104285־ מד"ר־א   מבחן סופי   27 ביוני 2011   בהצלחה!

3 שעות. חומר פתוח (מותר להשתמש בספרים ובמחברות).
מחשבונים: מותר אבל לא מומלץ. <u>מחשבים אסורים</u>.
9 שאלות. 12 נקודות לכל אחת ($\max = 108$).
רשמו תשובות סופיות ודרך פתרון (בקצרה).
<u>תשובה סופיות בלבד לא תזכה בנקודות.</u>

<u>מספר גירסה</u>:

1. יהי $x(t)$ פתרון של המשוואה

$$x' = x^6 + x$$

המקיים את תנאי ההתחלה $x(0) = 1$ ומוגדר בקטע מקסימלי אפשרי (t^-, t^+). מצא t^- ו־ t^+ (אינטגרלים בתשובה OK רק אם מתכנסים).
ציירו את הגרף של $x(t)$. כמה יש ל־ $x(t)$ נקודות פיתול?

Solution. It is an autonomous equation with two singular points: $x = 0$ and $x = -1$.
Since $x(0) = 1$ and $x'(0) > 0$ the solution $x(t)$ is an increasing function defined on the interval $(-\infty, t^+)$, tending to ∞ as $t \to t^+$ and tending to 0 as $t \to -\infty$. Here

$$t^+ = \int_1^\infty \frac{dx}{x^6 + x}$$

which is a finite number.

If t_1 is an inflection point then

$$x''(t_1) = 0 \Rightarrow (6x^5(t_1) + 1)x'(t_1) = 0 \Rightarrow x^5(t_1) + 1 = 0 \Rightarrow x(t_1) < 0.$$

But we know that for <u>our</u> solution $x(t) > 0$ for any t. Therefore there are no inflection points.

2. יהי $x(t)$ פתרון של המשוואה

$$x'' = \frac{1}{\sqrt{x}}$$

המקיים את תנאי ההתחלה $x(0) = 1, \quad x'(0) = -1$ ומוגדר בקטע מקסימלי אפשרי. מצא t_1 כך ש־ $x(t_1) = 2$. אינטגרלים בתשובה OK.

Solution. The solution $x(t)$ decreases in the interval $t \in [0, t^*]$, takes its minimal value x_{min} at the time t^*, after that (when $t > t^*$) it increases.
The theorem on energy gives

$$\frac{1}{2}(x')^2 - 2\sqrt{x} = C = const.$$

The constant C can be found from the initial conditions: $C = 1/2 - 2 = -3/2$. We obtain

$$(x')^2(t) = 4\sqrt{x}(t) - 3.$$

Since $x'(t^*) = 0$, substituting to this equation $t = t^*$ we obtain $x_{min} = 9/16$. On the time-interval $t \in [0, t^*]$ the finction $x(t)$ decreases, consequently

$$x'(t) = -\sqrt{4\sqrt{x} - 3}, \quad t \in [0, t^*].$$

It follows

$$t^* = \int_1^{9/16} - \frac{dx}{\sqrt{4\sqrt{x} - 3}} = \int_{9/16}^1 \frac{dx}{\sqrt{4\sqrt{x} - 3}}.$$

When $t > t^*$ the function $x(t)$ increases, consequently

$$x'(t) = \sqrt{4\sqrt{x} - 3}, \quad t > t^*.$$

It allows to find

$$t_1 - t^* = \int_{9/16}^2 \frac{dx}{\sqrt{4\sqrt{x} - 3}}$$

and obtain the final answer

$$t_1 = \int_{9/16}^1 \frac{dx}{\sqrt{4\sqrt{x} - 3}} + \int_{9/16}^2 \frac{dx}{\sqrt{4\sqrt{x} - 3}}.$$

3. יהי $x(t)$ פתרון של המשוואה

$$x'' = (12x-3)^5$$

המקיים את תנאי ההתחלה $x(0)=0,\ x'(0)=v_0$ ומוגדר בקטע מקסימלי אפשרי. מצא v_0 כך ש־ $\lim_{t\to\infty} x'(t)=0$ (עם הוכחה). אין אינטגרלים בתשובה.

