SYMPLECTIC SINGULARITIES OF VARIETIES: THE METHOD OF ALGEBRAIC RESTRICTIONS

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Abstract. We study germs of singular varieties in a symplectic space. In [A1] V. Arnol'd discovered so called "ghost" symplectic invariants which are induced purely by singularity. We introduce algebraic restrictions of differential forms to singular varieties and show that this ghost is exactly the invariants of the algebraic restriction of the symplectic form. This follows from our generalization of Darboux-Givental' theorem from non-singular submanifolds to arbitrary quasi-homogeneous varieties in a symplectic space. Using algebraic restrictions we introduce new symplectic invariants and explain their geometric meaning. We prove that a quasi-homogeneous variety N is contained in a non-singular Lagrangian submanifold if and only if the algebraic restriction of the symplectic form to N vanishes. The method of algebraic restriction is a powerful tool for various classification problems in a symplectic space. We illustrate this by complete solutions of symplectic classification problem for the classical A, D, E singularities of curves, the S_5 singularity, and for regular union singularities.

1. Introduction and main results

1.1. **Starting points.** The starting points for this paper are as follows:

• the classical Darboux-Givental' theorem on non-singular submanifolds of a symplectic manifold (proved by A. Givental' and firstly published in [AG]);

• the works [A1], [A2] in which V. Arnol'd studied singular curves in symplectic and contact spaces and introduced the local symplectic and contact algebras.

• the work [Z] developing the local contact algebra.

The work [Z] is based on the notion of the algebraic restriction of a contact structure to a subset N of a contact manifold. The present work is based on a similar notion of the algebraic restriction to N of a symplectic structure, and we show that like in the contact case it is a powerful tool for the study of singular submanifolds of a symplectic manifold.

1.2. **Darboux-Givental' theorem.** A diffeomorphism $\Phi : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ of a symplectic space $(\mathbb{R}^{2n}, \omega)$ is called a symplectomorphism if it preserves the symplectic form ω: $\Phi^*\omega = \omega$. Two subsets $N_1, N_2 \subset \mathbb{R}^{2n}$ are called symplectomorphic if there exists a symplectomorphism which brings N_2 to N_1.

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Convention. Throughout the paper all objects are germs at 0 of a fixed category which is either C^∞ or real-analytic.

Theorem 1.1. (Darboux-Givental' theorem, see [AG]).
(i) *Let N be a non-singular submanifold of $\mathbb{R}^{2n}$ and let ω_0, ω_1 be symplectic forms on $\mathbb{R}^{2n}$ with the same restriction to TN. There exists a local diffeomorphism Φ such that $\Phi(x) = x$ for any $x \in N$ and $\Phi^*\omega_1 = \omega_0$.*
(ii) (corollary of (i)) *Two equal-dimensional non-singular submanifolds N_1, N_2 of a symplectic space $(\mathbb{R}^{2n}, \omega)$ are symplectomorphic if and only if the restrictions of the symplectic form ω to TN_1 and TN_2 are diffeomorphic.*

Let $\mathrm{Symp}(\mathbb{R}^{2n})|_{\mathbb{R}^r} = \{\omega|_{T\mathbb{R}^r} : \ \omega \in \mathrm{Symp}(\mathbb{R}^{2n})\}$, where $\mathrm{Symp}(\mathbb{R}^{2n})$ denotes the set of all symplectic 2-forms on $\mathbb{R}^{2n}$. Theorem 1.1, (ii) reduces the classification of germs of non-singular r-dimensional submanifolds of a symplectic manifold with respect to the group of symplectomorphisms to the classification of the set $\mathrm{Symp}(\mathbb{R}^{2n})|_{\mathbb{R}^r}$ with respect to the group of all local diffeomorphisms of $\mathbb{R}^r$. This reduction is completed by an explicit description of $\mathrm{Symp}(\mathbb{R}^{2n})|_{\mathbb{R}^r}$.

Theorem 1.2. (see [AG]). *The set $\mathrm{Symp}(\mathbb{R}^{2n})|_{\mathbb{R}^r}$ consists of closed 2-forms on $\mathbb{R}^r$ of rank $\geq 2(r-n)$.*

1.3. **The problem of symplectic classification of singular varieties.** The present work is devoted to the following problem.

Problem A. To classify with respect to the group of symplectomorphisms the class of all varieties in a symplectic space $(\mathbb{R}^{2n}, \omega)$ which are diffeomorphic to a fixed singular variety N.

We give a method for solving this problem for any quasi-homogeneous variety N based on generalization of Theorem 1.1 from non-singular submanifolds to arbitrary quasi-homogeneous varieties. We recall the definition of a quasi-homogeneous variety in section 2.6. The simplest example is

$$N = A_k = \{x \in \mathbb{R}^{2n} : \ x_1^{k+1} - x_2^2 = x_{\geq 3} = 0\}, \quad k \geq 1, \tag{1.1}$$

which is a cusp if k is even and the union of two non-singular curves if k is odd.

1.4. **Arnold's ghost invariant.** A natural symplectic invariant of a singular variety N is the restriction of the symplectic 2-form to the regular part N^{reg} of N. This invariant is not complete - there are other independent and much more involved invariants. To explain this, in the work [A1] V. Arnol'd solved the classification Problem **A** for the simplest case when the restriction of the symplectic structure to N^{reg} vanishes – case $N = A_{2\ell}$. Arnol'd proved that if $\ell \geq 2$ then there are exactly $2\ell + 1$ singularities (orbits). Describing this result Arnol'd wrote
"...something nontrivial remains from the symplectic structure at the singular points of the curve. It would be interesting to describe this ghost of the symplectic structure in terms of the local algebra of the singularity."

1.5. **Our approach.** We believe that in the present paper this objective has been reached: the ghost is exactly the singularity of the **algebraic restriction** of the symplectic structure to A_k. The algebraic restrictions are introduced in the beginning of section 2. The results of section 2 give a method (the method of algebraic restrictions) for solving Problem **A** for many types of singularities. The main results are Theorems **A - C** (proved in section 3) and **D**. Theorem **A** is the base for the method – it is a generalization of Theorem 1.1 from non-singular submanifolds to arbitrary quasi-homogeneous varieties N: one has to replace the pullback by the algebraic restriction. Theorem **B** states that the symplectic form has zero algebraic restriction to N if and only if N is contained in a non-singular Lagrangian submanifold. We introduce the index of non-isotropness and the symplectic multiplicity of N and show how these symplectic invariants can be calculated using the algebraic restrictions (Theorems **C** and **D**). In section 2 we also illustrate the method of algebraic restrictions showing that the results in [A1], devoted to Problem **A** with $N = A_k$, are almost immediate corollaries of Theorems **A** - **D**. Of course these theorems can be applied to many much more involved singularities. In the present work, using the method of algebraic restrictions, we continue [A1] solving Problem **A** for the case that N is one of the classical D_k, E_6, E_7, E_8 singularities of planar curves (sections 4, 5), we also solve Problem **A** for the case $N = S_5 = \{x_1^2 - x_2^2 - x_3^2 = x_2x_3 = x_{\geq 4} = 0\}$ (section 6) and for the case that N is a regular union singularity, i.e. $N = N_1 \cup \cdots \cup N_s$, where N_i is a non-singular submanifold and the sum $T_0N_1 + \cdots + T_0N_s$ is direct (section 7).

2. The method of algebraic restrictions

2.1. **Definition of algebraic restrictions.** Given a germ of a non-singular manifold M denote by $\Lambda^p(M)$ the space of all germs at 0 of differential p-forms on M. Given a subset $N \subset M$ introduce the following subspaces of $\Lambda^p(M)$:

$$\Lambda^p_N(M) = \{\omega \in \Lambda^p(M): \quad \omega(x) = 0 \text{ for any } x \in N\};$$

$$\mathcal{A}^p_0(N, M) = \{\alpha + d\beta : \quad \alpha \in \Lambda^p_N(M), \ \beta \in \Lambda^{p-1}_N(M).\}$$

The relation $\omega(x) = 0$ means that the p-form ω annihilates any p-tuple of vectors in T_xM, i.e. all coefficients of ω in some (and then any) local coordinate system vanish at the point x.

It is easy to check that in the case that N is a non-singular submanifold of $\mathbb{R}^m$ the restriction of ω to TN can be defined in the following algebraic way.

Proposition 2.1. *If N is a non-singular submanifold of M then a p-form ω on M has zero restriction to TN if and only if $\omega \in \mathcal{A}^p_0(N, M)$. Therefore the restriction of ω to TN can be defined as the equivalence class of ω in the space $\Lambda^p(M)$, where the equivalence is as follows: ω is equivalent to $\widetilde{\omega}$ if $\omega - \widetilde{\omega} \in \mathcal{A}^p_0(N, M)$.*

Proof. Take local coordinates $x = (x_1, \ldots, x_k), y = (y_1, \ldots, y_l)$ on M such that N is described by the equations $x = 0$. A p-form ω has zero restriction to TN if and only if it can be written in the form $\sum x_i\alpha_i + \sum dx_i \wedge \mu_i$, where α_i are p-forms and μ_i are $(p-1)$-forms. It remains to note that $dx_i \wedge \mu_i = d(x_i\mu_i) - x_i d\mu_i$. □

Note now that Proposition 2.1 involves no structure of N. Allowing N to be any subset of M and calling the equivalence classes by algebraic restrictions (we believe this name is natural) we get the following definition, generalizing the definition in [Z] of the algebraic restriction to N of a 1-form.

Definition 2.2. Let N be a subset of M and let $\omega \in \Lambda^p(M)$. The algebraic restriction of ω to N is the equivalence class of ω in $\Lambda^p(M)$, where the equivalence is as follows: ω is equivalent to $\widetilde{\omega}$ if $\omega - \widetilde{\omega} \in \mathcal{A}^p_0(N, M)$.

Notation. The algebraic restriction of a p-form ω on M to a subset $N \subset M$ will be denoted by $[\omega]_N$. Writing $[\omega]_N = 0$ (or saying that ω has zero algebraic restriction to N) we mean that $[\omega]_N = [0]_N$, i.e. $\omega \in A_0^p(N, M)$.

It is clear that if $\omega \in \mathcal{A}_0^p(N, M)$ then $d\omega \in \mathcal{A}_0^{p+1}(N, M)$. This allows to define the differential of an algebraic restriction: $d[\omega]_N = [d\omega]_N$. Another well-defined operation is the external multiplication: $[\omega_1]_N \wedge [\omega_2]_N = [\omega_1 \wedge \omega_2]_N$, where ω_1 and ω_2 are differential forms of any degrees. This operation is well-defined due to the following almost obvious proposition.

Proposition 2.3. *Let $N \subset \mathbb{R}^m$ and let ω be a p-form on $\mathbb{R}^m$ such that $[\omega]_N = 0$. Let μ be any q-form on $\mathbb{R}^m$. Then $[\omega \wedge \mu]_N = 0$.*

Proof. It suffices to write ω in the form $\alpha + d\beta$ with α and β vanishing at any point of N and to note that $d\beta \wedge \mu = d(\beta \wedge \mu) + (-1)^q \beta \wedge d\mu$. □

2.2. **Example: algebraic restrictions of 2-forms to A_k.** The set of algebraic restrictions of p-forms on $\mathbb{R}^m$ to any variety $N \subset \mathbb{R}^m$ is a vector space if p is fixed. Let us calculate this space for the case $p = 2$ and $N = A_k = (1.1)$. Since the functions $x_{\geq 3}$ have zero algebraic restriction to A_k then by Proposition 2.3 the algebraic restriction to A_k of any 2-form can be represented by a 2-form of the form $f(x_1, x_2)dx_1 \wedge dx_2$. Let $H = x_1^{k+1} - x_2^2$. We will use again (several times) Proposition 2.3. Since $[dH]_{A_k} = 0$ then $[dH \wedge dx_1]_{A_k} = [dH \wedge dx_2]_{A_k} = 0$. It follows that if $f(x_1, x_2)$ belongs to the gradient ideal of H then $[f(x_1, x_2)dx_1 \wedge dx_2]_{A_k} = 0$. The gradient ideal is (x_2, x_1^k). Consequently the algebraic restriction to A_k of any 2-form on $\mathbb{R}^{2n}$ can be represented by a 2-form of the form $\sum_{i=0}^{k-1} c_i x_1^i dx_1 \wedge dx_2$. It is easy to show that if such a 2-form has zero algebraic restriction to A_k then $c_0 = \cdots = c_{k-1} = 0$. We obtain:

the dimension of the space of algebraic restrictions to A_k of all 2-forms on $\mathbb{R}^{2n}$ is equal to k. This space is spanned by the algebraic restrictions

$$[A_k]^i = [x_1^i dx_1 \wedge dx_2]_{A_k}, \quad i = 0, \ldots, k-1. \tag{2.1}$$

2.3. **The action of the group of diffeomorphisms.** Let M and $\widetilde{M}$ be non-singular equal-dimensional manifolds and let $\Phi : \widetilde{M} \to M$ be a local diffeomorphism. Let N be a subset of M. It is clear that $\Phi^* \mathcal{A}_0^p(N, M) = \mathcal{A}_0^p(\Phi^{-1}(N), \widetilde{M})$. Therefore the action of the group of diffeomorphisms can be defined as follows: $\Phi^*([\omega]_N) = [\Phi^*\omega]_{\Phi^{-1}(N)}$, where ω is an arbitrary p-form on M. Let $\widetilde{N} \subset \widetilde{M}$. Two algebraic restrictions $[\omega]_N$ and $[\widetilde{\omega}]_{\widetilde{N}}$ are called **diffeomorphic** if there exists a local diffeomorphism from $\widetilde{M}$ to M sending the first algebraic restriction to the second one. This of course requires that the same diffeomorphism sends $\widetilde{N}$ to N.

If $M = \widetilde{M}$ and $N = \widetilde{N}$ then the definition of diffeomorphic algebraic restrictions reduces to the following one: two algebraic restrictions $[\omega]_N$ and $[\widetilde{\omega}]_N$ are diffeomorphic if there exists a local symmetry Φ of N (i.e. a local diffeomorphism preserving N) such that $[\Phi^*\omega]_N = [\widetilde{\omega}]_N$.

2.4. **Reduction theorem.** If a set $N \subset \mathbb{R}^m$ is contained in a non-singular submanifold $M \subset \mathbb{R}^m$ then the classification of algebraic restrictions to N of p-forms on $\mathbb{R}^m$ reduces to the classification of algebraic restrictions to N of p-forms on M. At first note that the algebraic restrictions $[\omega]_N$ and $[\omega|_{TM}]_N$ can be identified:

Proposition 2.4. *Let N be the germ at 0 of a subset of $\mathbb{R}^m$ contained in a non-singular submanifold $M \subset \mathbb{R}^m$ and let ω_1, ω_2 be p-forms on $\mathbb{R}^m$. Then $[\omega_1]_N = [\omega_2]_N$ if and only if $\left[\omega_1|_{TM}\right]_N = \left[\omega_2|_{TM}\right]_N$.*

Proof. Take local coordinates in which $M = \{x \in \mathbb{R}^n : \ x_1 = \cdots = x_s = 0\}$. Then $[x_1]_N = \cdots = [x_s]_N = 0$ and Proposition 2.4 follows from Proposition 2.3. □

The following, less obvious statement, means that the *orbits* of the algebraic restrictions $[\omega]_N$ and $[\omega|_{TM}]_N$ also can be identified.

