## 1. Complex numbers

1.1. Complex plane, $Rez$, $Imz$, $|z|$, $\bar{z}$, $Argz$.

1.2 Trigonometric form.

1.3 Elementary operations, $|z_1 z_2|$, $|z^n|$, $Argz^n$.

1.4 Solving $z^n = w$.

## 2. Polynomials

2.1 Real and complex roots of a *real* degree $n$ polynomial.

2.2 Multiplicities of roots. Finding the multiplicity of a root by factorization and using derivatives.

2.3 If $z$ is a root then $\bar{z}$ is also a root of the same multiplicity.

2.4 The sum of multiplicities is $n$ (main theorem of algebra).

2.5. Viete's theorem: the sum and the product of the roots of a polynomial (repeated according to their multiplicities)

## 3. Systems of linear equations. Rank of a matrix

3.1 Examples with unique/infinitely many/no solutions.

3.2 Matrices. The space $\mathbb{R}^n$ (= vectors=columns=$n \times 1$ matrices). Multiplication of a matrix by a vector, writing a system of linear equations in the form $Ax = b$.

3.3. Tsura medureget, dirug.

3.4 One of the definitions of rank: the number of nonzero rows in tsura medureget. Theorem: tsura medureget is not unique, but rank is well-defined. If $A$ is an $m \times n$ matrix then $rankA \leq min(m, n)$.

3.5. Solution of an arbitrary system $Ax = b$ where $A$ is an $m \times n$ matrix and $b \in \mathbb{R}^m$. Distinguishing the cases of (a) unique (b) infinitely many (c) no solutions. The number of parameters in the set of all solutions. Which conclusions can be made if we know $rankA$ only? (the cases $rankA < m,\ rankA = m,\ rankA > m$).

3.6. The set of all solutions of the homogeneous system $Ax = 0$ in terms of $rankA$.

3.7. Relation between solution of linear systems $Ax = b$ and $Ax = 0$.

## 4. Operations on matrices. Inverse matrix. Determinants

4.1. $A+B$, $AB$.

4.2. $AB \neq BA$, some cases when $AB = BA$. $A(BC) = (AB)C$.

4.3. The matrix $I$. If $A$ and $B$ are square matrices and $AB = I$ then $BA = I$. Definition of invertible matrix.

Theorem: an $n \times n$ matrix $A$ is invertible if and only if $rankA = n$.

Solving $AC = B$ with respect to $C$ if $A$ is a square invertible matrix.

Finding $A^{-1}$ by elementary transformations on rows (making the same operations on the rows of $I$).

4.3. Determinants and the ways of their calculations.

Theorem: for an $n \times n$ matrix $A$ the following are equivalent:
(a) $detA \neq 0$;
(b) $A$ is invertible;
(c) $rankA = n$;
(d) the system $Ax = b$ has a solution (unique) for any $b \in \mathbb{R}^n$.

4.4. Solving $Ax = b$ with a square matrix $A$ via determinants.

4.5. Finding the inverse matrix $A^{-1}$ via determinants.

4.6. Symmetric, anti-symmetric, triangular matrices. $detA^t = detA$.

## 5. Subspaces of $\mathbb{R}^n$. Linear independence, basis, dimension (all in $\mathbb{R}^n$)

5.1. Lines through the origin of $\mathbb{R}^2$. Two ways of presenting them:
(a) by one equation
(b) the set of points $tv, t \in \mathbb{R}$, $v$ is a certain vector in $\mathbb{R}^2$.

Lines through the origin of $\mathbb{R}^3$. Two ways of presenting them:
(a) by two equations
(b) the set of points $tv, t \in \mathbb{R}$, $v$ is a certain vector in $\mathbb{R}^3$.

Planes through the origin of $\mathbb{R}^3$. Two ways of presenting them:
(a) by one equation
(b) the set of points $t_1v_1 + t_2v_2, t_1, t_2 \in \mathbb{R}$, $v_1$ and $v_2$ are certain vectors in $\mathbb{R}^3$.

5.2. Definition of linear independent vectors in $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^n$. Definition of $span(v_1, ..., v_k)$ of $k$ vectors in $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^n$.

Theorem: $m < n$ vectors in $R^n$ never span $R^n$, and $m > n$ vectors are always linearly dependent.

5.3. Definition of basis of $\mathbb{R}^n$. Standard basis. Examples of non-standard bases.

5.4. Definition of a subspace $V \subset R^n$. Several equivalent definitions of a basis and $dim$ of $V$.

Theorem: if $v_1, ..., v_k \in V$ and $k < dimV$ then (a) $v_1, ..., v_k$ do not span $V$. If $k > dimV$ then (b) $v_1, ..., v_k$ are linearly dependent. If $k = dimV$ then (a) holds if and only if (b) holds.

5.5. Subspaces $V \subset \mathbb{R}^n$ given by

(a) a system $Ax = 0$ where $A$ is an $m \times n$ matrix

(b) $V = span(v_1, ..., v_k)$ where $v_1, ..., v_k \in \mathbb{R}^n$.

Finding $dimV$ and an example of a basis of $V$ for each of these cases. From (a) to (b) and from (b) to (a).

