Math 106A. Fall 2008. M. Zhitomirskii

Exam 1. October 22, 2008

11 problems, each requires one to four minutes if you chose a good way to solve. Show all your work. Problems 1 - 3 are warming up. Problems 7-10 are more difficult than problems 4 - 6. Problem 11 is a bonus problem. No tricks are required, only the methods and the theorems we did in the class, and the simplest linear algebra.

Each problem solved **by 10:50 am: 10 points**. Each problem solved after 10:50 am and **by Friday, Oct 24, 9:30 am : 3 points** provided you explain me your solution, with all details.

The max grade is 110 points (perfect solutions of problems 1-10 and the bonus problem 11 by 10:50 am).

The test in available in my homepage. Good luck!

Name (Last, First) ____________________________

Grade ______________

Problem 1: __________ (max 10 pts)

Problem 2: __________ (max 10 pts)

Problem 3: __________ (max 10 pts)

Problem 4: __________ (max 10 pts)

Problem 5: __________ (max 10 pts)

Problem 6: __________ (max 10 pts)

Problem 7: __________ (max 10 pts)

Problem 8: __________ (max 10 pts)

Problem 9: __________ (max 10 pts)

Problem 10: __________ (max 10 pts)

Problem 11: __________ (max 10 pts)

1. Find an eigenvector of the matrix $\begin{pmatrix} 3 & 5 \\ 0 & 2 \end{pmatrix}$ which is not proportional to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

2. Find an eigenvector of the matrix $\begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ corresponding to the eigenvalue $\lambda = 3$.

3. Find n and an $n \times n$ matrix A such that the system $y_1'' = y_2$, $y_2'' = y_1$ can be transferred to the first order system $X' = AX$.

4. Let $\begin{pmatrix} sin\,t \\ cos\,t \end{pmatrix}$ be the solution of the system $X' = AX$, where A is a certain 2×2 matrix. Note that this solution satisfies the initial condition $X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Find the solution of the same system satisfying the initial condition $X(\pi) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

5. Let A be a 3×3 matrix with eigenvalues

$$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = -3$$

and with corresponding eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}.$$

Find the solution $X(t)$ of the system $X' = AX$ satisfying the initial condition $X(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

6. Let A be a 4×4 matrix with eigenvalues

$$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = -1, \quad \lambda_4 = -2$$

and corresponding eigenvectors

$$v_1 = \begin{pmatrix} 11 \\ 14 \\ -15 \\ -131 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 28 \\ 24 \\ 13 \\ 98 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 2 \\ 5 \\ 1 \\ 4 \end{pmatrix}.$$

Let $X(t)$ be the solution $X(t)$ of the system $X' = AX$ satisfying the initial condition $X(0) = \begin{pmatrix} a \\ b \\ 5 \\ 8 \end{pmatrix}$. Under which (necessary and sufficient) condition on a, b one has $X(t) \to 0$ as $t \to \infty$?

7. Let A be a 2×2 matrix with eigenvalues $\lambda_1 = 1, \lambda_2 = -1$ and corresponding eigenvectors $v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $v_2 = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$, and let $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ be the solution of the system $X' = AX$ satisfying the initial condition $X(0) = \begin{pmatrix} -1 \\ a \end{pmatrix}$. Under which (necessary and sufficient) condition on a there exists time $T \in (-\infty, \infty)$ such that $x_2(T) = 0$?

8. Let A be a 2×2 matrix with eigenvalues $\lambda_1 = -1, \lambda_2 = -3$ and corresponding eigenvectors $v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $v_2 = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$, and let $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ be the solution of the system $X' = AX$ satisfying the initial condition $X(0) = \begin{pmatrix} a \\ 1 \end{pmatrix}$. Under which (necessary and sufficient) condition on a one has $x_2(t) < 0$ for sufficiently big t (i.e. there exists T such that $x_2(t) < 0$ as $t > T$)?

9. Give an example of a 2×2 matrix A such that the system $X' = AX$ has the phase portrait

10. There exists a number b such that the phase portrait of the system $X' = \begin{pmatrix} 2 & a \\ b & 5 \end{pmatrix} X$ is a saddle for $a > 4$ and a source or a sink for $a < 4$ or, vice a versa, a saddle for $a < 4$ and a source or a sink for $a > 4$. Find this number b and determine the type of the phase portrait (saddle, sink, source) for $a < 4$ and for $a > 4$. (Hint: using the fact that the sum of the eigenvalues is the trace and their product is the determinant saves at least 5 min and reduces to zero the probability of computational error).

11. For the system $x_1' = x_1^2 - x_2^2$, $x_2' = x_2^2$, find two straight line phase curves which do NOT belong to the x_1-axes.