Solution of Problem 3

Problem 3. Set up (but do not compute) an iterated integral for the volume of the region $W \subset \mathbb{R}^3$ determined by the conditions

$$W = \{(x, y, z) : \ 3z^2 \le x + 2y \le z, \quad x \ge 0, \quad y \ge 0\}.$$

Hint. Find the projection of W to the (x, y)-plane, i.e. the region

$$D = \{(x, y) : \quad \text{there exits } z \text{ such that } (x, y, z) \in W\} \subset \mathbb{R}^2.$$

If you sketch the region D, it will be easy to describe W as an elementary region (start with describing D as an elementary region).

Many students did not follow the hint and tried to describe W as an elementary region starting with z. This required:

a) Finding the limits for z (numbers)

b) Computing the section of W by the plane $z = z^*$.

Many students did a) well ($0 \le z \le 1/3$), but b) is not simple and almost nobody did it right.

On the other hand, following my hint is a much simpler way. A point (x, y) belongs to the projection D if and only if $x \ge 0, y \ge 0$, and there exists z such that $z \ge x + 2y$ and $z \le \sqrt{\frac{x+2y}{3}}$. Such z exists if and only if

$$x + 2y \le \sqrt{\frac{x + 2y}{3}}$$

which is equivalent to the condition $x + 2y \le 1/3$. Therefore the projection D is described by the conditions

$$D : \ x + 2y \le 1/3, \ x \ge 0, \ y \ge 0$$

and we see that D is a triangle with vertices at $(0, 0), (1/3, 0)$, and $(0, 1/6)$. It is very easy to describe this traingle as an elementary region, for exampe

$$0 \le x \le 1/3, \ 0 \le y \le 1/6 - x/2.$$

The limits for z are clear: $x + 2y \le z \le \sqrt{\frac{x+2y}{3}}$. It follows that the volume of W is the iterated integral

$$\int_0^{1/3} \int_0^{1/6-x/2} \int_{x+2y}^{\sqrt{\frac{x+2y}{3}}} dzdydx.$$