5.6. Solving the following problems:

(a) given a vector $v \in \mathbb{R}^n$ and vectors $v_1, ..., v_k \in \mathbb{R}^n$ determine whether $v \in span(v_1, ..., v_k)$.

(b) given linearly independent vectors $v_1, ..., v_k$ find vectors $v_{k+1}, ..., v_n$ such that $v_1, ..., v_n$ is a basis.

5.7. Two more (equivalent) definitions of a rank of an $m \times n$ matrix:

(a) the dim of the subspace of $\mathbb{R}^n$ spanned by rows

(b) the dim of the subspace of $\mathbb{R}^m$ spanned by columns.

## 6. Vector spaces

6.1. Definition of a vector space. Main examples for this course: $\mathbb{R}^n$, the space $Mat(m, n)$ of $m \times n$ matrices, the space $P_m[t]$ of polynomials of degree $\leq m$.

6.2. Subspaces of a vector space. Examples of subspaces of $Mat(m, n)$ and $P_m[t]$.

6.3. Linear dependence, span, basis, dimension. Definitions and main theorems: generalization of topic 5.

6.4. From an $n$-dimensional space $V$ to $\mathbb{R}^n$. Coordinate vector.

6.5. Finding dimension and an example of a basis of certain subspaces of $Mat(m, n)$ and $P_m[t]$.

## 7. Linear transformations

7.1. Definition of a linear transformation $T$ from a space $V$ to a space $W$. Examples of linear transformations:

(a) from $\mathbb{R}^p$ to $R^m$, $Mat(m,n)$, $P_m[t]$

(b) from $Mat(m,n)$ to $\mathbb{R}^p$, $Mat(\tilde{m},\tilde{n})$, $P_k[t]$

(c) from $P_k[t]$ to $\mathbb{R}^p$, $Mat(m,n)$.

7.2. Representative matrix $[T]_e$ of a linear transformation $T$ in a standard basis $e$. Finding $T(v)$ by $[T(v)]_e = [T]_e[v]_e$, where $[v]_e$ is the vector of coordinates in the standard basis $e$.

7.3. Composition of linear transformations. Representative matrix of composition = product of representative matrices.

7.4. Kernel and image of a linear transformation, finding them via a matrix of the transformation in standard basis.

7.5. 1-1 and onto linear transformations.

1-1 $\iff$ $ker = \{0\}$.

Theorem: If $T : V \to W$, $dimV = n, dimW = m$, then:

(a) if $n > m$ then $T$ is not 1-1.

(b) If $n < m$ then $T$ is not onto.

(c) If $n = m$ then $T$ is 1-1 if and only if it is onto.

## 8. Linear operators

8.1. Definition: a linear operator is a linear transformation from a vector space $V$ to the same vector space $V$.

8.2. Examples when the representative matrix of $T$ in the standard basis is not diagonal, and in certain non-standard basis it is diagonal (as a motivation for nonstandard basis). Definition: a basis is "good" if the matrix of $T$ in this basis is diagonal. Definition: if a "good" basis exists then the operator $T$ is called diagonalizable, and its matrix (in any basis) is called diagonalizable.

8.3. Definition of eigenvalues, eigenspaces, and eigenvectors of a linear operator $T : V \to V$. A number $\lambda$ is an eigenvalue if $T(v) = \lambda v$ for some nonzero vector $v$. The set of all vectors $v$ satisfying this relation is a subspace of the space $V$. It is called eigenspace corresponding to $\lambda$. Nonzero vectors of this eigenspace are called eigenvectors corresponding to $\lambda$.

8.4. Definition of eigenvalues, eigenspaces, and eigenvectors of a square $n \times n$ matrix $A$. A number $\lambda$ is an eigenvalue of $A$ if $det(A - \lambda I) =$

0. The set of all solutions $x$ of the system $(A - \lambda I)x = 0$ is a subspace of $\mathbb{R}^n$ called the eigenspace corresponding to $\lambda$. Nonzero vectors of this eigenspace are called eigenvectors corresponding to $\lambda$.

8.5. Relation between the eigenvalues and eigenvectors of operators and matrices.

Let $V$ be a vector space, $dim V = n$, $T : V \to V$ is a linear operator, and let $A$ be the matrix of $T$ in ANY basis, say the standard basis $e$. Then:

(a) $\lambda$ is an eigenvalue of $T$ if and only if it is an eigenvalue of $A$

(b) $v$ is an eigenvector of $T$ if and only if its coordinate vector $[v]_e$ is an eigenvector of $A$

8.6. Finding eigenvalues. Characteristic polynomial and algebraic multiplicity of eigenvalues. complex eigenvalues. The sum and the product of the eigenvalues (repeated according to algebraic multiplicities).

8.7. Geometric multiplicity of eigenvalues. Theorem about diagonalizability (a matrix is diagonalizable over $\mathbb{R}$ if and only if all its eigenvalues are real and the geometric multiplicity of each of them coincides with the algebraic multiplicity). Particular case: the algebraic multiplicity is 1 for each of the eigenvalues.

8.8. Finding a "good basis" (if exists) for a linear operator $T : V \to V$, i.e. a basis $e$ of $V$ in which the representative matrix $[T]_e$ is diagonal.