Theorem 2.5. *Let N_1, N_2 be subsets of $\mathbb{R}^m$ contained in equal-dimensional non-singular submanifolds M_1, M_2 respectively. Let ω_1, ω_2 be two p-forms. The algebraic restrictions $[\omega_1]_{N_1}$ and $[\omega_2]_{N_2}$ are diffeomorphic if and only if the algebraic restrictions $\left[\omega_1|_{TM_1}\right]_{N_1}$ and $\left[\omega_2|_{TM_2}\right]_{N_2}$ are diffeomorphic.*

Proof. The "if" part follows from Proposition 2.4. To prove the "only if" part it suffices to prove the following: the restrictions of any p-form ω to TM_1 and TM_2 have diffeomorphic algebraic restrictions to any set $N \subset M_1 \cap M_2$. This statement easily follows from the following observations: (a) one can easily prove that there exists a local diffeomorphism of $\mathbb{R}^m$ sending M_1 to M_2 and preserving pointwise the set $M_1 \cap M_2$ (and consequently preserving pointwise N); (b) any local diffeomorphism Φ preserving N pointwise preserves the algebraic restriction to N of any p-form. The latter follows from Proposition 2.3 because Φ has the form $x_i \to x_i + \phi_i(x)$, where $\phi_i(x)$ are functions vanishing at points of N. □

2.5. **Example: classification of algebraic restrictions of 2-forms to A_k.** We continue Example 2.2. The curve A_k has a symmetry of the form $\Phi : (x_1, x_2) \to (x_1\phi^2, x_2\phi^{k+1})$ where $\phi = \phi(x_1, x_2)$ is any function such that $\phi(0) = 1$. In view of section 2.2 consider the symmetries

$$\Phi : \ (x_1, x_2) \to \left(x_1(1 + rx_1^s)^2, \ x_2(1 + rx_1^s)^{k+1}\right), \quad r \in \mathbb{R}, \ s \geq 1.$$

It is easy to calculate

$$\Phi^*(x_1^p dx_1 \wedge dx_2) = \left(\left(x_1^p + \widetilde{r}x_1^{p+s} + o(||(x_1, x_2)||^{p+s}\right)\right) dx_1 \wedge dx_2, \ \ \widetilde{r} = r(2p+2s+k+3).$$

Along with results of section 2.2 this implies

$$(\Phi)^* \left([A_k]^p\right) \in [A_k]^p + \widetilde{r} \cdot [A_k]^{p+s} + \text{span}\left([A_k]^{p+s+1}, \ldots, [A_k]^{k-1}\right).$$

Since r and $s \geq 1$ are arbitrary it follows that any algebraic restriction of the affine space $[A_k]^p + \text{span}\left([A_k]^{p+1}, \ldots, [A_k]^{k-1}\right)$ is diffeomorphic to $[A_k]^p$. Therefore any *non-zero* algebraic restriction to A_k of a 2-form on $\mathbb{R}^{2n}$ is diffeomorphic to $r \cdot [A_k]^p$, where $r \neq 0$ and $p \in \{0, ..., k-1\}$. The factor r can be reduced to 1 due to the scale symmetries $(x_1, x_2) \to (t^2 x_1, t^{k+1} x_2)$ and $(x_1, x_2) \to (x_1, -x_2)$. The algebraic restrictions $[A_k]^i$ and $[A_k]^j$ with $i < j \leq k-1$ are not diffeomorphic because, as it is easy to prove, $[A_k]^i$ cannot be represented by a 2-form with zero i-jet. Therefore we obtain the following result:

any non-zero algebraic restriction to A_k of a 2-form on $\mathbb{R}^m$ is diffeomorphic to one and only one of the algebraic restrictions (2.1).

2.6. **Relative cohomology groups.** The name "algebraic restriction" was introduced in [Z], but the differential subcomplex of the de Rham complex related to the spaces $A_0^p(N,M)$ and the corresponding relative cohomology groups

$$H^p(N,\mathbb{R}^m) = \frac{\{\omega \in \mathcal{A}_0^p(N,\mathbb{R}^m) :\ d\omega = 0\}}{\{d\alpha :\ \alpha \in \mathcal{A}_0^{p-1}(N,\mathbb{R}^m)\}}$$

were studied much earlier, see [R], [Sa1], [B], [Se], [Gr1], [Gr2]. See also the work [DJZ] and other references there. The main purpose of the mentioned works was to express certain local properties of N in terms of vanishing of some of the relative cohomology groups. In the present work we will use the main result in this direction which can be called the relative Poincare lemma.

Definition 2.6. The germ at 0 of a set $N \subset \mathbb{R}^m$ is called quasi-homogeneous if there exist a local coordinate system $x_1,\dots,x_m$ and positive numbers $\lambda_1,\dots,\lambda_m$ such that the following holds: if a point with coordinates $x_i = a_i$ belongs to N then for any $t \in [0,1]$ the point with coordinates $x_i = t^{\lambda_i} a_i$ also belongs to N.

Theorem 2.7. (see [R]). *If $N \subset \mathbb{R}^m$ is a quasi-homogeneous subset then $H^p(N,\mathbb{R}^m) = \{0\}$ for any $p \geq 1$.*

2.7. **Generalization of Darboux-Givental' theorem.** The method of algebraic restrictions is based on the following theorem.

Theorem A. (cf. Theorem 1.1).

(i) *Let N be a quasi-homogeneous subset of $\mathbb{R}^{2n}$. Let ω_0, ω_1 be symplectic forms on $\mathbb{R}^{2n}$ with the same algebraic restriction to N. There exists a local diffeomorphism Φ such that $\Phi(x) = x$ for any $x \in N$ and $\Phi^*\omega_1 = \omega_0$.*

(ii) (corollary of (i)) *Two quasi-homogeneous subsets N_1, N_2 of a fixed symplectic space $(\mathbb{R}^{2n},\omega)$ are symplectomorphic if and only if the algebraic restrictions of the symplectic form ω to N_1 and N_2 are diffeomorphic.*

Theorem **A** generalizes Theorem 1.1 since any non-singular submanifold is quasi-homogeneous and, as we explained in Proposition 2.1, the algebraic restriction of a p-form ω to a non-singular submanifold N can be identified with $\omega|_{TN}$.

Remark. Our proofs in section 3 show that in Theorem **A** and in its corollaries – Theorems **B**, **C**, **D** below – the assumption that N is quasi-homogeneous can be replaced by the condition $H^2(N,\mathbb{R}^{2n}) = \{0\}$. This condition follows from the quasi-homogeneity of N (see Theorem 2.7), but in general it is weaker than the quasi-homogeneity. It is possible that $H^2(N,\mathbb{R}^{2n}) = \{0\}$ but one of the other cohomology groups is not trivial and consequently N is not quasi-homogeneous, see [Gr1]. See also [DJZ] where there are examples of non-quasi-homogeneous varieties N such that all cohomology groups are trivial. If $H^2(N,\mathbb{R}^{2n}) \neq \{0\}$ then the conclusion of Theorem **A**, (i) remains the same if the symplectic forms ω_1, ω_2 satisfy the additional assumption that $\omega_1 - \omega_2$ has zero class in $H^2(N,\mathbb{R}^{2n})$. The proof is the same as that of Theorem **A**, (i) in section 3. Nevertheless, we believe that for a certain class of varieties N such that $H^2(N,\mathbb{R}^{2n}) \neq \{0\}$ the algebraic restriction $[\omega]_N$ remains to be a complete symplectic invariant unless $[\omega]_N = 0$.

2.8. **Application to Problem A.** Let us fix the following notations:

- $\left[\Lambda^2(\mathbb{R}^{2n})\right]_N$: the vector space consisting of algebraic restrictions to a subset $N \subset \mathbb{R}^{2n}$ of all 2-forms on $\mathbb{R}^{2n}$;

• $\left[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})\right]_N$: the subspace of $\left[\Lambda^2(\mathbb{R}^{2n})\right]_N$ consisting of algebraic restrictions to N of all closed 2-forms on $\mathbb{R}^{2n}$;

• $\left[\mathrm{Symp}(\mathbb{R}^{2n})\right]_N$: the open set in $\left[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})\right]_N$ consisting of algebraic restrictions to N of all symplectic 2-forms on $\mathbb{R}^{2n}$.

Theorem **A** reduces problem **A** for quasi-homogeneous N to the following

Problem B. To classify the algebraic restrictions of set $\left[\mathrm{Symp}(\mathbb{R}^{2n})\right]_N$ with respect to the group of symmetries of N.

In fact, assume that problem **B** is solved, i.e. we have a final list of normal forms $[\theta_1]_N, ..., [\theta_s]_N \in \left[\mathrm{Symp}(\mathbb{R}^{2n})\right]_N$ for algebraic restrictions, where θ_i are certain 2-forms (some of them might depend on parameters). The 2-forms θ_i representing the algebraic restrictions might be not symplectic and even not closed. But we know that there exist symplectic forms ω_i such that $[\omega_i]_N = [\theta_i]_N$. Now, given a fixed symplectic space $(\mathbb{R}^{2n}, \omega_0)$ take local diffeomorphisms Φ_i of $\mathbb{R}^{2n}$ sending ω_i to ω_0 (the existence of such diffeomorphism follows from the classical Darboux theorem). Consider the varieties $N^i = \Phi_i^{-1}(N)$. By Theorem **A** the tuple $N^1, \ldots, N^s$ is a final list of normal forms for problem **A**.

2.9. **Arnold's ghost invariant in terms of algebraic restrictions.** As we mentioned in section 1.4, for the case $N = A_k =$(1.1) Problem **A** was studied by V. Arnol'd in [A1] (for even k). In fact, the classification results in [A1] and the ghost invariant are already obtained by our method in examples given in sections 2.2 and 2.5. Since A_k is contained in a non-singular 2-manifold then Proposition 2.4 implies that the algebraic restriction to A_k of any 2-form on $\mathbb{R}^{2n}$ can be realized by a symplectic form provided $n \geq 2$. Therefore the results of sections 2.2, 2.5 imply that in the classification Problem **B** with $N = A_k \subset \mathbb{R}^{2n\geq 4}$ there are exactly $k+1$ orbits - the orbits of the k algebraic restrictions (2.1) and the orbit of the zero algebraic restriction.

This complete solution of Problem **B** can be easily transferred to solution of Problem **A** – the classification of symplectic A_k-singularities. The algebraic restrictions $[A_k]^i$ are represented by 2-forms which are not symplectic, but since they belong to $\left[\mathrm{Symp}(\mathbb{R}^{2n})\right]_{A_k}$ then they also can be represented by symplectic forms. For example the zero algebraic restriction can be represented by a symplectic form

$$\theta^k = dx_1 \wedge dx_3 + dx_2 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n}$$

and $[A_k]^i$ with $i < k$ can be represented by the symplectic form

$$\theta^i = x_1^i dx_1 \wedge dx_2 + \theta^k, \quad 1 \leq i \leq k-1.$$

Given a symplectic form ω fix a local diffeomorphism Φ_i bringing the symplectic form θ^i to ω, $i = 0, 1, \ldots, k$. Let $A_k^i = \Phi_i^{-1}(A_k)$. By Theorem **A** any singular curve in the symplectic space $(\mathbb{R}^{2n}, \omega)$ which is diffeomorphic to A_k is symplectomorphic to one and only one of the curves $A_k^0, \ldots, A_k^k$. This gives us the classification result obtained in [A1].

The geometric meaning of this classification, explained in [A1], is also one of the applications of the method of algebraic restrictions, as it will be showed below.

2.10. **The geometric meaning of the zero algebraic restriction.** Theorem 1.1 easily implies that if N_1, N_2 are any diffeomorphic subsets of non-singular Lagrangian submanifolds in a fixed symplectic space then N_1 and N_2 are symplectomorphic. How to check if a subset of a symplectic manifold is contained in a non-singular Lagrangian submanifold?

Theorem B. *A quasi-homogeneous set N of a symplectic space $(\mathbb{R}^{2n}, \omega)$ is contained in a non-singular Lagrangian submanifold if and only if the symplectic form ω has zero algebraic restriction to N.*

Example 2.8. Let C be a curve in a symplectic space $(\mathbb{R}^{2n \geq 4}, \omega)$ which is diffeomorphic to A_k. Let A_k^k be the curve defined in section 2.9. By Theorem **B** the curve C is contained in a non-singular Lagrangian submanifold if and only if it is symplectomorphic to A_k^k.

Arnol'd also introduced a symplectic invariant characterizing how far is a curve of the class A_k from the closest non-singular Lagrangian submanifold. In the next subsection we show that this invariant can be generalized and expressed in terms of algebraic restrictions.

2.11. **Index of isotropness.** In terms of algebraic restrictions one can express the following symplectic invariant. Given a differential form germ ω with zero $(k-1)$-jet and non-zero k-jet we will say that k is the order of vanishing of ω. If $\omega(0) \neq 0$ then the order of vanishing is 0. If $\omega = 0$ or, in the C^∞-category, ω has the zero Taylor expansion, then the order of vanishing is ∞.

Definition 2.9. Let N be a subset of a symplectic space $(\mathbb{R}^{2n}, \omega)$. The index of isotropness of N is the maximal order of vanishing of the 2-forms $\omega|_{TM}$ over all non-singular submanifolds M containing N.

It is easy to prove that an equivalent definition is as follows: the index of isotropness is the maximal order of tangency between non-singular submanifolds containing N and non-singular isotropic submanifolds of the same dimension. The index of isotropness is equal to 0 if N is not contained in any non-singular submanifold which is tangent to some isotropic submanifold of the same dimension. If N is contained in a non-singular Lagrangian submanifold then the index of isotropness is ∞. (In the analytic category "if" can be replaced by "if and only if").

Theorem C. *The index of isotropness of a quasi-homogeneous variety N in a symplectic space $(\mathbb{R}^{2n}, \omega)$ is equal to the maximal order of vanishing of closed 2-forms representing the algebraic restriction $[\omega]_N$.*

Example 2.10. (cf. results in [A1]). Let A_k^i be the curves in a symplectic space $(\mathbb{R}^{2n}, \omega)$ defined in section 2.9. By Theorem **C** the index of isotropness of A_k^i is equal to i if $i \leq k-1$ and the index of isotropness of A_k^k (the curve which is contained in a non-singular Lagrangian submanifold) is ∞.

2.12. **Symplectic multiplicity.** One more invariant which can be effectively described in terms of algebraic restrictions is the symplectic multiplicity of a variety in a symplectic space. This invariant, generalizing the symplectic defect of a parametrized curve [IJ1], is defined below. At first let us fix the definition of a variety and one of equivalent definitions of the (usual) multiplicity of a variety. Recall that the zero set of an ideal I in the ring of function germs $(\mathbb{R}^m, 0) \to \mathbb{R}$

is the subset of $\mathbb{R}^m$ consisting of points at which any function in I vanishes. The ideal has the property of zeros if it contains any function vanishing on its zero set. Throughout the paper *by a variety in* $\mathbb{R}^m$ *we mean the zero set of a* k*-generated ideal having the property of zeros,* $k \geq 1$.

Definition 2.11 (cf. [T], [AVG])**.** Denote by $\mathrm{Var}(k,m)$ the space of all varieties described by k-generated ideals. Given $N \in \mathrm{Var}(k,m)$ denote by (N) the orbit of N with respect to the group of local diffeomorphisms. The multiplicity (or Tjurina number) of N is the codimension of (N) in $\mathrm{Var}(k,m)$.