Solution. The main point in this problem is to observe that

$$x''<0 \ \ as \ \ x<1/4; \quad x''=0 \ \ as \ \ x=1/4; \quad x''>0 \ \ as \ \ x>1/4.$$

It means that the problem is almost identical to the problem about the pendulum: the point $x=1/4$ plays the same role as the angle $\theta=\pi$. Repeating (with few modifications) the proof for the pendulum (not that simple, but given in the lectures, and students could use any notes) one can prove that there is a positive critical velocity $v_{0,crit}$ such that if $v_0=v_{0,crit}$ then $x(t)$ is an increasing fnction tending to $1/4$ as $t\to\infty$ and $x'(t)\to 0$ as $t\to\infty$, exactly as for the pendulum. This critical velocity $v_{0,crit}$ is exactly the answer to the problem because one can prove that the following:

- if $0<v_0<v_{0,crit}$ then $x(t)$ increases to $x_{max}<1/4$ and after that start to decrease so that
$\lim_{t\to\infty} x(t)=\lim_{t\to\infty} x'(t)=-\infty$;
- if $v_0\le 0$ then $x(t)$ decreases all the time and as above
$\lim_{t\to\infty} x(t)=\lim_{t\to\infty} x'(t)=-\infty$;
- if $v_0>v_{0,crit}$ then $x(t)$ increases all the time and
$\lim_{t\to\infty} x(t)=\lim_{t\to\infty} x'(t)=\infty$.

The critical velocity $v_{0,crit}$ can be found from the energy equation

$$\frac{1}{2}(x')^2-\int_0^{x(t)}(12s-3)^5ds=\frac{1}{2}v_0^2 \Leftrightarrow \frac{1}{2}(x')^2-\frac{(12x(t)-3)^6}{12\cdot 6}+\frac{3^6}{12\cdot 6}=\frac{1}{2}v_0^2.$$

Take the limit as $t\to\infty$. If $v_0=v_{0,crit}$ then $x(t)\to 1/4,\ x'(t)\to 0$ and we obtain $\frac{1}{2}v_{0,crit}^2=\frac{3^6}{12\cdot 6}$ from where

$$v_{0,crit}=\sqrt{\frac{2\cdot 3^6}{12\cdot 6}}=\sqrt{\frac{3^6}{6^2}}=\frac{3^3}{6}=9/2.$$

4. יהי $x(t)$ פתרון של המשוואה

$$x' = (x^2 + x - 12)^5 (t^2 - 1)^7$$

המקיים את תנאי ההתחלה $x(0) = 0$ ומוגדר לכל $t \in \mathbf{R}$.
מצא $\lim_{t\to\infty} x(t)$, $\lim_{t\to-\infty} x(t)$ (<u>עם הוכחה</u>) וציירו את הגרף של $x(t)$.
אין אינטגרלים בתשובה.

Solution. The function $x^2 + x - 12$ vanishes at points $x = 3$ and $x = -4$. Therefore $x^*(t) \equiv 3$ and $x^{**}(t) \equiv -4$ are constant solutions. It follows that our solution $x(t)$ takes values between -4 and 3 for any t. Therefore $x^2(t) + x(t) - 12 < 0$ for any t. It follows that the sign of $x'(t)$ is opposite to the sign of $(t^2 - 1)^7$. Therefore $x(t)$ decrease if $t > 1$ and if $t < -1$ and $x(t)$ increase if $t \in (-1, 1)$. ASt points $t = \pm 1$ the solution $x(t)$ has max and min respectively. It also follows that there are <u>finite</u> limits $\lim_{t\to\infty} x(t) = A$ and $\lim_{t\to-\infty} x(t) = B$. To find A and B write the relation between t and $x(t)$:

$$\int_0^{x(t)} \frac{ds}{s^2 + s - 12} = \int_0^t (s^2 - 1)^7.$$

It follows

$$\int_0^A \frac{ds}{s^2 + s - 12} = \int_0^B \frac{ds}{s^2 + s - 12} = \infty.$$

We know that A and B are finite numbers. It follows that both A and B are the roots of the equation $s^2 + s - 12 = 0$. now it follows that $A = -4$ and $B = 3$.

