ 0 & d-7 & e \\ 0 & 0 & 0 \end{pmatrix}$$

The rank depends on the number $r = be - c(d-7)$, whether it is equal to 0 or not. If $r \neq 0$ then the rank is equal to 2 and consequently the geometric multiplicity of the eigenvalues 7 is equal to 1. If $r = 0$ then the rank is equal to 1 and consequently the geometric multiplicity of the eigenvalues 7 is equal to 2.

Case 4. $a = d \neq 7$, i.e. the matrix has the form $\begin{pmatrix} a & b & c \\ 0 & a & e \\ 0 & 0 & 7 \end{pmatrix}$.
In this case we have two eigenvalues: a and 7. The eigenvalue 7 has algebraic multiplicity 1 and consequently the geometric multiplicity 1. The eigenvalue a has algebraic multiplicity 2 and consequently its geometric multiplicity is either 1 or 2. To see when it is 1 and when it is 2 one has to compute the rank of the matrix

$$\begin{pmatrix} a & b & c \\ 0 & a & e \\ 0 & 0 & 7 \end{pmatrix} - aI = \begin{pmatrix} 0 & b & c \\ 0 & 0 & e \\ 0 & 0 & 7-a \end{pmatrix}.$$

It depends on the number b: if $b \neq 0$ then the rank is 2 and consequently the geometric multiplicity of the eigenvalues a is equal to 1. If $b = 0$ then the rank is 1 and consequently the geometric multiplicity of the eigenvalues a is equal to 2.

Case 5 (the last possibility). $a = d = 7$, i.e. the matrix has the form $\begin{pmatrix} 7 & b & c \\ 0 & 7 & e \\ 0 & 0 & 7 \end{pmatrix}$. In this case there is only one eigenvalue 7. Its algebraic multiplicity equals 3 and consequently its geometric multiplicity is either 1 or 2 or 3. To see when it is 1, when 2, and when 3, one has to compute the rank of the matrix

$$\begin{pmatrix} 7 & b & c \\ 0 & 7 & e \\ 0 & 0 & 7 \end{pmatrix} - 7I = \begin{pmatrix} 0 & b & c \\ 0 & 0 & e \\ 0 & 0 & 0 \end{pmatrix}.$$

If $e \neq 0$ and $b \neq 0$ then the rank equals 2 and consequently the geometric multiplicity of the eigenvalue 7 is 1.
If $e \neq 0$ and $b = 0$ then the rank equals 1 and consequently the geometric multiplicity of the eigenvalue 7 is 2.
If $e \neq 0$ and at leat one of the numbers b, c is different from 0 then the rank equals 1 and consequently the geometric multiplicity of the eigenvalue 7 is 2.
finally, if $e = b = c = 0$ then the rank equals 0 and consequently the geometric multiplicity of the eigenvalue 7 is 3.

Theorem 4. Let A be an $n \times n$ and let λ_i be an eigenvalue of algebraic multiplicity 2.

1. If the geometric multiplicity of λ_i is 1 then there exist a couple of vectors $v_i \neq 0, v_i' \neq 0$ such that

$$Av_i = \lambda_i v_1, \quad Av_i' = \lambda_i v_i' + v_i$$

Any vectors $v_i m v_i'$ satisfying these conditions are linearly independent.

Here v_i is an eigenvector corresponding to λ_i and v_i' can be called a vector associated with v_i.

2. If the geometric multiplicity of λ_i is 2 then there exists two linearly independent eigenvectors $v_i^{(1)}, v_i^{(2)}$ corresponding to λ_i.

The role of the associated eigenvector v_i' for solving the system $X' = AX$ is as follows.

Theorem 5. Let A be an $n \times n$ and let λ_i be an eigenvalue of algebraic multiplicity 2 and geometric multiplicity 1. Let v_i be an eigenvector corresponding to λ_i. The, as we know, the vector function $X(t) = e^{\lambda_i t} v_i$ is a solution of the system $X' = AX$. Another, linearly independent solution is

$$X(t) = e^{\lambda_i t}\big(tv_i + v_i'\big),$$

where v_i' is a vector associated with the eigenvector v_i.