To make this definition precise one should associate with N a map germ $H : (\mathbb{R}^m, 0) \to (\mathbb{R}^k, 0)$ whose k components are generators of the ideal of functions vanishing on N. Then the orbit (N) can be identified with the orbit of H with respect to the V-equivalence, see [AVG]. Recall from [AVG] that the V-equivalence of two map germs $H, \tilde{H} : (\mathbb{R}^m, 0) \to (\mathbb{R}^k, 0)$ means the existence of a local diffeomorphism Φ and a germ M of a map from $\mathbb{R}^m$ to the manifold of non-singular $k \times k$ matrices such that $\tilde{H} = M \cdot H(\Phi)$.

A variety $N \in \mathrm{Var}(k,m)$ is called a complete intersection singularity if k is the depth of the ideal of functions vanishing on N. (In the holomorphic category this means that k is the codimension of N in $\mathbb{C}^m$). *If* N *is not a complete intersection singularity then its multiplicity is* ∞. This follows from the fact that the set of k-tuples of function germs generating an ideal of depth $\neq k$ has infinite codimension in the space of all k-tuples of function germs.

In view of Definition 2.11 we define the symplectic multiplicity of a variety in a symplectic space as follows.

Definition 2.12. Let N be a variety in a symplectic space $(\mathbb{R}^{2n}, \omega)$. Let (N) be the orbit of N with respect to the group of local diffeomorphisms and let $(N)^{\mathrm{symp}}$ be the orbit of N with respect to the group of local symplectomorphisms. The symplectic multiplicity of N is the codimension of $(N)^{\mathrm{symp}}$ in (N).

To make this definition precise take, as above, a map germ $H : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^k, 0)$ whose components generate the ideal of functions vanishing on N. Let $(H)_V$ be the orbit of H with respect to the V-equivalence and let $(H)_{V,\mathrm{symp}}$ be the orbit of H with respect to the V-symplectic-equivalence. The V-symplectic-equivalence is defined in the same way as the V-equivalence; the only difference is that we require that Φ (the change of coordinates in the source space) is a local symplectomorphism. The codimension of $(N)^{\mathrm{symp}}$ in (N) is the codimension of $(H)_{V,\mathrm{symp}}$ in $(H)_V$.

The classical Darboux theorem implies another equivalent definition of the symplectic multiplicity of $N \subset (\mathbb{R}^{2n}, \omega)$: it is the codimension of the orbit of ω with respect to the group of local symmetries of N in the space of all closed 2-forms. Therefore Theorem **A** implies the following statement.

Theorem D. (corollary of Theorem **A**). *The symplectic multiplicity of a quasi-homogeneous variety in a symplectic space* $(\mathbb{R}^{2n}, \omega)$ *is equal to the codimension of the orbit of the algebraic restriction* $[\omega]_N$ *with respect to the group of local symmetries of* N *in the space* $\left[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})\right]_N$.

Example 2.13. Let A_k^i be the curves in a symplectic space $(\mathbb{R}^{2n}, \omega_0)$ defined in section 2.9. In section 2.5 we proved that the algebraic restriction $c_0[A_k]^0 + \cdots + c_{k-1}[A_k]^{k-1}$ is diffeomorphic to $[A_k]^p$ if and only if $c_1 = \cdots = c_{p-1} = 0$ and $c_p \neq 0$. Therefore by Theorem **D** the symplectic multiplicity of the curve A_k^i is equal to i.

This holds for all $i \leq k$ (the curve A^k_k corresponds to the zero algebraic restriction, i.e. to the case $c_0 = \cdots = c_{k-1} = 0$).

2.13. **The dimension of the space** $\big[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})\big]_N$**.** In view of results of the previous subsections it is worth to present several general results on the number

$$s(N) = \dim\big[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})\big]_N.$$

Theorem 2.14. *Let N be a quasi-homogeneous variety in a symplectic space of dimension $2n$ such that $s(N) < \infty$. The symplectic multiplicity of N does not exceed $s(N)$. It is equal to $s(N)$ if and only if N is contained in a non-singular Lagrangian submanifold.*

Proof. The first statement is a corollary of Theorem **D**. The second statement follows from Theorems **B**, **D** and the following statement: if $a \in \big[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})\big]_N$ and $a \neq 0$ then the orbit of a with respect to the group of symmetries of N has dimension ≥ 1. To prove this statement it suffices to note that in the quasi-homogeneous coordinates (see Definition 2.6) the flow $x_i \to e^{-\lambda_i t}x_i$ preserves N and brings a to a family of algebraic restrictions a_t such $a_t \to 0$ as $t \to \infty$. $\square$

It is easy to prove that if N is a stratified submanifold of dimension ≥ 2 (i.e. at least one of the strata has dimension ≥ 2) then the space consisting of the pullbacks to the regular part N^{reg} of N of all possible closed 2-forms on $\mathbb{R}^{2n}$ is infinite-dimensional. Since two 2-forms on $\mathbb{R}^{2n}$ with the same algebraic restriction to N have the same pullback to N^{reg} (see Proposition 2.1) then we obtain

Proposition 2.15. *If N is a stratified submanifold of dimension bigger than* 1 *then $s(N) = \infty$.*

Within 1-dimensional stratified submanifolds N consider at first the case that N is a complete intersection singularity.

Proposition 2.16 (real-analytic category; corollary of results by Greuel [Gr1])**.** *Let $N \subset \mathbb{R}^{2n}$ be a one-dimensional complete intersection singularity with finite Tjurina number (multiplicity) $\tau(N)$. If N is quasi-homogeneous then $s(N) = \tau(N)$.*

In fact, Greuel proved a much more general statement in the holomorphic category [Gr1]: if $N \subset \mathbb{C}^k$ is an isolated complete intersection singularity of dimension m then the Milnor number of N is equal to the dimension of the space $[\Lambda^m(\mathbb{C}^k)]_N/d\big([\Lambda^{m-1}(\mathbb{C}^k)]_N\big)$. Greuel also proved [Gr1] that for any quasi-homogeneous isolated complete intersection singularity the Milnor number is equal to $\tau(N)$. In the case $m = 1$ these results of Greuel imply that for any N satisfying the assumptions in Proposition 2.16 one has $\tau(N) = \dim\ [\Lambda^1(\mathbb{R}^{2n})]_N/d\big([\Lambda^0(\mathbb{R}^{2n})]_N\big)$. Now we use one more time the quasi-homogeneity of N. By Theorem 2.7 one has $H^2(N, \mathbb{R}^{2n}) = \{0\}$. This implies that the space $[\Lambda^1(\mathbb{R}^{2n})]_N/d\big([\Lambda^0(\mathbb{R}^{2n})]_N\big)$ is isomorphic to $\big[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})\big]_N$. Consequently $s(N) = \tau(N)$.

We do not know a direct proof of Theorem 2.16. We neither know if the assumption that N is quasi-homogeneous can be removed. Our results in section 4.1 show that it can be removed if N is a planar curve.

Conjecturally $s(N) < \infty$ for **any** 1-dimensional stratified submanifold $N \subset \mathbb{R}^{2n}$.

Example 2.17. Let $N_1, ..., N_p, p \geq 2$ be non-singular 1-dimensional submanifolds of $\mathbb{R}^{2n}$ such that $\dim(T_0N_1 + \cdots + T_0N_p) = p$. Let $N = N_1 \cup \cdots \cup N_p$. The ideal of functions vanishing on N is k-generated with $k = p(p-1)/2 + 2n - p$. One has $k > \operatorname{codim} N = 2n - 1$ unless $p = 2$. Therefore if $p \geq 3$ then N is not a complete intersection singularity and the multiplicity of N is ∞. On the other hand $s(N) < \infty$ for any p. Our results in section 7 imply that two closed 2-forms have the same algebraic restriction to N if and only if they have the same restriction to the p-space $T_0N_1 + \cdots + T_0N_p$. Therefore $s(N) = p(p-1)/2$.

2.14. **Calculation of the set** $\left[\mathrm{Symp}(\mathbb{R}^{2n})\right]_N$**.** The space $\left[\Lambda^2(\mathbb{R}^{2n})\right]_N$ can be calculated using Proposition 2.3, see section 6.1. In this subsection we present a simple way for transitions $\left[\Lambda^2(\mathbb{R}^{2n})\right]_N \to \left[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})\right]_N \to \left[\mathrm{Symp}(\mathbb{R}^{2n})\right]_N$. At first let us distinguish the case where two or all of these spaces coincide.

Proposition 2.18. *Let $N \subset \mathbb{R}^{2n}$. If N is contained in a non-singular 2-dimensional submanifold then $\left[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})\right]_N = \left[\Lambda^2(\mathbb{R}^{2n})\right]_N$. If N is contained in a non-singular n-dimensional submanifold then $\left[\mathrm{Symp}(\mathbb{R}^{2n})\right]_N = \left[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})\right]_N$.*

The first statement follows from Proposition 2.4 and the fact that any 2-form on a 2-manifold is closed. The second statement follows from Theorem 2.19 below.

This transition $\left[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})\right]_N \to \left[\mathrm{Symp}(\mathbb{R}^{2n})\right]_N$ is equivalent to distinguishing closed 2-forms θ on $\mathbb{R}^{2n}$ whose algebraic restrictions to $N \subset \mathbb{R}^{2n}$ is realizable by symplectic structure, i.e. $[\theta]_N = [\omega]_N$ for some symplectic form ω.

Theorem 2.19. *Let $N \subset \mathbb{R}^{2n}$. Let r be the minimal dimension of non-singular submanifolds of $\mathbb{R}^{2n}$ containing N. Let M be one of such r-dimensional submanifolds. The algebraic restriction $[\theta]_N$ of a closed 2-form θ is realizable by a symplectic form on $\mathbb{R}^{2n}$ if and only if $\mathrm{rank}(\theta|_{T_0M}) \geq 2r - 2n$.*

Theorem 2.19 is an almost obvious corollary of Theorem 1.2, Proposition 2.4 and the following lemma.

Lemma 2.20. *Let $N \subset \mathbb{R}^m$. Let $W \subseteq T_0\mathbb{R}^m$ be the tangent space to some (and then any) non-singular submanifold containing N of minimal dimension within such submanifolds. If ω is a p-form with zero algebraic restriction to N then $\omega|_W = 0$.*

Proof. Fix a non-singular submanifold M containing N of minimal dimension within such submanifolds (then $W = T_0M$). By Proposition 2.4 the form $\omega|_{TM}$ also has zero algebraic restriction to N and consequently it can be expressed in the form $\alpha + d\beta$, where α and β are forms on M vanishing at any point of N. Since N is not contained in any non-singular hypersurface of M then any function vanishing on N has zero 1-jet at 0. It follows that $d\beta(0) = 0$ and then $(\omega|_{TM})(0) = 0$. $\Box$

Now we give an algorithm for the transition $\left[\Lambda^2(\mathbb{R}^{2n})\right]_N \to \left[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})\right]_N$ under the assumptions that N is quasi-homogeneous and the space $\left[\Lambda^2(\mathbb{R}^{2n})\right]_N$ is finite-dimensional. (See section 6.1 where this algorithm is realized for the case $N = S_5$). Take any basis $a_1, \dots, a_k$ of $\left[\Lambda^2(\mathbb{R}^{2n})\right]_N$ and consider the algebraic restrictions $da_1, \dots, da_k \in \left[\Lambda^3(\mathbb{R}^{2n})\right]_N$. Let p be the dimension of the vector space over $\mathbb{R}$ spanned by these algebraic restrictions. The case $p = 0$ is *not* excluded. Change the order in the tuple $a_1, \dots, a_k$ so that

(a) the algebraic restrictions $da_1, \dots, da_p$ are linearly independent.

Replace now the algebraic restrictions $a_i, p < i \leq k$ by $a_i + \sum_{j=1}^p k_{ij}a_j$ with suitable $k_{ij} \in \mathbb{R}$ so that

(b) $da_{p+1} = \cdots = da_k = 0$.

Theorem 2.21. *Let N be a quasi-homogeneous subset of $\mathbb{R}^{2n}$ and let $a_1, \ldots, a_k$ be a basis of $\left[\Lambda^2(\mathbb{R}^{2n})\right]_N$ satisfying (a) and (b). Then $a_{p+1}, \ldots, a_k$ is a basis of the space $\left[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})\right]_N$.*

Proof. To conclude that the algebraic restrictions $a_{p+1}, \ldots, a_k$ span the space $\left[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})\right]_N$ we do not need the assumption that N is quasi-homogeneous. In fact, the algebraic restriction to N of any closed 2-form ω can be expressed in the form $[\omega]_N = c_1 a_1 + \cdots + c_k a_k$, and taking the differential of this relation we obtain $0 = c_1 da_1 + \cdots + c_p da_p$. By (a) $c_1 = \cdots = c_p = 0$, i.e. $[\omega]_N \in \mathrm{span}(a_{p+1}, \ldots, a_k)$.

The quasi-homogeneity of N is required in order to prove that $a_{p+1}, \ldots, a_k \in \left[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})\right]_N$, i.e. that the algebraic restrictions $a_{i>p}$ can be represented by closed 2-forms. In what follows $i = p+1, \ldots, k$. Take any 2-forms ω_i representing a_i. Since N is quasi-homogeneous then by Theorem 2.7 the cohomology group $H^3(N, \mathbb{R}^m)$ vanishes. This means that any closed 3-form with zero algebraic restriction to N, in particular the 3-forms $d\omega_i$, is a differential of some 2-form with zero algebraic restriction to N. Therefore $d\omega_i = d\widetilde{\omega}_i$, where $[\widetilde{\omega}_i]_N = 0$. The 2-form $\omega_i - \widetilde{\omega}_i$ is closed because $d\omega_i = d\widetilde{\omega}_i$. It represents the algebraic restriction a_i: since $[\widetilde{\omega}_i]_N = 0$ then $a_i = [\omega_i]_N = [\omega_i - \widetilde{\omega}_i]_N$. $\square$

3. Proof of Theorems A, B, and C

In section 3.1 we reduce Theorem **A**, (i) to the case that the symplectic forms ω_0 and ω_1 in this theorem satisfy the condition $(\omega_0 - \omega_1)(0) = 0$. In this case Theorem **A**, (i) can be easily proved by the homotopy method (section 3.2). Theorem **B** is proved in section 3.3 using Theorem **A**, and Theorem **C** is proved in section 3.4 using Theorem **B**. Throughout the proof we use the following lemma.

Lemma 3.1. *Let ω be a closed 2-form on $\mathbb{R}^m$ with zero algebraic restriction to $N \subset \mathbb{R}^m$. Let $M \subseteq \mathbb{R}^m$ be a non-singular submanifold containing N of minimal possible dimension within such submanifolds. There exists a closed 2-form θ on $\mathbb{R}^m$ such that $\theta|_{TM} = \omega|_{TM}$, $[\theta]_N = 0$, and $\theta(0) = 0$.*

Proof. Let $\mu = \omega|_{TM}$. By Lemma 2.20 one has $\mu(0) = 0$. Let $\pi : \mathbb{R}^{2n} \to M$ be a submersion which is the identity on M. Let $\theta = \pi^*\mu$. Then θ is a closed 2-form which vanishes at 0 and whose restriction to TM coincides with that of ω. Since $[\omega]_N = 0$ and $\omega|_{TM} = \theta|_{TM}$ then by Proposition 2.4 we obtain $[\theta]_N = 0$. $\square$

3.1. Reduction of Theorem A, (i) to the case $(\omega_0 - \omega_1)(0) = 0$.

Take a non-singular submanifold M as in Lemma 3.1. By this lemma there exists a closed 2-form θ such that $\theta|_{TM} = \omega_0|_{TM} - \omega_1|_{TM}$, $[\theta]_N = 0$ and $\theta(0) = 0$. Set $\widetilde{\omega} = \omega_1 + \theta$. Then $\omega_0, \omega_1, \widetilde{\omega}$ have the following properties: (a) $\widetilde{\omega}$ is symplectic (since $\theta(0) = 0$); (b) $\widetilde{\omega}|_{TM} = \omega_0|_{TM}$; (c) $[\widetilde{\omega}]_N = [\omega_1]_N$, $(\widetilde{\omega} - \omega_1)(0) = 0$. By Theorem 1.1 there exists a local diffeomorphism preserving M pointwise (and consequently preserving N pointwise) and bringing $\widetilde{\omega}$ to ω_0. Therefore Theorem **A**, (i) for the forms ω_0 and ω_1 will be proved if we prove it for the forms ω_1 and $\widetilde{\omega}$.