5. למטריצה 3×3 ממשית A יש וקטור עצמי
$(1+2i, 3+i, 7+2i)$ המתאים לערך עצמי $2i$
וקטור עצמי $(1,1,1)$ המתאים לערך עצמי -5.
יהי $x(t)$ הפתרון של המערכת $x' = Ax$
המקיים את תנאי ההתחלה $x(0) = (a, b, 2)$ ומוגדר לכל $t \in \mathbf{R}$.
מצא תנאי על $a, b \in \mathbf{R}$ כך שהפתרון של המערכת $x' = Ax$
המקיים את תנאי ההתחלה $x(0) = (a, b, 2)$
(א) הוא וקטור־פונקציה מחזורית
(ב) $lim_{t\to\infty} x(t) = 0 \in \mathbf{R}^3$.
אין אינטגרלים, מספרים מרוכבים ו־ span בתשובה.

Solution. In the case of diagonalizable matrix (over $\mathbf{C}$) nd only one couple of pure imaginary conjugate eigenvalues $\pm\alpha i$ the solution $x(t)$ is periodic if and only if $x(0)$ belongs to the subspace of $\mathbf{C}^3$ spaned by the eigenvectors corresponding to these eigenvalues and eigenvectors corresponding to the eigenvalue 0 (if 0 is an eigenvalue). In this problem there are no zero eigevalue. Therefore the solution is periodic if and only if $(a, b, 2)$ belongs to the span of the vectors $v_1 = (1 + 2i, 3 + i, 7 + 2i)$ (corresponding to the eigenvalue $2i$) and the conjugate vector $v_2 = (1-2i, 3-i, 7-2i)$ corresponding to the eigenvalue $-2i$). Since a and b are real numbers, this condition is equivalent to the condition that the vector $(a, b, 2)$ belongs to the span of the vectors Rev_1 and Imv_1, i.e. to the span of the vectors $(1, 3, 7)$ and $(2, 1, 2)$. If a student cannot determine what does it mean for a, b then his/her grade for the first course of algebra should be cancelled. Otherwise it is easy to get the answer: $12a - b = 10 = 0$.

The second question. In the case of diagonalizable matrix the solution $x(t)$ tends to 0 as $t \to \infty$ if and only if the vector $x(0) = (a, b, 2)$ belongs to the subspace of $\mathbf{C}^3$ spanned by the eigenvectors corresponding to the eigenvalues with negative real part. In the problem there is only one such eigenvalue -5 and the corresponding eigeibector is $(1, 1, 1)$. Therefore $X(t) \to 0$ as $t \to \infty$ if and only if the vectors $(a, b, 2)$ and $(1, 1, 1)$ are proportional which means $a = b = 2$.

6. יהי

$$J_\lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad A_\mu = \begin{pmatrix} 1 & 1 \\ \mu & -11 \end{pmatrix}.$$

אם $\mu = -36$ אז A דומה ל־ J_{-5}.

ב־ fig.1 מוצגת התמונה הפזית של המערכת $x' = J_\lambda x$ כאשר $\lambda < 0$.

ציירו את התמונה הפזית של המערכת $x' = A_\mu x$ כאשר

(א) $\mu = -35$ (ב) $\mu = -36$ (ג) $\mu = -37$

Solution. If $\mu = -35$ then the eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = -6$, the corresponding eigenvectors are $v_1 = (1, -5)$ and $v_2 = (1, -7)$. The phase portrait is a stable node with invariants lines ℓ_1 and ℓ_2 spanned by v_1 and v_2; the phase curves which do not belong to these lines are tangent to the line ℓ_1.

If $\mu = -36$ there is only one eigenvalue $\lambda = -5$ and in this case the matrix A is similar to the matrix J_{-5}. Therefore the phase portrait is the same as one showed for J_{-5} with the only difference that the invariant line (which is unique!) is not the x_1-axes as for J_{-5} but the line spanned by an eigenvector of A, it is the line $\ell = span(1, -6)$.

If $\mu = -37$ then the eigenvalues are $-5 \pm i$ and there are no invariant lines at all. The phase portrait is a stabel focus. The velocity vector at the point, say $(1, 1)$ is $(1, -37)$ - it allows to determine that the spirals approach $0 \in \mathbf{R}^2$ clockwise.

