Math 106A. Fall 2008. M. Zhitomirskii

Exam 1. October 22, 2008

Problem 1. Find an eigenvector of the matrix $\begin{pmatrix} 3 & 5 \\ 0 & 2 \end{pmatrix}$ which is not proportional to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Solution. The matrix is triangular, therefore its eigenvalues are the diagonal entries 3 and 2. The eigenvector corresponding to the eigenvalue 3 is proportional to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The eigenvector corresponding to the eigenvalue 2 is proportional to $\begin{pmatrix} 1 \\ -5 \end{pmatrix}$.

Problem 2. Find an eigenvector of the matrix $\begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ corresponding to the eigenvalue $\lambda = 3$.

Solution. It is a non-zero solution of the system $\begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. It is $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ or a proportional vector.

Problem 3. Find n and an $n \times n$ matrix A such that the system $y_1'' = y_2$, $y_2'' = y_1$ can be transferred to the first order system $X' = AX$.

Solution. One has $n = 4$ because the new functions are $y_1(t), y_2(t)$ and their first order derivatives $y_1'(t), y_2'(t)$. The matrix A is not unique. It depends on the order (numeration) of the new functions. For example, introducing $x_1 = y_1, x_2 = y_1', x_3 = y_2, x_4 = y_2'$ we obtain $x_1' = x_2, x_2' = x_3, x_3' = x_4, x_4' = x_1$ and $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$. Another possibility is to introduce $x_1 = y_1, x_2 = y_2, x_3 = y_1', x_4 = y_2'$. In this case $x_1' = x_3, x_2' = x_4, x_3' = x_2, x_4' = x_1$ and $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Problem 4. Let $\begin{pmatrix} sin\, t \\ cos\, t \end{pmatrix}$ be the solution of the system $X' = AX$, where A is a certain 2×2 matrix. Note that this solution satisfies the initial condition $X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Find the solution of the same system satisfying the initial condition $X(\pi) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Solution. I assumed that the students use the shift of time: if $X(t)$ is a solution of an autonomous system then so is $\tilde{X}(t) = X(t+s)$, for any $s \in \mathbb{R}$. In particular, if $X(t)$ is a solution satisfying the initial condition $X(0) = X_0$ then $\tilde{X}(t) = X(t-\pi)$ is a solution of the same system satisfying the initial condition $X(\pi) = X_0$. Using this, we obtain an immediate answer $\begin{pmatrix} sin(t-\pi) \\ cos(t-\pi) \end{pmatrix} = \begin{pmatrix} -sint \\ -cost \end{pmatrix}$.

For this particular exercise (rather than for the general case!) this solution can be obtained by another way: since the system is linear, multiplying solution $X(t)$ by -1 we obtain another solution $-X(t)$, and one can check that he is lucky: the vector function $-X(t)$ takes at the point $t = \pi$ exactly what we need. Of course students who did so got max number of points, but take into account that next time there could be no such a luck.

Problem 5. Let A be a 3×3 matrix with eigenvalues

$$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = -3$$

and with corresponding eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}.$$

Find the solution $X(t)$ of the system $X' = AX$ satisfying the initial condition $X(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Solution. Any solution has the form $C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2 + C_3 e^{\lambda_3 t} v_3$. To find C_1, C_2, C_3 substitute $t = 0$. We obtain

$$C_1 + C_2 + C_3 = 1, C_1 = 0, C_2 + C_3 = 0$$

and it follows $C_1 = 0, C_2 = -1/2, C_3 = 1/2$.

Problem 6. Let A be a 4×4 matrix with eigenvalues

$$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = -1, \quad \lambda_4 = -2$$

and corresponding eigenvectors

$$v_1 = \begin{pmatrix} 11 \\ 14 \\ -15 \\ -131 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 28 \\ 24 \\ 13 \\ 98 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 2 \\ 5 \\ 1 \\ 4 \end{pmatrix}.$$

Let $X(t)$ be the solution $X(t)$ of the system $X' = AX$ satisfying the initial condition $X(0) = \begin{pmatrix} a \\ b \\ 5 \\ 8 \end{pmatrix}$. Under which (necessary and sufficient) condition on a, b one has $X(t) \to 0$ as $t \to \infty$?

Solution. $X(t) \to 0$ as $t \to \infty$ if and only if the vector $X(0)$ belongs the the span of the eigenvectors corresponding to the negative eigenvalues, i.e.

$$\begin{pmatrix} a \\ b \\ 5 \\ 8 \end{pmatrix} \in span\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 1 \\ 4 \end{pmatrix}. \right\}$$

This means that

$$a = k_1 + k_2, \quad b = 2k_1 + 5k_2, \quad 5 = 0 \cdot k_1 + 1 \cdot k_2, \quad 8 = 3k_1 + 4k_2$$

for some real numbers k_1, k_2. The last two equations imply $k_2 = 5, k_1 = -4$. By the first two equations $a = 6, b = 17$.