3.2. Proof of Theorem A, (i) in the case $(\omega_0 - \omega_1)(0) = 0$.

We will prove the existence of a family of diffeomorphisms Φ_t preserving pointwise N and bringing the form $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$ to the form ω_0, for any $t \in [0,1]$. This family will be found within families satisfying the ODE $\frac{d\Phi_t}{dt} = V_t(\Phi_t)$, $\Phi_0 = id$, where V_t is a family of vector fields on $\mathbb{R}^{2n}$ vanishing at any point of N. (The latter implies that Φ_t preserves N pointwise). Let $\mathcal{L}_V$ be the Lie derivative along a vector field V.

The requirement $\Phi_t^*\omega_t = \omega_0$ is equivalent to the condition $\mathcal{L}_{V_t}\omega_t + \frac{d\omega_t}{dt} = 0$. Since ω_t is a closed 2-form we obtain the equation

$$d(V_t \rfloor(\omega_0 + t(\omega_1 - \omega_0))) = \omega_0 - \omega_1 \tag{3.1}$$

with respect to the family V_t under the constraint that V_t vanishes at points of N. Since N is quasi-homogeneous then by Theorem 2.7 $\omega_0 - \omega_1 = d\beta$, where β is a 1-form vanishing at any point of N. Therefore to solve (3.1) it suffices to solve the equation

$$V_t \rfloor(\omega_0 + t(\omega_1 - \omega_0)) = \beta. \tag{3.2}$$

This equation can be treated as a square system of linear equations parametrized by a point $x \in \mathbb{R}^{2n}$ close to 0 and $t \in [0, 1]$. The assumption $(\omega_0 - \omega_1)(0) = 0$ implies $(\omega_0 + t(\omega_1 - \omega_0))(0) = \omega_0(0)$. The form ω_0 is symplectic and consequently the 2-form $(\omega_0 + t(\omega_1 - \omega_0))$ has maximal rank $2n$ for any t at any point x close to 0. Therefore for any such t and x the matrix of the linear system (3.2) is non-degenerate and consequently (3.2) has a unique solution V_t. It vanishes at any point of N since so does the 1-form β.

3.3. **Proof of Theorem B.** The "if" part of Theorem **B** follows from Proposition 2.4. Let us prove the "only if" part: if $[\omega]_N = 0$ then N is contained in a non-singular Lagrangian submanifold. Fix a non-singular submanifold M and a closed 2-form θ as in Lemma 3.1. Since $\theta(0) = 0$ then the form $\omega - \theta$ is symplectic. The manifold M is isotropic with respect to $\omega - \theta$. By Theorem **A** ,(i) there exists a local diffeomorphism sending $\omega - \theta$ to ω preserving N. It sends M to a non-singular submanifold $\widetilde{M}$ which contains N and which is isotropic with respect to ω.

3.4. **Proof of Theorem C.** We have to prove the following two statements:

1. If $M \subset (\mathbb{R}^{2n}, \omega)$ is a non-singular submanifold containing N and such that the restriction $\omega|_{TM}$ has zero k-jet, $k \geq 0$, then there exists a closed 2-form $\widetilde{\omega}$ on $\mathbb{R}^{2n}$ with zero k-jet such that $[\omega]_N = [\widetilde{\omega}]_N$;
2. If $\widetilde{\omega}$ is a closed 2-form on $\mathbb{R}^{2n}$ with zero k-jet, $k \geq 0$, such that $[\omega]_N = [\widetilde{\omega}]_N$ then there exists a non-singular submanifold $M \subset \mathbb{R}^{2n}$ containing N such that the restriction $\omega|_{TM}$ has zero k-jet.

To prove the first statement fix a submersion $\pi : \mathbb{R}^{2n} \to M$ which is the identity on M and set $\widetilde{\omega} = \pi^*(\omega|_{TM})$. Then $\widetilde{\omega}$ is a closed 2-form on $\mathbb{R}^{2n}$ with zero k-jet. The forms ω and $\widetilde{\omega}$ have the same restriction to TM and by Proposition 2.4 the same algebraic restriction to N. Therefore $\widetilde{\omega}$ is a required closed 2-form.

To prove the second statement consider the form $(\omega - \widetilde{\omega})$. It is symplectic and it has zero algebraic restriction to N. By Theorem **B** N is contained in a non-singular submanifold M such that $(\omega - \widetilde{\omega})|_{TM} = 0$. Since $\widetilde{\omega}$ has zero k-jet then its restriction to TM and consequently the restriction of ω to TM also has zero k-jet.

4. Symplectic classification of singular planar quasi-homogeneous curves

By a singular planar quasi-homogeneous curve in $\mathbb{R}^{2n}$ we mean a curve given *in suitable coordinates* by the equations

$$N = \{H(x_1, x_2) = x_{\geq 3} = 0\} \subset \mathbb{R}^{2n} \tag{4.1}$$

where the function germ $H(x_1, x_2)$ satisfies the following conditions:

1. $H(0)=0$, $dH(0)=0$;

2. the property of zeros: the ideal of functions on $\mathbb{R}^2$ vanishing at any point of the set $\{H=0\}$ is generated by H;

3. the function $H(x_1,x_2)$ is a quasi-homogeneous polynomial. This means that there exist positive numbers λ_1,λ_2 (weights of quasi-homogeneity) and a positive number d (degree of quasi-homogeneity) such that $H(x_1,x_2)$ is a linear combination of monomials $x_1^{\alpha_1}x_2^{\alpha_2}$ satisfying the condition $\alpha_1\lambda_1+\alpha_2\lambda_2=d$.

The classical examples are the simple function germs A_k, D_k, E_6, E_7, E_8, see [AVG]. In section 4.1 we prove that the vector space $\left[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})\right]_N$ can be identified with the local algebra of the function $H(x_1,x_2)$. In section 4.2 we use this result and Theorems **C** and **D** to give a simple way of calculating the index of isotropness and the symplectic multiplicity of any planar quasi-homogeneous curve. In sections 4.3 - 4.4 we use the method of algebraic restrictions to present a complete symplectic classification of the A_k, D_k, E_6, E_7, E_8 singularities.

4.1. **The space of algebraic restrictions and the local algebra of H.** Theorem 4.2 below generalizes Example 2.2.

Definition 4.1. (see [AVG]). The factor space $\Lambda^0(\mathbb{R}^2)/(\nabla H)$ is called the *local algebra of* H and the dimension of this factor space is called the *multiplicity* of H.

Theorem 4.2 (cf. Theorem 2.16)**.** *Let $N=\{H(x_1,x_2)=x_{\geq 3}=0\}$ be a planar quasi-homogeneous curve where the function $H=H(x_1,x_2)$ has a finite multiplicity μ and let the tuple $f_1,f_2,\dots,f_\mu$ be a basis of the local algebra of H* [1] *such that $f_1(0)\neq 0, f_{\geq 2}(0)=0$.*

(i) $\left[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})\right]_N$ is a μ-dimensional vector space spanned by the algebraic restrictions $a_i=[f_i dx_1\wedge dx_2]_N$, $i=1,\dots,\mu$.
(ii) If $n\geq 2$ then $\left[\mathrm{Symp}(\mathbb{R}^{2n})\right]_N=\left[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})\right]_N$. The manifold $\left[\mathrm{Symp}(\mathbb{R}^{2})\right]_N$ consists of algebraic restrictions of the form $\left\{c_1a_1+\cdots+c_\mu a_\mu,\ c_1\neq 0\right\}$.

The second statement is a corollary of the first one and results in section 2.14. The first statement follows from Lemma 4.3 below and Proposition 2.4.

Lemma 4.3. *Let $H(x_1,x_2)$ be a quasi-homogeneous polynomial with the property of zeros. A 2-form $f(x_1,x_2)dx_1\wedge dx_2$ has zero algebraic restriction to the curve $\{H(x_1,x_2)=0\}$ if and only if $f\in(\nabla H)$.*

Proof. Since the function H has the property of zeros then for some function germs $A(x_1,x_2), B_1(x_1,x_2), B_2(x_1,x_2)$ one has

$$f dx_1\wedge dx_2 = HA dx_1\wedge dx_2 + d\left(H(B_1dx_1+B_2dx_2)\right).$$

It is easy to see that this condition is equivalent to the condition $f\in(H,\nabla H)$, where $(H,\nabla H)$ is the ideal generated by the H and its first order partial derivatives. It is clear that any quasi-homogeneous polynomial belongs to its gradient ideal, therefore $(\nabla H,H)=(\nabla H)$. □

Remark. If H is not quasi-homogeneous then, as we see from the proof of Lemma 4.3, the space $\left[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})\right]_N$ can be identified with the space $\Lambda^0(\mathbb{R}^2)/(\nabla H,H)$.

[1]after factorization of these function germs by the ideal (∇H).

The dimension τ of the latter space is called the Tjurina number (or the multiplicity) of *the curve* $\{H = 0\}$ (see Definition 2.11). By Saito's theorem [Sa1] $\tau < \mu$. [2]

4.2. **The index of isotropness and the symplectic multiplicity.** The index of isotropness and the symplectic multiplicity are defined in sections 2.11 and 2.12.

Theorem 4.4. *Let $N = \{H(x_1, x_2) = x_{\geq 3} = 0\}$ be a singular planar quasi-homogeneous curve in a symplectic space $(\mathbb{R}^{2n}, \omega)$. Let μ be the multiplicity of the function H.*

(i) The index of isotropness of N does not exceed $(\mu - 1)$ unless N is contained in a non-singular Lagrangian submanifold (in the latter case the index is ∞).

(ii) The symplectic multiplicity of N does not exceed μ. It is equal to μ if and only if N is contained in a non-singular Lagrangian submanifold.

The second statement is a direct corollary of Theorems 2.14 and 4.2. (It is also a direct corollary of Theorems **B** and 4.2). The first statement follows from the following corollary of Theorems 2.5, **C** and Lemma 4.3 allowing to calculate the index of isotropness for any planar quasi-homogeneous curve.

Notation. Given a 2-form ω on $\mathbb{R}^{2n}$ denote by $F_\omega = F_\omega(x_1, x_2)$ a function germ such that the pullback of ω to the 2-plane $x_{\geq 3} = 0$ has the form $F_\omega dx_1 \wedge dx_2$.

Theorem 4.5 (Corollary of Theorems 2.5, **C** and Lemma 4.3)**.**
Let N be as in Theorem 4.4. The index of isotropness of N is the maximal p such that $F_\omega \in (\nabla H) + \mathcal{M}^p$, where $\mathcal{M}$ denotes the maximal ideal in the ring of function germs on $\mathbb{R}^2$ (if $F_\omega \in (\nabla H)$ then $p = \infty$, if $F_\omega(0) \neq 0$ then $p = 0$).

Proof of Theorem 4.4, (i). If N is not contained in a non-singular Lagrangian submanifold then by Theorem **B** $[\omega]_N \neq 0$ and then by Proposition 2.4 and Lemma 4.3 one has $F_\omega \notin (\nabla H)$. Since $\mathcal{M}^\mu \subset (\nabla H)$ (see [AVG]) then $F_\omega \notin (\nabla H) + \mathcal{M}^\mu$ and by Theorem 4.5 the index of isotropness does not exceed $(\mu - 1)$. $\square$

The following theorem gives a simple way for calculation of the symplectic multiplicity of any planar quasi-homogeneous curve.

Theorem 4.6. *Let N be as in Theorem 4.4. The symplectic multiplicity of N is equal to* $\dim \Lambda^0(\mathbb{R}^2)/(\nabla H, F_\omega)$, *where $(\nabla H, F_\omega)$ is the ideal generated by the function germs $\partial H/\partial x_1, \partial H/\partial x_2, F_\omega$.*

Example 4.7. Consider the curve

$$C: \quad \{p_1^2 p_2 - p_2^3 = 0,\ q_1 = p_2^3,\ q_2 = 0\} \subset (\mathbb{R}^4, \omega_0 = dp_1 \wedge dq_1 + dp_2 \wedge dq_2).$$

This is a planar quasi-homogeneous curve diffeomorphic to the curve D_4: in the local coordinates $x_1 = p_1, x_2 = p_2, x_3 = q_1 - p_2^3, x_4 = q_2$ it takes the form $H(x_1, x_2) = x_1^2 x_2 - x_2^3 = x_3 = x_4 = 0$. In the same coordinates the form ω_0 takes the form $dx_1 \wedge (dx_3 + 3x_2^2 dx_2) + dx_2 \wedge dx_4$. The restriction of this form to the 2-surface $x_3 = x_4 = 0$ is $3x_2^2 dx_1 \wedge dx_2$. The ideal $(\nabla H, 3x_2^2) = (x_1 x_2, x_1^2 - 3x_2^2, x_2^2)$ coincides with the ideal $(x_1 x_2, x_1^2, x_2^2)$. By Theorems 4.5 and 4.6 the index of isotropness of C is equal to 2 and the symplectic multiplicity of C is equal to 3.

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[2] This means that if H is not quasi-homogeneous then the multiplicity of the curve $\{H = 0\}$ is smaller than the multiplicity of the function H. The number $\mu - \tau$ is called the degree of non-quasi-homogeneity of H, see [V].

The proof of Theorem 4.6 consists of several steps. At first we use Theorems **D** and 2.5 reducing Theorem 4.6 to the following proposition.

Proposition 4.8. *Let $H(x_1, x_2)$ be a quasi-homogeneous polynomial of finite multiplicity having the property of zeros. The codimension in the space $\left[\Lambda^2(\mathbb{R}^2)\right]_{\{H=0\}}$ of the orbit of the algebraic restriction $[F(x_1, x_2)dx_1 \wedge dx_2]_{\{H=0\}}$ with respect to the group of symmetries of the curve $\{H = 0\}$ is equal to the dimension of the factor space $\Lambda^0(\mathbb{R}^2)/(\nabla H, F)$.*

Notation. Given an algebraic restriction $a \in \left[\Lambda^2(\mathbb{R}^{2n})\right]_N$ denote by $T(a)$ the tangent space at a to the orbit of a with respect to the group of symmetries of N.

Proposition 4.9. *Let H be as in Proposition 4.8 and let $a \in \left[\Lambda^2(\mathbb{R}^2)\right]_{\{H=0\}}$. Then* $\dim T(a) = \dim\ \left(\Lambda^0(\mathbb{R}^2) \cdot a\right)$.

If a is represented by 2-form $F(x_1, x_2)dx_1 \wedge dx_2$ then by Theorem 4.2 one has $\mathrm{codim}\ \left(\Lambda^0(\mathbb{R}^2) \cdot a\right) = \dim\Lambda^0(\mathbb{R}^2)/(\nabla H, F)$. Therefore Proposition 4.9 and Theorem 4.2 imply Proposition 4.8 and consequently Theorem 4.6. The proof of Proposition 4.9 requires certain techniques related to quasi-homogeneous algebraic restrictions, therefore it is postponed to section 5.