Note that in this problem we meet the following bifurcation: when the parameter μ is a bit bigger than -36 one has two invariant lines very close one to the other; when $\mu = -36$ these two invariant lines paste together to one line; after that, when μ is a bit smaller than -36 there are no invariant lines at all.

Such bifurcations, when two objects meet together and then "kill each other" and disappear, are typical for bifurcation theory and can be met in a number of applications.

7. יהי

$$T^{-1}AT = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

מצא את הפתרון של המערכת

$$x'(t) = Ax(t) + \begin{pmatrix} sin^{17}t \\ 0 \end{pmatrix}, \quad x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

המקיים את תנאי ההתחלה $x_1(3) = 1, x_2(3) = 0$. **אינטגרלים בתשובה** OK.

Solution. The eigenvectors of the matrix A are the columns of the matrix T. Therefore the general solution of the homogeneous system is as follows

$$x(t) = \begin{pmatrix} 2e^t & 3e^{-t} \\ e^t & 2e^{-t} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix},$$

and the solution of the given system can be found within vector-functions of the form

$$x(t) = \begin{pmatrix} 2e^t & 3e^{-t} \\ e^t & 2e^{-t} \end{pmatrix} \begin{pmatrix} C_1(t) \\ C_2(t) \end{pmatrix}. \tag{1}$$

Substituting it to the given system one obtains a system of linear equations with respect to $C_1'(t)$ and $C_2'(t)$:

$$\begin{pmatrix} 2e^t & 3e^{-t} \\ e^t & 2e^{-t} \end{pmatrix} \begin{pmatrix} C_1'(t) \\ C_2'(t) \end{pmatrix} = \begin{pmatrix} sin^{17}t \\ 0 \end{pmatrix}.$$

This system can be easily solved:

$$C_1'(t) = 2e^t sin^{17}t, \quad C_2'(t) = -e^t sin^{17}t.$$

To find the initial conditions for $C_1(t)$ and $C_2(t)$ substitute $t = 3$ to (1):

$$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} C_1(3) \\ C_2(3) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

It follows

$$C_1(0) = 2, \ C_2(0) = -1$$

and consequently

$$C_1(t) = 2\int_3^t e^s \cdot sin^{17}sds, \quad C_2(t) = -\int_3^t e^s \cdot sin^{17}sds. \tag{2}$$

It remains to substitute (2) to (1).

8. מצא את כל הנקודות הסינגולריות היציבות אסימפטוטית של המערכת

$$x_1' = sinx_2, \quad x_2' = sinx_1 + sinx_2$$

Solution. The singular (= equilibrium) points are the solution of the system $sinx_2 = 0,\ sinx_1 + sinx_2 = 0$ which is

$$A_{m,n} = (\pi m, \pi n), \quad m.n \in \mathbf{Z}.$$

The Jacobi matrix at the point $A_{m.n}$ is the matrix

$$\begin{pmatrix} 0 & (-1)^n \\ (-1)^m & (-1)^n \end{pmatrix}.$$

It has

$$trace = (-1)^n, \quad det = (-1)^{m+n+1}.$$

If m is even and n is odd then $trasce < 0$ and $det > 0$ and consequently $A_{m,n}$ is asymptotically stabe. Otherwise either $trace > 0$ or $det < 0$ and $A_{m,n}$ is not asymptotically stable. Therefore the asymptotically stabe singular points are points

$$A_{2k,2\ell+1}, \quad k, \ell \in \mathbf{Z}$$

and there are no other asymptotically stabel singular points.

9. מצא את קבוצת כל הפתרונות הממשיים של המשוואה

$$x^{(3)}(t) = x'''(t) = x(t) + e^{-t/2} sin(bt)$$

ל־ **(א)** $b = 1$ **(ב)** $b = \frac{\sqrt{3}}{2}$.

אין אינטגרלים ומספרים מרוכבים בתשובה.

Solution. It is important whether or not the number $-1/2 + bi$ is a root of the characteristic polynomial $P(\lambda) = \lambda^3 - 1$. The roots of $P(\lambda)$ are $1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$. It remains to use the formulas and make computations with complex numbers. I do not write computations here.