In view of Theorems 4 and 5 we define one solution associated with an eigenvalue of algebraic multiplicity 1 and two solutions associated with an eigenvalue of multiplicity 2 as follows:

Definition 6.

1. A solution of the system $X' = AX$ associated with eigenvalue λ_i of algebraic multiplicity 1 is $e^{\lambda_i t}v_i$ where v_i is an eigenvector corresponding to λ_i.

2. A couple of solutions of the system $X' = AX$ associated with eigenvalue λ_i of algebraic multiplicity 2 and geometric multiplicity 1 is the couple

$$e^{\lambda_i t}v_i, \quad e^{\lambda_i t}\big(tv_i + v_i'\big),$$

where v_i' is a vector associated with the eigenvector v_i. See the first statement of Theorem 4.

3. A couple of solutions of the system $X' = AX$ associated with eigenvalue λ_i of algebraic multiplicity 2 and geometric multiplicity 2 is the couple

$$e^{\lambda_i t}v_i^{(1)}, \quad e^{\lambda_i t}v_i^{(2)}$$

where $v_i^{(1)}$ and $v_i^{(2)}$ are linearly independent vectors corresponding to λ_i. See the second statement of Theorem 4.

Theorem 7. Let A be an $n \times n$ matrix such that each of its eigenvalues has algebraic multiplicity 1 or 2. Collect the solutions of the system $X' = AX$ associated with all the eigenvalues (see Definition 6). We obtain n solutions. They are linearly independent and consequently these n solutions is a basis of the vector space of all solutions.

Theorem 7 allows to solve any system $X' = AX$ where $n \times n$ matrix A has eigenvalues of algebraic multiplicities ≤ 2.

Example. Let

$$A = \begin{pmatrix} 4 & 1 & 2 \\ 0 & 7 & a \\ 0 & 0 & 7 \end{pmatrix}.$$

Let us find abasis of the space of all solutions of the system $X' = AX$.

The matrix A has eigenvalue 4 of algebraic multiplicity 1 and the eigenvalue 7 of algebraic multiplicity 2. Compute the eigenvector corresponding to the eigenvalue 4: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Therefore the solution corresponding to the eigenvalue 4 is $e^{4t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

The geometric multiplicity of the eigenvalue 7 is 1 if $a \neq 0$ and is 2 if $a = 0$. Therefore we must consider the following two cases.

The case $a \neq 0$. In this case the geometric multiplicity of the eigenvalue 7 is 1 and we must compute an eigenvector v and an associated vector v'. The eigenvector v is a solution of the system

$$\begin{pmatrix} -3 & 1 & 2 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} \cdot v = 0$$

and we can take $v = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$. The associated eigenvector v' is a solution of the system

$$\begin{pmatrix} -3 & 1 & 2 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} \cdot v' = v = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

and we can take for example $v' = \begin{pmatrix} 2/a \\ 1 \\ 3/a \end{pmatrix}$. We obtain the following couple of solutions corresponding to the eigenvalue 7: $e^{7t}\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$ and $e^{7t}\left(t \cdot \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 2/a \\ 1 \\ 3/a \end{pmatrix} \right)$. We obtain the following basis of the space of all solutions:

$$e^{4t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e^{7t}\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \quad e^{7t}\begin{pmatrix} 2/a + t \\ 1 + 3t \\ 3/a \end{pmatrix}$$

The case a=0. In this case the geometric multiplicity of the eigenvalue 7 is 2 and there are two linearly independent eigenvectors corresponding to this eigenvalue. They are solutions of the system

$$\begin{pmatrix} -3 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot v = 0$$

and we can take for example

$$v^{(1)} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \quad v^{(2)} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}.$$

We obtain a basis of the space of all solutions:

$$e^{4t}\begin{pmatrix}1\\0\\0\end{pmatrix}, \quad e^{7t}\begin{pmatrix}1\\3\\0\end{pmatrix}, \quad e^{7t}\begin{pmatrix}2\\0\\3\end{pmatrix}$$