4.3. **Symplectic A-D-E classification.** Continuing results of section 2.9 we give a complete solution of Problem **A** with $N = \{H(x_1, x_2) = x_{\geq 3} = 0\}$ where $H(x_1, x_2)$ is a function representing one of the classical singularities A_k, D_k, E_6, E_7, E_8, see Table 1. Theorems **A** and 2.5 reduce Problem **A** to classification of algebraic restrictions of the space $\left[\Lambda^2(\mathbb{R}^2)\right]_{\{H=0\}}$ with respect to the group of symmetries of the curve $\{H = 0\} \subset R^2$. This classification involves functions and families of functions given in the second column of Table 1.

$H(x_1, x_2)$	$F_i(x_1, x_2),\ i = 0, 1, \ldots, \mu$
$A_k : x_1^{k+1} - x_2^2$ $k \geq 1$	$F_0 = 1$ $F_i = x_1^i,\ i = 1, \ldots, k-1$ $F_k = 0$
$D_k : x_1^2 x_2 - x_2^{k-1}$ $k \geq 4$	$F_0 = 1$ $F_i = bx_1 + x_2^i,\ \ i = 1, \ldots, k-4$ $F_{k-3} = (\pm 1)^k x_1 + bx_2^{k-3}$, $F_{k-2} = x_2^{k-3},\ F_{k-1} = x_2^{k-2},\ F_k = 0$
$E_6 : x_1^3 - x_2^4$	$F_0 = 1,\ F_1 = \pm x_2 + bx_1,\ F_2 = x_1 + bx_2^2$, $F_3 = x_2^2 + bx_1x_2,\ F_4 = \pm x_1x_2,\ F_5 = x_1x_2^2,\ F_6 = 0$
$E_7 : x_1^3 - x_1x_2^3$	$F_0 = 1,\ F_1 = x_2 + bx_1,\ F_2 = \pm x_1 + bx_2^2$, $F_3 = x_2^2 + bx_1x_2,\ F_4 = \pm x_1x_2 + bx_2^3$, $F_5 = x_2^3,\ F_6 = x_2^4,\ F_7 = 0$
$E_8 : x_1^3 - x_2^5$	$F_0 = \pm 1,\ F_1 = x_2 + bx_1,\ F_2 = x_1 + b_1x_2^2 + b_2x_2^3$ $F_3 = \pm x_2^2 + bx_1x_2,\ F_4 = \pm x_1x_2 + bx_2^3$, $F_5 = x_2^3 + bx_1x_2^2,\ F_6 = x_1x_2^2,\ F_7 = \pm x_1x_2^3,\ F_8 = 0$

Table 1. Classification of the algebraic restrictions to A_k, D_k, E_6, E_7, E_8.

Theorem 4.10. *Fix a function $H = H(x_1, x_2)$ in Table 1. Let $\mathcal{F}_i = [F_i dx_1 \wedge dx_2]_{\{H=0\}}$, where the functions F_i are given in the row of H.*

(i) Any algebraic restriction $a \in \left[\Lambda^2(\mathbb{R}^2)\right]_{\{H=0\}}$ is diffeomorphic to one of the normal forms $\mathcal{F}_i$, $i = 0, \dots, \mu$, where μ is the multiplicity of H.

(ii) The singularity classes defined by the normal forms $\mathcal{F}_0, \dots, \mathcal{F}_\mu$ are disjoint;

(iii) The singularity class defined by the normal form $\mathcal{F}_i$ has codimension i;

(iv) The parameters b, b_1, b_2 in the normal forms are moduli.

The second statement is proved in section 4.4, the other statements – in section 5. Let us transfer the normal forms $\mathcal{F}_i$ to symplectic normal forms following the algorithm in section 2.8. Fix any symplectic form, for example,

$$\omega_0 = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n.$$

If $n \geq 2$ then the algebraic restriction $[F_i(x_1, x_2)dx_1 \wedge dx_2]_N$ can be realized by the symplectic form $\omega_i = F_i dx_1 \wedge dx_2 + dx_1 \wedge dx_3 + dx_2 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n}$ which can be brought to ω_0 by the change of coordinates

$x_1 = p_1, \ x_2 = p_2, \ x_3 = q_1 - \int_0^{p_2} F_i(p_1, t)dt, \ x_4 = q_2,$

$x_5 = p_3, \ x_6 = q_3, \dots, \ x_{2n-1} = p_n, \ x_{2n} = q_n.$

The given change of coordinates brings $N =$(4.1) to the form

$$N^i = \left\{ H(p_1, p_2) = q_1 - \int_0^{p_2} F_i(p_1, t)dt = q_{\geq 2} = p_{\geq 3} = 0 \right\} \subset \left(\mathbb{R}^{2n}, \omega_0\right). \tag{4.2}$$

Theorems **A**, (ii), 2.5 and 4.10 imply the following complete symplectic classification of the A_k, D_k, E_6, E_7, E_8 singularities.

Theorem 4.11. *Fix a function $H = H(x_1, x_2)$ in Table 1. Any curve in the symplectic space $(\mathbb{R}^{2n}, \omega_0)$, $n \geq 2$, which is diffeomorphic to the curve $N : H(x_1, x_2) = x_{\geq 3} = 0$ can be reduced by a symplectomorphism to one and only one of the normal forms N^i, $i = 0, ..., \mu$, given by (4.2), where F_i are the functions in Table 1 and μ is the multiplicity of H. The parameters b, b_1, b_2 are symplectic moduli. The codimension of the symplectic singularity class defined by the normal form N^i in the class of all curves diffeomorphic to N is equal to i.*

If $n = 1$, i.e in the 2-dimensional case, the symplectic classification is much simpler. Theorems 4.2, (ii) and 4.10 along with Theorem **A**, (ii) imply the following

Theorem 4.12. *Let $H(x_1, x_2)$ be one of the functions in Table 1. All curves in the symplectic plane $(\mathbb{R}^2, dp \wedge dq)$ which are diffeomorphic to the curve $\{H = 0\}$ are symplectomorphic unless $H = E_8$. Any curve in $(\mathbb{R}^2, dp \wedge dq)$ which is diffeomorphic to $E_8 : \{x_1^3 - x_2^5 = 0\}$ is symplectomorphic to one of the curves $p^3 \pm q^5 = 0$.*

Remark. It is easy to prove that the curves $p^3 \pm q^5 = 0$ are not symplectomorphic. The statement of Theorem 4.12 also follows from the works [V] and [Gi]. It is also contained in the works [IJ1], [IJ2] along with other results on classification of curves in $\mathbb{R}^2$ with respect to volume-preserving diffeomorphisms.

4.4. **Distinguishing normal forms (proof of Theorem 4.10, (ii)).** The normal form N^i in Theorem 4.11 corresponds to the normal form $\mathcal{F}_i$ in Theorem 4.10. Using Table 1 and Theorems 4.5 and 4.6 it is easy to calculate the index of isotropness and the symplectic multiplicity of all singularities, see Table 2. They do not depend on the parameters of the normal forms except for the case $D_k^i, 2 \leq i \leq k-4$, when the index of isotropness is different for $b \neq 0$ and for $b = 0$.

As we see from Table 2, either the index of isotropness or the symplectic multiplicity distinguishes all normal forms except for the following two couples: (α) E_6^3 and E_6^4; (β) E_8^5 and E_8^6. To distinguish these normal forms we will distinguish the corresponding normal forms for algebraic restrictions:

(α) $[(x_2^2 + bx_1x_2)dx_1 \wedge dx_2]_{\{H=0\}}$ and $[\pm x_1x_2dx_1 \wedge dx_2]_{\{H=0\}}$, $H = x_1^3 - x_2^4$;

(β) $[(x_2^3 + bx_1x_2^2)dx_1 \wedge dx_2]_{\{H=0\}}$ and $[x_1x_2^2dx_1 \wedge dx_2]_{\{H=0\}}$, $H = x_1^3 - x_2^5$.

These couples can be distinguished as follows. Let $a = [F(x_1, x_2)dx_1 \wedge dx_2]_{\{H=0\}}$, where H is any quasi-homogeneous polynomial. Consider the ideal $(\nabla H, F)$. We will say that this ideal is associated with a. The associated ideals are invariantly related to algebraic restrictions: if $a, \widetilde{a} \in \left[\Lambda^2(\mathbb{R}^{2n})\right]_N$ are diffeomorphic then the associated ideals are diffeomorphic. This follows from Lemma 4.3 and the observation that any diffeomorphism sending a 2-form $Fdx_1 \wedge dx_2$ to $\widetilde{F}dx_1 \wedge dx_2$ sends the ideal generated by F to the ideal generated by $\widetilde{F}$. Therefore to distinguish the couples $(\alpha), (\beta)$ it suffices to distinguish the couples of associated ideals. In the case (α) the associated ideals are $I_\alpha^{(1)} = \left(x_1^2,\ x_2^3,\ x_2^2 + bx_1x_2\right)$ and $I_\alpha^{(2)} = \left(x_1^2,\ x_2^3,\ x_1x_2\right)$. In the case (β) they are $I_\beta^{(1)} = \left(x_1^2, x_2^4, x_2^3 + bx_1x_2^2\right)$ and $I_\beta^{(2)} = \left(x_1^2, x_2^4, x_1x_2^2\right)$. It is easy to prove that $I_\alpha^{(1)}$ is not diffeomorpic to $I_\alpha^{(2)}$ and $I_\beta^{(1)}$ is not diffeomorpic to $I_\beta^{(2)}$ (to prove this it suffices to consider the 2-jets of functions in the ideals $I_\alpha^{(1)}$ and $I_\alpha^{(2)}$ and the 2-jets of functions in the ideals $I_\beta^{(1)}$ and $I_\beta^{(2)}$).

Normal form	index of isotr.	sympl. multip.
A_k^i, $0 \leq i \leq k-1$	i	i
A_k^k	∞	k
D_k^0	0	0
D_k^1	1	2
D_k^i, $2 \leq i \leq k-4$	$b \neq 0:\ 1$; $b = 0:\ i$	$i+1$
D_k^{k-3}	1	$k-2$
D_k^{k-2}	$k-3$	$k-2$
D_k^{k-1}	$k-2$	$k-1$
D_k^k	∞	k
E_6^0	0	0
E_6^1	1	2
E_6^2	1	3
E_6^3	2	4
E_6^4	2	4
E_6^5	3	5
E_6^6	∞	6

Normal form	index of isotr.	sympl. multip.
E_7^0	0	0
E_7^1	1	2
E_7^2	1	3
E_7^3	2	4
E_7^4	2	5
E_7^5	3	5
E_7^6	4	6
E_7^7	∞	7
E_8^0	0	0
E_8^1	1	2
E_8^2	1	4
E_8^3	2	4
E_8^4	2	5
E_8^5	3	6
E_8^6	3	6
E_8^7	4	7
E_8^8	∞	8

Table 2. Symplectic invariants of A_k, D_k, E_6, E_7, E_8 singularities.

5. Proof of Proposition 4.9 and Theorem 4.10

Throughout this section, including formulations of the statements, $H = H(x_1, x_2)$ is a quasi-homogeneous polynomial with respect to the weights λ_1, λ_2. Any quasi-homogeneity should be understood as that with respect to the weights λ_1, λ_2. We also assume that H has the property of zeros and a finite multiplicity μ.

Proposition 4.9 is proved in sections 5.1 - 5.3. The main ingredients are the structure of the algebra of infinitesimal symmetries of the curve $\{H = 0\}$ (section 5.1) and the quasi-homogeneous algebraic restrictions (section 5.2). The same ingredients are used for the proof of Theorem 4.10, statements (i), (iii), (iv) in sections 5.4 - 5.6 (Theorem 4.10, (ii) is already proved in section 4.4).

5.1. **The infinitesimal symmetries of the curve $\{H = 0\}$.** An infinitesimal symmetry of the curve $\{H = 0\}$ is a vector field tangent to this curve. The space of all infinitesimal symmetries is an algebra with respect to the Lie bracket. It will be denoted by $\mathrm{Symm}^{\mathrm{inf}}(\{\mathrm{H} = 0\})$. [3] Consider the following Euler vector field E and the Hamiltonian vector field $\mathcal{H}$ related to H via the volume form $dx_1 \wedge dx_2$:

$$E = \lambda_1 x_1 \partial/\partial x_1 + \lambda_2 x_2 \partial/\partial x_2, \quad \mathcal{H} = (\partial H/\partial x_2)\partial/\partial x_1 - (\partial H/\partial x_1)\partial/\partial x_2.$$

The following lemma was used in many works, see for example [A1], [L].

Lemma 5.1. *Any vector field $V \in \mathrm{Symm}^{\mathrm{inf}}(\{\mathrm{H} = 0\})$ has the form $V = g_1 E + g_2 \mathcal{H}$ for some functions g_1, g_2.*

Proof. Since H has the property of zeros then $V(H) = RH$ for some function R. One has $E(H) = \delta \cdot H$, where δ is the degree of quasi-homogeneity of H. Let $V_1 = V - RE/\delta$. Then $V_1(H) = 0$. Let $V_1 = A\partial/\partial x_1 + B\partial/\partial x_2$, then $(Adx_2 - Bdx_1) \wedge dH = 0$. Since H has a finite multiplicity then the form dH has the division property (see, for example [M]) and this relation implies $Adx_2 - Bdx_1 = R_1 dH$ for some function R_1. This can be written in the form $V_1 = -R_1 \cdot \mathcal{H}$. We obtain $V = R_1 \cdot \mathcal{H} - RE/\delta$. □

By the following lemma the Hamiltonian part of the algebra $\mathrm{Symm}^{\mathrm{inf}}(\{\mathrm{H} = 0\})$ leads to the symmetries preserving any algebraic restriction in $\left[\Lambda^2(\mathbb{R}^2)\right]_{\{H=0\}}$. In what follows $\mathcal{L}_V$ denotes the Lie derivative along the vector field V.

Lemma 5.2. $\mathcal{L}_{g\mathcal{H}}(a) = 0$ *for any* $g \in \Lambda^0(\mathbb{R}^2)$ *and any* $a \in \left[\Lambda^2(\mathbb{R}^2)\right]_{\{H=0\}}$.

Proof. Let $F(x_1, x_2)$ be any function. Let $\theta = \mathcal{L}_{g\mathcal{H}}(F dx_1 \wedge dx_2)$. We have to prove $[\theta]_{\{H=0\}} = 0$. Note that $\mathcal{H} \rfloor dx_1 \wedge dx_2 = dH$. This implies $\theta = d(gF\mathcal{H} \rfloor dx_1 \wedge dx_2) = d(gF dH) = d(-Hd(gF))$. □

Recall that $T(a)$ denotes the tangent space at a to the orbit of an algebraic restriction a. Lemmas 5.1 and 5.2 imply the following statement.

Proposition 5.3. *Let* $a \in \left[\Lambda^2(\mathbb{R}^2)\right]_{\{H=0\}}$. *Then* $T(a) = \left\{\mathcal{L}_{gE}(a), \quad g \in \Lambda^0(\mathbb{R}^2)\right\}$.

[3]another notation for the same algebra is $Derlog(\{H = 0\})$, see for example [Sa2].

5.2. **Quasi-homogeneous algebraic restrictions.** Now we will calculate the tangent space $T(a)$ more explicitly. This requires working with quasi-homogeneous algebraic restrictions. The possibility to define quasi-homogeneous algebraic restrictions follows from the following lemma.

Notation. Given a function $F = F(x_1, x_2)$ denote by $F^{(\delta)}$ the quasi-homogeneous part of degree δ of its Taylor series with respect to the weights λ_1, λ_2.