Problem 7. Let A be a 2×2 matrix with eigenvalues $\lambda_1 = 1, \lambda_2 = -1$ and corresponding eigenvectors $v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $v_2 = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$, and let $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ be the solution of the system $X' = AX$ satisfying the initial condition $X(0) = \begin{pmatrix} -1 \\ a \end{pmatrix}$. Under which (necessary and sufficient) condition on a there exists time $T \in (-\infty, \infty)$ such that $x_2(T) = 0$?

Solution. Drawing the phase portrait (saddle) we see that the phase curve containing the point $\begin{pmatrix} -1 \\ a \end{pmatrix}$ intersects the x_1-axes if and only if the vector $X(0) = \begin{pmatrix} -1 \\ a \end{pmatrix}$ belongs to the sector between the rays $\{x_2 = 3x_1, x_1 < 0\}$ and $x_2 = -(5/2)x_1, x_1 < 0\}$ which means $-3 < a < 5/2$.

Problem 8. Let A be a 2×2 matrix with eigenvalues $\lambda_1 = -1, \lambda_2 = -3$ and corresponding eigenvectors $v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $v_2 = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$, and let $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ be the solution of the system $X' = AX$ satisfying the initial condition $X(0) = \begin{pmatrix} a \\ 1 \end{pmatrix}$. Under which (necessary and sufficient) condition on a one has $x_2(t) < 0$ for sufficiently big t (i.e. there exists T such that $x_2(t) < 0$ as $t > T$)?

Solution. Drawing the oriented phase portrait (sink) and taking into account that the phase curves are tangent to the line $x_2 = 3x_1$ (corresponding to the eigenvalue -1) we see that the phase curve containing the point $\begin{pmatrix} a \\ 1 \end{pmatrix}$ approaches the origin from the semiplane $\{x_2 < 0\}$ if and only if the point $\begin{pmatrix} a \\ 1 \end{pmatrix}$ is located to the left of the line $x_2 = (-5/2)x_1$ which means $a < -(2/5)$.

Problem 9. Give an example of a 2×2 matrix A such that the system $X' = AX$ has the phase portrait (it was a figure in the test).

Solution. According to the phase portrait one of the eigenvalues is zero, the corresponding eigenvector is proportional to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, the other eigenvalue is negative, the corresponding eigenvector is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Therefore

$$A \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad A \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda \end{pmatrix}, \quad \lambda < 0$$

and it follows that $A = \begin{pmatrix} 0 & 0 \\ \lambda & \lambda \end{pmatrix}$, $\lambda < 0$.

Problem 10. There exists a number b such that the phase portrait of the system $X' = \begin{pmatrix} 2 & a \\ b & 5 \end{pmatrix} X$ is a saddle for $a > 4$ and a source or a sink for $a < 4$ or, vice a versa, a saddle for $a < 4$ and a source or a sink for $a > 4$. Find this number b and determine the type of the phase portrait (saddle, sink, source) for $a < 4$ and for $a > 4$. (Hint: using the fact that the sum of the eigenvalues is the trace and their product is the determinant saves at least 5 min and reduces to zero the probability of computational error).

Solution. The given condition imply that for the required number b, the number $\lambda_1\lambda_2$ is positive when $a > 4$ and negative when $a < 4$ or vice a versa. In any case it follows that $\lambda_1\lambda_2 = 0$ as $a = 4$ and consequently $detA = det\begin{pmatrix} 2 & 4 \\ b & 5 \end{pmatrix} X = 0$ from where $b = 5/2$. So we fix $b = 5/2$. Now, the sink is impossible for any a because $\lambda_1 + \lambda_2 = trace = 2 + 5 = 7 > 0$. Therefore saddle or source. If saddle then $\lambda_1\lambda_2 < 0$, consequently $detA < 0$, i.e. $a > 4$. And if source then $\lambda_1\lambda_2 > 0$, consequently $detA > 0$, i.e. $a < 4$.

Problem 11. For the system $x_1' = x_1^2 - x_2^2,\ x_2' = x_2^2$, find two straight line phase curves which do NOT belong to the x_1-axes.

Solution. A straight line $x_2 = Cx_1$ or its part is a straight line phase curve if the velocity vector at any point of this line, i.e. at any point $A = \begin{pmatrix} x_1 \\ Cx_1 \end{pmatrix}$ with any x_1, belongs to this line, i.e. proportional to $\begin{pmatrix} 1 \\ C \end{pmatrix}$. The velocity vector at the point A is the vector $\begin{pmatrix} x_1^2 - C^2x_1^2 \\ C^2x_1^2 \end{pmatrix}$. We obtain the following (necessary and sufficient) condition for the slope C:

$$\frac{C^2x_1^2}{x_1^2 - c^2x_1^2} = C, \quad \text{for any } x_1.$$

This condition is equivalent to the following equation for C: $\frac{C^2}{1-C^2} = C$ or equivalently $C^2 = C(1 - C^2)$. Since we do not allow our line to belong to the x_1-axes, one has $C \neq 0$ and canceling C we obtain a quadratic equation $C = 1 - C^2$. Solving it we obtain $C = (1 \pm \sqrt{5})/2$ which gives us four straight line phase curves (two rays in the line $x_2 = \frac{(1+\sqrt{5})}{2}x_1$ and two rays in the line $x_2 = \frac{(1-\sqrt{5})}{2}x_1$.