Lemma 5.4. *If* $[Fdx_1 \wedge dx_2]_{\{H=0\}} = 0$ *then* $[F^{(\delta)}dx_1 \wedge dx_2]_{\{H=0\}} = 0$ *for any* δ.

Proof. Follows from Lemma 4.3 and the observation that $\partial H/\partial x_1, \partial H/\partial x_2$ are also quasi-homogeneous polynomials with respect to the weights λ_1, λ_2. □

Lemma 5.4 allows to define quasi-homogeneous algebraic restrictions as follows.

Definition 5.5. Let $F = F(x_1, x_2)$ and $a = [Fdx_1 \wedge dx_2]_{\{H=0\}}$. The algebraic restriction $a^{(\delta)} = [F^{(\delta-\lambda_1-\lambda_2)}dx_1 \wedge dx_2]_{\{H=0\}}$ will be called the quasi-homogeneous degree δ part of a. If $a = a^{(\delta)}$ then a is called quasi-homogeneous of degree δ.

Why $F^{(\delta-\lambda_1-\lambda_2)}$, not $F^{(\delta)}$ in the definition of $a^{(\delta)}$? This is so in order to have

Lemma 5.6. *If an algebraic restriction* $a \in \left[\Lambda^2(\mathbb{R}^2)\right]_{\{H=0\}}$ *is quasi-homogeneous of degree* δ *then* $\mathcal{L}_E(a) = \delta \cdot a$.

Proof. Let $a = [Fdx_1dx_2]_{\{H=0\}}$. Calculate the Lie derivative

$$\mathcal{L}_E\left(Fdx_1 \wedge dx_2\right) = d\left(E \rfloor Fdx_1 \wedge dx_2\right) = \mathcal{L}_E F dx_1 \wedge dx_2 + F\mathcal{L}_E(dx_1 \wedge dx_2).$$

It remains to note that $\mathcal{L}_E(dx_1 \wedge dx_2) = (\lambda_1+\lambda_2)dx_1 \wedge dx_2$ and $\mathcal{L}_E F = (\delta-\lambda_1-\lambda_2)F$ since F is quasi-homogeneous of degree $\delta - \lambda_1 - \lambda_2$. □

Lemma 5.7. *For any* $a \in \left[\Lambda^2(\mathbb{R}^2)\right]_{\{H=0\}}$ *the sum* $\sum_{\delta\in\mathbb{R}} a^{(\delta)}$ *is finite.*

Proof. Obviously $a^{(\delta)} = 0$ if $\delta < \lambda_1 + \lambda_2$ or if $\delta \neq \alpha_1\lambda_1 + \alpha_2\lambda_2$ for some positive integers α_1, α_2. Therefore we have to prove that $a^{(\delta)} = 0$ for sufficiently big δ. Let $\delta > \lambda_1 + \lambda_2 + \mu$, where μ is the multiplicity of H. Then $a^{(\delta)}$ has the form $[Fdx_1 \wedge dx_2]_{\{H=0\}}$, where the function F has zero μ-jet. Any such function belongs to the gradient ideal (∇H), see [AVG]. By Lemma 4.3 one has $a^{(\delta)} = 0$. □

5.3. **Proof of Proposition 4.9.** In view of Proposition 5.3 let us calculate the Lie derivative $\mathcal{L}_{gE}(a)$ for quasi-homogeneous g and a.

Lemma 5.8. *If* $a \in \left[\Lambda^2(\mathbb{R}^2)\right]_{\{H=0\}}$ *and* $g \in \Lambda^0(\mathbb{R}^2)$ *are quasi-homogeneous of degrees* δ_1 *and* δ_2 *then* $\mathcal{L}_{gE}(a) = (\delta_1 + \delta_2)ga$.

Proof. For any $\omega \in \Lambda^2(\mathbb{R}^2)$, and $g \in \Lambda^0(\mathbb{R}^2)$ and any vector field V on $\mathbb{R}^2$ one has $\mathcal{L}_{gV}\omega = g \cdot (\mathcal{L}_V(\omega)) + (\mathcal{L}_V(g)) \cdot \omega$. Therefore $\mathcal{L}_{gE}(a) = g \cdot \mathcal{L}_E(a) + (\mathcal{L}_E(g)) \cdot a$. One has $\mathcal{L}_E(g) = \delta_2 g$. By Lemma 5.6 $\mathcal{L}_E(a) = \delta_1 a$. □

Consider the linear operator

$$Q : \left[\Lambda^2(\mathbb{R}^2)\right]_{\{H=0\}} \to \left[\Lambda^2(\mathbb{R}^2)\right]_{\{H=0\}}, \quad Q(a) = \sum_{\delta\in\mathbb{R}} \delta \cdot a^{(\delta)}.$$

It is well-defined by Lemma 5.7. Proposition 5.3 and Lemma 5.8 imply

Proposition 5.9. $T(a) = Q\left(\Lambda^0(\mathbb{R}^2) \cdot a\right)$.

Since $a^{(0)} = 0$ (moreover $a^{(\delta)} = 0$ for $\delta < \lambda_1 + \lambda_2$) the operator Q is non-singular and consequently $\dim T(a) = \dim \left(\Lambda^0(\mathbb{R}^2) \cdot a\right)$.

5.4. **Proof of Theorem 4.10, (i).** The normal forms in Theorem 4.10 follow from Propositions 5.10 and 5.11 below. To formulate these propositions it is convenient to use the following notation.

Notation. Denote by $o(\delta)$ the subspace of the space $\left[\Lambda^2(\mathbb{R}^2)\right]_{\{H=0\}}$ consisting of algebraic restrictions without quasi-homogeneous terms of degree $\leq \delta$.

Proposition 5.10. *Let $a_1, \dots, a_\mu$ be a basis of the space $\left[\Lambda^2(\mathbb{R}^2)\right]_{\{H=0\}}$ consisting of quasi-homogeneous algebraic restrictions of degrees $\delta_1 \leq \delta_2 \leq \cdots \leq \delta_\mu$. Let $a = c_1a_1 + \cdots + c_\mu a_\mu$. If a_s belongs to the affine space $g\cdot(c_1a_1+\cdots+c_{s-1}a_{s-1})+o(\delta_s)$ for some function g such that $g(0) = 0$ then a is diffeomorphic to an algebraic restriction in the affine space $c_1a_1 + \cdots + c_{s-1}a_{s-1} + o(\delta_s)$.*

Proof. Let us show that a symmetry Ψ of the curve $\{H = 0\}$ reducing a to the required normal form is contained in the flow Φ^t of the vector field gE. Since $g(0) = 0$ and the degrees of quasi-homogeneity of $a_{\geq s}$ are not less than δ_s then by Lemma 5.8 one has $\mathcal{L}_{gE}a \in \mathcal{L}_{gE}(c_1a_1 + \cdots + c_{s-1}a_{s-1}) + o(\delta_s)$. Lemma 5.8 and the assumption of Proposition 5.10 imply $\mathcal{L}_{gE}a \in \delta_s a_s + o(\delta_s)$. It follows

$$d(\Phi^t)^*a/dt \in (\Phi^t)^*(\delta_s a_s + o(\delta_s)).$$

Note now that for any t the diffeomorphism Φ^t preserves the x_1 and the x_2-axes and since $g(0) = 0$ then Φ^t has identity linear approximation. These properties imply that Φ^t preserves the affine space $\delta_s a_s + o(\delta_s)$ and consequently

$$d(\Phi^t)^*a/dt \in \delta_s a_s + o(\delta_s).$$

Since $\Phi_0 = id$ it follows $(\Phi^t)^*a = a + t\delta_s a_s + o(\delta_s)$. Let $t_0 = -c_s/\delta_s$. Then $\Psi = \Phi^{t_0}$ is the required symmetry. □

To prove Theorem 4.10,(i) for all singularities except D_k it suffices to use the following corollary of Proposition 5.10.

Proposition 5.11. *Let $a_1, \dots, a_\mu$ be a basis of the space $\left[\Lambda^2(\mathbb{R}^2)\right]_{\{H=0\}}$ consisting of quasi-homogeneous algebraic restrictions of degrees $\delta_1 < \delta_2 < \cdots < \delta_\mu$. Any algebraic restriction of the form $a = c_pa_p + \cdots + c_\mu a_\mu$ with $c_p \neq 0$ is diffeomorphic to an algebraic restriction of the form $\widetilde{a} = c_pa_p + \widetilde{c}_{p+1}a_{p+1} + \cdots + \widetilde{c}_\mu a_\mu$, where $\widetilde{c}_i = 0$ for all $i \geq p+1$ such that $a_i \in \Lambda^0(\mathbb{R}^2)\cdot a_p$.*

Proof. By Lemma 5.4 and the assumption $\delta_i < \delta_{i+1}$ any algebraic restriction in the space $o(\delta_s)$ is a linear combination of $a_{s+1}, \dots, a_\mu$. Therefore to prove Proposition 5.11 it suffices to prove that if $a_s = g \cdot a_p$ for some function g then the algebraic restriction a is diffeomorphic to an algebraic restriction in the affine space $c_1a_1 + \cdots + c_{s-1}a_{s-1} + o(\delta_s)$. This follows from Proposition 5.10 since $g(0) = 0$ (if we had $g(0) \neq 0$ then by Lemma 5.4 a_p and a_s would be proportional). □

The proof of Theorem 4.10,(i) requires, except Propositions 5.10 and 5.11, the following lemma.

Lemma 5.12. *Let $a = [x_1^{\alpha_1}x_2^{\alpha_2}dx_1 \wedge dx_2]_{\{H=0\}}$ and $c \neq 0$. Then $c \cdot a$ is diffeomorphic to $\pm a$. If the curve $\{H = 0\}$ admits a symmetry $(x_1, x_2) \to (-x_1, x_2)$ or $(x_1, x_2) \to (x_1, -x_2)$ which changes the sign of the monomial $x_1^{\alpha_1}x_2^{\alpha_2}$ then the algebraic restrictions $\pm a$ are diffeomorphic.*

Proof. The first statement follows from the fact that the group of symmetries of the curve $\{H = 0\}$ includes the scale transformations $(x_1, x_2) \to (t^{\lambda_1} x_1, t^{\lambda_2} x_2)$. The second statement is obvious. □

Theorem 4.10, (i) for the A_k, E_6, E_7, E_8 singularities (respectively D_k singularities) is a direct corollary of Proposition 5.11 (respectively Proposition 5.10), Lemma 5.12, the obvious implication

$$g \in \Lambda^0(\mathbb{R}^2) \cdot f \implies [g dx_1 \wedge dx_2]_{\{H=0\}} \in \Lambda^0(\mathbb{R}^2) \cdot [f dx_1 \wedge dx_2]_{\{H=0\}}$$

and the relations in the last column of Table 3. In this table we use the notation

$$[f(x_1, x_2)] = [f dx_1 \wedge dx_2]_{\{H=0\}}.$$

H	λ_1, λ_2	Basis of $\left[\Lambda^2(\mathbb{R}^2)\right]_{\{H=0\}}$	Relations following from Lemma 4.3
$x_1^{k+1} - x_2^2$	$2, k+1$	$[1], [x_1], \ldots, [x_1^{k-1}]$	
$x_1^2 x_2 - x_2^{k-1}$	$k-2, 2$	$[1], [x_2], \ldots, [x_2^{\ell}], [x_1]$, $[x_2^{\ell+1}], [x_2^{\ell+2}], \ldots, [x_2^{k-2}]$ $\ell = [(k-1)/2]$	$[x_2^j] = (b[x_1] + [x_2^i]) \cdot x_2^{j-i}$ $(b \in \mathbb{R}, j > i)$; $[x_2^{k-2}] = \frac{2x_2}{k-1}[x_1]$
$x_1^3 - x_2^4$	$4, 3$	$[1], [x_2], [x_1]$, $[x_2^2], [x_1 x_2], [x_1 x_2^2]$	
$x_1^3 - x_1 x_2^3$	$3, 2$	$[1], [x_2], [x_1], [x_2^2]$ $[x_1 x_2], [x_2^3], [x_2^4]$	$[x_2^3] = 3x_1 \cdot [x_1]$
$x_1^3 - x_2^5$	$5, 3$	$[1], [x_2], [x_1], [x_2^2]$, $[x_1 x_2], [x_2^3], [x_1 x_2^2], [x_1 x_2^3]$	

Table 3. From Propositions 5.10, 5.11 to Theorem 4.10, (i).

All relations in the last column of Table 3 are obvious corollaries of Lemma 4.3. In the second column of Table 3 we give the weights λ_1, λ_2 of quasi-homogeneity of the function H. In the third column we present a basis of the space $\left[\Lambda^2(\mathbb{R}^{2n})\right]_N$ satisfying the assumption of Proposition 5.11 for all singularities except D_k with even k. For the latter singularities the basis in Table 3 satisfies the assumption of Proposition 5.10. The construction of such a basis, for any H, is very simple. One has to take the *monomial* basis $f_1, \ldots, f_\mu$ of the local algebra of H, to calculate the degrees of these monomials with respect to the weights λ_1, λ_2 and to rearrange them so that the degrees form a non-decreasing sequence. Then, replacing f_i by the algebraic restriction $[f_i]$ we obtain a required basis.

Example 5.13. Consider the case $H = D_k = x_1^2 x_2 - x_2^{k-1}$. Decompose an algebraic restriction $a \in \left[\Lambda^2(\mathbb{R}^2)\right]_{\{H=0\}}$ by the basis in Table 3:

$$a = c_0[1] + c_1[x_2] + \cdots + c_{k-2}[x_2^{k-2}] + \alpha \cdot [x_1]. \tag{5.1}$$

Propositions 5.10, 5.11 and Lemma 5.12 imply that if the coefficients c_i and α satisfy the condition given in the first column of Table 4 then a is diffeomorphic to the normal form in Theorem 4.10, which we present again in the second column of Table 4. Note that the first column contains *all* possible cases.

$c_0 \neq 0$	$\mathcal{F}_0 : [1]$
$c_0 = \cdots = c_{i-1} = 0,\, c_i \neq 0, i \leq k-4$	$\mathcal{F}_i : [bx_1 + x_2^i]$
$c_0 = \cdots = c_{k-4} = 0, \alpha \neq 0$	$\mathcal{F}_{k-3} : [(\pm 1)^{k-1} x_1 + bx_2^{k-3}]$
$c_0 = \cdots = c_{k-4} = 0, \alpha = 0, c_{k-3} \neq 0$	$\mathcal{F}_{k-2} : [x_2^{k-3}]$
$c_0 = \cdots = c_{k-3} = 0, \alpha \neq 0, c_{k-2} \neq 0$	$\mathcal{F}_{k-1} : [x_2^{k-2}]$
$c_0 = \cdots = c_{k-2} = 0, \alpha = 0$	$\mathcal{F}_k : [0]$

Table 4. The correspondence between the normal forms in Theorem 4.10 for the case $H = D_k$ and the coefficients in (5.1).

5.5. **Proof of Theorem 4.10, (iii).** Let $a \in \left[\Lambda^2(\mathbb{R}^2)\right]_{\{H=0\}}$. Take a basis $[f_1], \ldots, [f_\mu]$ of $\left[\Lambda^2(\mathbb{R}^2)\right]_{\{H=0\}}$ as in Table 3. Let $a = c_1[f_1] + \cdots + c_\mu[f_\mu]$. Tracing the proof of Theorem 4.10,(i) we can check that the normal form $\mathcal{F}_i$ holds if exactly i of the coefficients $c_1, ..., c_\mu$ are equal to 0 (see Example 5.13 where this follows from Table 4). By Theorem 4.10, (ii) "if" can be replaced by "if and only if".

5.6. **Proof of Theorem 4.10, (iv).** Any normal form with parameters in Theorem 4.10 has the form $a_0 + b_1 a_1 + \cdots + b_s a_s$ where a_i are algebraic restrictions, b_i are parameters, $s \leq 2$. To prove that the parameters are moduli we have to prove

$$a_i \notin T\left(a_0 + b_1 a_1 + \cdots + b_s a_s\right). \tag{5.2}$$

Proposition 5.9 allows to calculate this tangent space explicitly and to check (5.2) for each of the normal forms in Theorem 4.10. As an example consider the most difficult case – the only normal form with two parameters – the normal form

$$\mathcal{F}_2 = [x_1 + b_1 x_2^2 + b_2 x_2^3]$$

for the case $H = E_8 = x_1^3 - x_2^5$. We continue to use the notation $[f] = [f dx_1 \wedge dx_2]_{\{H=0\}}$ from the previous subsection. We have to prove

$$[x_2^2], [x_2^3] \notin T(\mathcal{F}_2). \tag{5.3}$$

By Proposition 5.9 one has

$$T(\mathcal{F}_2) = \Big\{ \sum_r r \cdot [g \cdot \mathcal{F}_2]^{(r)},\ g \in \Lambda^0(\mathbb{R}^2) \Big\}, \tag{5.4}$$

where ${}^{(r)}$ denotes the quasi-homogeneous part of degree r with respect to the weights $\lambda_1 = 5, \lambda_2 = 3$. Lemma 4.3 implies the relations

$$x_1^{\geq 2} x_2^{\alpha} \mathcal{F}_2 = x_1^{\alpha} x_2^{\geq 4} \mathcal{F}_2 = x_1 x_2^2 \mathcal{F}_2 = 0$$

for any $\alpha \geq 0$, and the relations

$$x_1 \mathcal{F}_2 = b_1 [x_1 x_2^2] + b_2 [x_1 x_2^3],\ x_2 \mathcal{F}_2 = [x_1 x_2] + b_1 [x_2^3],$$

$$x_2^2 \mathcal{F}_2 = [x_1 x_2^2],\ x_1 x_2 \mathcal{F}_2 = b_1 [x_1 x_2^3],\ x_2^3 \mathcal{F}_2 = [x_1 x_2^3].$$

These relations and (5.4) imply

$$T(\mathcal{F}_2) = \operatorname{span}\left(5[x_1] + 6b_1[x_2^2] + 9b_2[x_2^3],\ 8[x_1x_2] + 9b_1[x_2^3],\ [x_1x_2^2],\ [x_1x_2^3]\right).$$

Since the algebraic restrictions $[x_1], [x_1x_2], [x_2^2], [x_2^3], [x_1x_2^2], [x_1x_2^3]$ are linearly independent (see the last row of Table 3) it is clear that (5.3) holds for any b_1, b_2.

6. Symplectic S_5-singularities

Denote by (S_5) the class of varieties in a fixed symplectic space $(\mathbb{R}^{2n}, \omega)$ which are diffeomorphic to

$$S_5 = \{x \in \mathbb{R}^{2n \geq 4} : \ x_1^2 - x_2^2 - x_3^2 = x_2 x_3 = x_{\geq 4} = 0.\} \tag{6.1}$$

We will use the method of algebraic restrictions to obtain a complete classification of symplectic singularities in (S_5). In section 6.1 we calculate the manifold $[\mathrm{Symp}(\mathbb{R}^{2n})]_{S_5}$ and classify its algebraic restrictions. This allows us to decompose (S_5) onto symplectic singularity classes, section 6.2. In section 6.3 we transfer the normal forms for algebraic restrictions to symplectic normal forms. In section 6.4 we give an equivalent definition of the symplectic singularity classes in canonical terms. Some of the proofs are contained in sections 6.5, 6.6.

6.1. **Algebraic restrictions and their classification.** One has the relations

$$[d(x_2 x_3)]_N = [x_2 dx_3 + x_3 dx_2]_N = 0 \tag{6.2}$$

$$[d(x_1^2 - x_2^2 - x_3^2)]_N = 2 \cdot [x_1 dx_1 - x_2 dx_2 - x_3 dx_3]_N = 0 \tag{6.3}$$

Multiplying these relations by suitable 1-forms we obtain the relations in Table 5.

	Relation	Proof
1.	$[x_2 dx_2 \wedge dx_3]_N = 0$	$(6.2) \wedge dx_2$
2.	$[x_3 dx_2 \wedge dx_3]_N = 0$	$(6.2) \wedge dx_3$
3.	$[x_1^2 dx_2 \wedge dx_3]_N = 0$	follows from rows 1. and 2. since $[x_1^2]_N = [x_2^2 + x_3^2]_N$
4.	$[x_1 dx_1 \wedge dx_2]_N = 0$	$(6.3) \wedge dx_2$ along with row 2.
5.	$[x_2^2 dx_1 \wedge dx_2]_N = 0$	$(6.3) \wedge x_2 dx_1$ (since $[x_2 x_3]_N = 0$)
6.	$[x_3^2 dx_1 \wedge dx_2]_N = 0$	$(6.2) \wedge x_3 dx_1$ (since $[x_2 x_3]_N = 0$)
7.	$[x_1 dx_1 \wedge dx_3]_N = 0$	$(6.3) \wedge dx_3$ along with row 1.
8.	$[x_2 dx_1 \wedge dx_3]_N = -[x_3 dx_1 \wedge dx_2]_N$	$(6.2) \wedge dx_1$
9.	$[x_3 dx_1 \wedge dx_3]_N = -[x_2 dx_1 \wedge dx_2]_N$	$(6.3) \wedge dx_1$

Table 5. Relations towards calculating $\left[\Lambda^2(\mathbb{R}^m)\right]_N$ for $N = S_5$.

Table 5 and Proposition 2.3 easily imply the following statements.

Proposition 6.1. *Any 2-form with zero 1-jet has zero algebraic restriction to S_5.*

Proposition 6.2. $[\Lambda^2(\mathbb{R}^{2n})]_{S^5}$ *is a 6-dimensional vector space spanned by the algebraic restrictions to S_5 of the 2-forms*

$\theta_1 = dx_1 \wedge dx_2, \ \theta_2 = dx_2 \wedge dx_3, \ \theta_3 = dx_3 \wedge dx_1, \theta_4 = x_2 dx_1 \wedge dx_2,$

$\sigma_1 = x_3 dx_1 \wedge dx_2, \ \sigma_2 = x_1 dx_2 \wedge dx_3.$

Proposition 6.2 and results of section 2.14 (Theorems 2.19 and 2.21) imply the following description of the space $[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})]_{S^5}$ and the manifold $[\mathrm{Symp}(\mathbb{R}^{2n}]_{S_5}$.

Theorem 6.3. *The space $[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})]_{S^5}$ has dimension 5. It is spanned by the algebraic restrictions to S_5 of the 2-forms*

$$\theta_1, \ldots, \theta_4, \ \theta_5 = \sigma_1 - \sigma_2.$$

If $n \geq 3$ then $[\mathrm{Symp}(\mathbb{R}^{2n})]_{S_5} = [\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})]_{S^5}$. The manifold $[\mathrm{Symp}(\mathbb{R}^4)]_{S_5}$ is an open part of the 5-space $[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^4)]_{S^5}$ consisting of algebraic restrictions of the form $[c_1\theta_1 + \cdots + c_5\theta_5]_{S_5}$ such that $(c_1, c_2, c_3) \neq (0,0,0)$.

Remark. The fact that $\dim[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})]_{S^5} = 5$ follows from Proposition 2.16 since S_5 is a complete intersection singularity of multiplicity 5.

Theorem 6.4.

(i) Any algebraic restriction in $[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})]_{S^5}$ can be brought by a symmetry of S_5 to one of the normal forms $[S_5]^i$ given in the second column of Table 6;

(ii) The codimension in $[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})]_{S^5}$ of the singularity class corresponding to the normal form $[S_5]^i$ is equal to i;

(iii) The singularity classes corresponding to the normal forms are disjoint;

(iv) The parameters c, c_1, c_2 of the normal forms $[S_5]^0, [S_5]^2, [S_5]^3$ are moduli.

Class	Normal forms for algebraic restrictions	cod	μ^{sym}	ind	Canonical definition
$(S_5)^0$ $2n \geq 4$	$[S_5]^0 : [\theta_2 + c_1\theta_1 + c_2\theta_3]_{S_5}$ $(c_1, c_2) \neq (0,0)$	0	2	0	$\omega\vert_W \neq 0$, $\ker\, \omega\vert_W \neq \ell_1^*, \ell_2^*, \ell_3^*$
$(S_5)^2$ $2n \geq 4$	$[S_5]^2 : [\theta_2 + c\theta_4]_{S_5}$	2	3	0	$\omega\vert_W \neq 0$, $\ker\, \omega\vert_W \in \{\ell_1^*, \ell_2^*, \ell_3^*\}$
$(S_5)^3$ $2n \geq 6$	$[S_5]^3 : [\theta_4 + c\theta_5]_{S_5}$	3	4	1	$\omega\vert_W = 0$, $[\omega]_N \neq 0$
$(S_5)^5$ $2n \geq 6$	$[S_5]^5 : [0]_{S_5}$	5	5	∞	$[\omega]_N = 0$

Table 6. Classification of symplectic S_5 singularities. *cod* – codimension of the classes; μ^{sym}– symplectic multiplicity; *ind* – the index of isotropness; W - the tangent space to a non-singular 3-dimensional manifold containing N; $\ell_1^*, \ell_2^*, \ell_3^*$ – the lines in W associated to the tangent lines to the strata of N.

6.2. **Symplectic singularity classes.** In the first column of Table 6 *by $(S_5)^i$ we denote a subclass of (S_5) consisting of $N \in (S_5)$ such that the algebraic restriction $[\omega]_N$ is diffeomorphic to some algebraic restriction of the normal form $[S_5]^i$.* Theorem **A**, Theorem 6.4 and Proposition 6.3 imply the following statement.

Proposition 6.5. *The classes $(S_5)^i$ are symplectic singularity classes, i.e. they are closed with respect to the action of the group of symplectomorphisms. The class (S_5) is the disjoint union of the classes $(S_5)^0, (S_5)^2, (S_5)^3, (S_5)^5$. The classes $(S_5)^0$ and $(S_5)^2$ are non-empty for any dimension $2n \geq 4$ of the symplectic space; the classes $(S_5)^3$ and $(S_5)^5$ are empty if $n = 2$ and not empty if $n \geq 3$.*

The following theorem explains why the given stratification of (S_5) is natural.

Theorem 6.6. *Fix $i \in \{0, 2, 3, 5\}$. All stratified submanifolds $N \in (S_5)^i$ have the same (a) symplectic multiplicity and (b) index of isotropness given in Table 6.*

Proof. The part (a) follows from Theorems **D** and 6.4 and the fact that the codimension in $[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})]_{S_5}$ of the orbit of an algebraic restriction $a \in [S_5]^i$ is equal to the sum of the number of moduli in the normal form $[S_5]^i$ and the codimension in $[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})]_{S_5}$ of the class of algebraic restrictions defined by this normal form.

The part (b) for the normal form $[S_5]^5$ follows from Theorem **B** (or from Theorem **C**). For the normal forms $[S_5]^0$ and $[S_5]^2$ it follows from Theorem **C** and Lemma 2.20. For $[S_5]^3$ the part (b) follows from Theorem **C** and Proposition 6.1. $\square$

6.3. **Symplectic normal forms.** Let us transfer the normal forms $[S_5]^i$ to symplectic normal forms using Theorem **A**, i.e. realizing the algorithm in section 2.8. Fix a family ω^i of symplectic forms on $\mathbb{R}^{2n}$ realizing the family $[S_5]^i$ of algebraic restrictions. We can fix, for example

$$\omega^0 = \theta_2 + c_1\theta_1 + c_2\theta_3 + dx_1 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n}, \quad (c_1, c_2) \neq (0,0)$$

$$\omega^2 = \theta_2 + c\theta_4 + dx_1 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n};$$

$$\omega^3 = \theta_4 + c\theta_5 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n};$$

$$\omega^5 = dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}.$$

Corollary 6.7. *Let ω be a symplectic form on $\mathbb{R}^{2n}, n \geq 3$ (resp. $n = 2$). Fix, for $i = 0,2,3,5$ (resp. for $i = 0,2$) a family Φ^i of local diffeomorphisms which bring the family of symplectic forms ω^i to the symplectic form ω: $(\Phi^i)^*\omega^i = \omega$. Consider the families $S_5^i = (\Phi^i)^{-1}(S_5)$. Any stratified submanifold of the symplectic space $(\mathbb{R}^{2n}, \omega)$ which is diffeomorphic to S_5 can be reduced by a local symplectomorphism to one and only one of the normal forms $S_5^i, i = 0,2,3,5$ (resp. $i = 0,2$). The parameters of the normal forms are moduli.*

Of course the normal forms S_5^i depend on the choice of the diffeomorphisms Φ^i in Corollary 6.7 and of the symplectic forms ω^i realizing the algebraic restrictions. For example, if ω is expressed in Darboux coordinates, $\omega = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n$ then a suitable choice of ω^i and Φ^i leads to the following normal forms:

S_5^0: $p_1^2 - p_2^2 - q_2^2 = 0,\ p_2q_2 = 0,\ q_1 = c_1p_2 + c_2q_2,\ p_{\geq 3} = q_{\geq 3} = 0,\ (c_1, c_2) \neq (0,0);$
S_5^2: $p_1^2 - p_2^2 - q_2^2 = 0,\ p_2q_2 = 0,\ q_1 = cp_2^2,\ p_{\geq 3} = q_{\geq 3} = 0;$
S_5^3: $p_1^2 - p_2^2 - p_3^2 = 0,\ p_2p_3 = 0,\ q_1 = p_2^2/2,\ q_2 = cp_1p_3,\ q_{\geq 3} = p_{\geq 4} = 0;$
S_5^5: $p_1^2 - p_2^2 - p_3^2 = 0,\ p_2p_3 = 0,\ q_{\geq 1} = p_{\geq 4} = 0.$

6.4. **Canonical definition of the classes $(S_5)^i$.** The classes $(S_5)^i$ can be distinguished geometrically, without using any local coordinate system. Let $N \in (S_5)$. Then N is the union of 4 non-singular 1-dimensional submanifolds (strata). Denote by $\ell_1(N), ..., \ell_4(N)$ the tangent lines at 0 to the strata. These lines span a 3-space $W = W(N)$. Equivalently $W(N)$ is the tangent space at 0 to some (and then any) non-singular 3-manifold containing N. The classes $(S_5)^i$ can be distinguished in terms of the restriction $\omega|_W$, where ω is the symplectic form, and the following three lines in the 3-space W associated with the lines $\ell_1(N), ..., \ell_4(N)$:

$$\ell_1^* = \ell_1^*(N) = (\ell_1(N) \oplus \ell_2(N)) \cap (\ell_3(N) \oplus \ell_4(N));$$

$$\ell_2^* = \ell_2^*(N) = (\ell_1(N) \oplus \ell_3(N)) \cap (\ell_2(N) \oplus \ell_4(N));$$

$$\ell_3^* = \ell_3^*(N) = (\ell_1(N) \oplus \ell_4(N)) \cap (\ell_2(N) \oplus \ell_3(N)).$$

The constructed lines $\ell_1^*, \ell_2^*, \ell_3^*$ are well-defined 1-dimensional subspaces of the 3-space W because W is spanned by any three of the lines $\ell_1(N), ..., \ell_4(N)$. For example, for $N = S_5 =$(6.1) it is easy to calculate

$$\ell_1^*(N) = \mathrm{span}(\partial/\partial x_1),\ \ell_{2,3}^*(N) = \mathrm{span}(\partial/\partial x_2 \pm \partial/\partial x_3). \tag{6.4}$$

Theorem 6.8. *A stratified submanifold $N \in (S_5)$ of a symplectic space $(\mathbb{R}^{2n}, \omega)$ belongs to the class $(S_5)^i$ if and only if the couple (N, ω) satisfies the condition in the last column of Table 6, the row of $(S_5)^i$.*

Remark. One can ask why this is a theorem, not the definition of $(S_5)^i$. Of course we could use the last column of Table 6 as the definition of the classes, but this way of exposition is not "honest": the geometric characterization of the classes was obtained as a result of analysis of normal forms for algebraic restrictions.

Proof of Theorem 6.8. The conditions on the pair (ω, N) in the last column of Table 6 are disjoint. This fact and Theorem 6.4, (i) reduce Theorem 6.8 to the following statement: the condition given in the last column of Table 6, the row of $(S_5)^i$, are satisfied for any $N \in (S_5)^i$. This statement is a corollary of the following claims:

1. Each of the conditions in the last column of Table 6 is invariant with respect to the action of the group of diffeomorphisms in the space of pairs (ω, N);
2. Each of these conditions depends only on the algebraic restriction $[\omega]_N$;
3. Take the simplest 2-forms ω^i representing the normal forms $[S_5]^i$ for algebraic restrictions: $\omega^0 = \theta_2 + c_1\theta_1 + c_2\theta_3,\ \omega^2 = \theta_2 + c\theta_4,\ \omega^3 = \theta_4 + c\theta_5,\ \omega^5 = 0$. The pair $(\omega = \omega^i, S_5)$ satisfies the condition in the last column of Table 6, the row of $(S_5)^i$.

The first statement is obvious, the second one follows from Lemma 2.20. To prove the third statement it suffices to note that in the case $N = S_5 = (6.1)$ one has $W = \mathrm{span}(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$ and the kernel of the restriction to W of the 2-form $\theta_2 + c_1\theta_1 + c_2\theta_3$ is the line spanned by the vector $\partial/\partial x_1 + c_2\partial/\partial x_2 - c_1\partial/\partial x_3$. This line coincides with one of the lines (6.4) if any only if $c_1 = c_2 = 0$. $\square$

Theorem 6.8 allows to distinguish the classes $(S_5)^0 \cup (S_5)^2$ and $(S_5)^3 \cup (S_5)^5$ in simple geometric terms: $N \in (S_5)^3 \cup (S_5)^5$ if and only if $\omega|_W = 0$. The geometric distinguishing of the classes $(S_5)^3$ and $(S_5)^5$ follows from Theorem **B** : $N \in (S_5)^5$ if and only if N it is contained in a non-singular Lagrangian submanifold. The following theorem gives a simple way to check the latter condition without using algebraic restrictions. Given a 2-form σ on a non-singular submanifold M of $\mathbb{R}^{2n}$ such that $\sigma(0) = 0$ and a vector $v \in T_0M$ we denote by $\mathcal{L}_v\sigma$ the value at 0 of the Lie derivative of σ along a vector field V on M such that $v = V(0)$. The assumption $\sigma(0) = 0$ implies that the choice of V is irrelevant.

Theorem 6.9. *Let $N \in (S_5)$ be a stratified submanifold of a symplectic space $(\mathbb{R}^{2n}, \omega)$. Let M^3 be any non-singular submanifold containing N and let σ be the restriction of ω to TM^3. Let $v_i^* \in \ell_i^*$ be non-zero vectors. The symplectic form ω has zero algebraic restriction to N if and only if $\sigma(0) = 0$ and $\mathcal{L}_{v_1^*}\sigma(v_2^*, v_3^*) = \mathcal{L}_{v_2^*}\sigma(v_3^*, v_1^*) = \mathcal{L}_{v_3^*}\sigma(v_1^*, v_2^*) = 0$.*

6.5. **Proof of Theorem 6.4.** We will prove statements (i) and (iv). Statements (ii) and (iii) follow from Theorem 6.8 which was proved in section 6.4 (using only the part (i) of Theorem 6.4). The first statement of Theorem 6.4 follows from the following lemmas.

Lemma 6.10. *If $(c_1, c_2, c_3) \neq (0, 0, 0)$ then the algebraic restriction of the form $[c_1\theta_1 + \cdots + c_5\theta_5]_{S_5}$ can be reduced by a linear symmetry of S_5 to an algebraic restriction of the same form with $c_2 = 1$.*

Lemma 6.11. *The algebraic restriction of the form $[c_4\theta_4 + c_5\theta_5]_{S_5}$ with $(c_4, c_5) \neq (0, 0)$ can be reduced by a linear symmetry of S_5 to an algebraic restriction of the same form with $c_4 = 1$.*

Lemma 6.12. *The algebraic restriction of the form $[c_1\theta_1 + \theta_2 + c_2\theta_3 + r_1\theta_4 + r_2\theta_5]_{S_5}$ can be reduced by a symmetry of S_5 to the algebraic restriction $[c_1\theta_1 + \theta_2 + c_2\theta_3]_{S_5}$.*

Lemma 6.13. *The algebraic restriction of the form* $[\theta_2 + c_4\theta_4 + c_5\theta_5]_{S_5}$ *can be reduced by a symmetry of* S_5 *to the algebraic restriction* $[\theta_2 + c_4\theta_4]_{S_5}$.

Proof of Lemmas 6.10 and 6.11. If $c_2 \neq 0$ in the case of Lemma 6.10 or $c_4 \neq 0$ in the case of Lemma 6.11 then the required normal forms are clear due to the scale symmetries of S_5 of the form $x_i \to kx_i$ and the involution $x_1 \to -x_1$. It is easy to check that a suitable permutation of some of the four strata of S_5 brings the case $c_2 = 0$ (resp. $c_4 = 0$) to the case $c_2 \neq 0$ (resp. $c_4 \neq 0$). $\square$

To prove Lemmas 6.12 and 6.13 we use the non-linear symmetries of S_5 generated by the Euler vector field $E = x_1\partial/\partial x_1 + x_2\partial/\partial x_2 + x_2\partial/\partial x_3$.

Notation. Denote by Ψ_j^t the flow of the vector field x_jE, $j = 1, 2, 3$.

Lemma 6.14. *Let* $a_i = [\theta_i]_{S_5}, i = 1, ..., 5$. *The algebraic restriction* $(\Psi_j^t)^*a_i$ *has the form given in Table 7 in the row of* a_i *and the column of* Ψ_j.

	Ψ_1^t	Ψ_2^t	Ψ_3^t
a_1	a_1	$a_1 + 3ta_4$	$a_1 + ta_5$
a_2	$a_2 - 2ta_5$	a_2	a_2
a_3	a_3	$a_3 + ta_5$	$a_3 + 3ta_4$
a_4	a_4	a_4	a_4
a_5	a_5	a_5	a_5

Table 7. The algebraic restrictions $(\Psi_j^t)^*a_i$.

Lemmas 6.12 and 6.13 are immediate corollaries of Lemma 6.14. In fact, Table 7 implies that if $c_1 \neq 0$ (respectively $c_2 \neq 0$) then the algebraic restriction $c_1a_1 + a_2 + c_2a_3 + c_4a_4 + c_5a_5$ reduces to the form $c_1a_1 + a_2 + c_2a_3$ by the symmetry $\Psi_1^t \circ \Psi_2^s$ (respectively $\Psi_1^t \circ \Psi_3^s$) with suitable t and s. The table also implies that the algebraic restriction $a_2 + c_4a_4 + c_5a_5$ reduces to $a_2 + c_4a_4$ by the symmetry Ψ_1^t with a suitable t.

Proof of Lemma 6.14. The calculation of algebraic restrictions $(\Psi_j^t)^*a_i$ is based on the observation that if ω_i is a 2-form representing the algebraic restriction a_i then by Proposition 6.1 $(\Psi_j^t)^*a_i$ depends only on the 1-jet of ω and the 2-jet of Ψ_j. For example $j^2\Psi_1^t : x_1 \to x_1+tx_1^2$, $x_2 \to x_2+tx_1x_2$, $x_3 \to x_3+tx_1x_3$ and consequently

$$j^1((\Psi_1^t)^*a_2 = [dx_2 \wedge dx_3 + 2tx_1dx_2 \wedge dx_3 - tx_3dx_1 \wedge dx_2 + tx_2dx_1 \wedge dx_3]_{S_5}.$$

Using the relation $[x_2dx_3]_{S_5} = -[x_3dx_2]_{S_5}$ (since $[x_2x_3]_{S_5} = 0$) we obtain that $(\Psi_1^t)^*a_2 = a_2 - 2ta_5$. The other boxes in Table 7 can be filled in by similar simple calculations (using some relations in Table 5, for example $[x_1dx_1 \wedge dx_3]_{S_5} = 0$). $\square$

Now we will prove statement (iv) of Theorem 6.4. The fact that the parameters c_1 and c_2 are moduli in the normal form $[S_5]^0$ and the parameter c is a modulus in the normal form $[S_5]^3$ follows from the reduction Theorem 2.5 and the structure of the group of linear symmetries of S_5 treated as a stratified submanifold of $\mathbb{R}^3$ – it is easy to see that it consists of the scale transformations $x_i \to kx_i$ and the permutations of the strata.

Remark. The existence of two moduli in the symplectic classification of stratified submanifolds $N \in (S_5)$ follows from the existence of two moduli in the classification of 5-tuples of lines (one-dimensional subspaces) in a 3-space with respect to the group of linear transformations of this space. One should associate to N the 3-space $W(N)$ and the lines $\ell_1(N), ..., \ell_4(N), \ker\omega|_W \subset W(N)$, see section 6.4.

It remains to prove that c is a modulus in the normal form $[S_5]^2$. As above, Theorem 2.5 allows us to treat S_5 as a stratified submanifold of $\mathbb{R}^3$. Any symmetry Φ of S_5 preserving each of the four strata has the form $x_i \to kx_i$, therefore Φ brings the algebraic restriction $[\theta_2 + c\cdot\theta_4]_{S_5}$ to an algebraic restriction of the form $[k^2\theta_2 + r_4\theta_4 + r_5\theta_5]_{S_5}$. Therefore it suffices to prove that c is an invariant with respect to the symmetries of S_5 of the form

$$\Phi: \ x_1 \to x_1 + \phi_1(x),\ x_2 \to x_2 + \phi_2(x),\ x_3 \to x_3 + \phi_3(x), \tag{6.5}$$

where ϕ_i are functions with zero 1-jet. Using Table 5 we obtain

$$\Phi^*[\theta_2 + c\cdot\theta_4]_{S_5} = [\theta_2 + (c-r)\theta_4 + \tilde{r}\cdot\theta_5]_{S_5},\quad r = \frac{\partial^2\phi_3}{\partial x_1\partial x_2}(0) + \frac{\partial^2\phi_2}{\partial x_1\partial x_3}(0)$$

(the number $\tilde{r}$ also can be calculated, but we do not need it). Now, to prove that c is a modulus, we have to show that $r = 0$ for any symmetry Φ of S_5 of form (6.5). The fact that Φ preserves the strata $x_1 = \pm x_2, x_3 = 0$ and $x_1 = \pm x_3, x_2 = 0$ implies that ϕ_3 belongs to the ideal $(x_3, x_1^2 - x_2^2)$ and ϕ_2 belongs to the ideal $(x_3, x_1^2 - x_3^2)$. It follows that $\frac{\partial^2\phi_3}{\partial x_1\partial x_2}(0) = \frac{\partial^2\phi_2}{\partial x_1\partial x_3}(0) = 0$ and consequently $r = 0$.

6.6. **Proof of Theorem 6.9.** By Proposition 2.4 and Lemma 2.20 it suffices to prove that if a closed 2-form σ on $\mathbb{R}^3$ vanishes at 0 then σ has zero algebraic restriction to $S_5 = \{x_1^2 - x_2^2 - x_3^2 = x_2x_3 = 0\} \subset \mathbb{R}^3$ if and only if

$$\mathcal{L}_{v_1^*}\sigma(v_2^*, v_3^*) = \mathcal{L}_{v_2^*}\sigma(v_3^*, v_1^*) = \mathcal{L}_{v_3^*}\sigma(v_1^*, v_2^*) = 0. \tag{6.6}$$

Let $\sigma = A_3(x)dx_1\wedge dx_2 + A_1(x)dx_2\wedge dx_3 + A_2(x)dx_3\wedge dx_1$. Then, by the closeness of σ, one has $\frac{\partial A_1}{\partial x_1}(0) + \frac{\partial A_2}{\partial x_2}(0) + \frac{\partial A_3}{\partial x_3}(0) = 0$. Using (6.4) it is easy to calculate that the intersection of this condition and (6.6) gives

$$\frac{\partial A_1}{\partial x_1}(0) = \frac{\partial A_2}{\partial x_2}(0) + \frac{\partial A_3}{\partial x_3}(0) = \frac{\partial A_2}{\partial x_3}(0) + \frac{\partial A_3}{\partial x_2}(0) = 0. \tag{6.7}$$

Let us show that (6.7) is equivalent to the condition $[\sigma]_N = 0$. By Proposition 6.1 $[\sigma]_N = 0$ if and only if $[j^1\sigma]_N = 0$. The functions $x_1^2 - x_2^2 - x_3^2$ and x_2x_3 have zero 1-jet. Therefore $[\sigma]_N = 0$ if and only if there exist $r_1,\dots,r_6 \in \mathbb{R}$ such that

$$j^1\sigma = d(x_1^2 - x_2^2 - x_3^2)\wedge(r_1dx_1 + r_2dx_2 + r_3dx_3) + d(x_2x_3)\wedge(r_4dx_1 + r_5dx_2 + r_6dx_3).$$

This relation is a system of 9 linear equations with respect to 6 unknowns $r_1, ..., r_6$. It is easy to check that it is solvable if and only if the condition (6.7) holds.

7. Classification of symplectic regular union singularities

By a regular union singularity in $\mathbb{R}^{2n}$ we mean the union

$$N = N_1 \cup \cdots \cup N_s,\ s \geq 2 \tag{7.1}$$

of germs of s non-singular submanifolds of $\mathbb{R}^{2n}$ (in what follows - strata) such that the dimension of the space

$$W = T_0N_1 + \cdots + T_0N_s \tag{7.2}$$

is equal to the sum of the dimensions of the strata, i.e. the sum (7.2) is direct. If the number of strata and their dimensions are fixed then all such N are diffeomorphic. The set $\left[\mathrm{Symp}(\mathbb{R}^{2n})\right]_N$ can be explicitly described (section 7.1). Using this description and Theorem **A** we classify all symplectic regular union singularities with three 1-dimensional strata (section 7.2), with two 2-dimensional isotropic strata (section 7.3), and with two 2-dimensional symplectic strata (section 7.4).

7.1. **Algebraic restrictions.** At first we describe the space $\left[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})\right]_N$. Throughout subsection 7.1 N is an arbitrary regular union singularity (7.1).

Theorem 7.1. *Two closed 2-forms ω_1, ω_2 have the same algebraic restriction to N if and only if they have the same restriction to the tangent bundle to each of the strata N_i and ω_1 and ω_2 have the same restriction to the space W.*

It follows that $\left[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})\right]_N$ is a finite dimensional vector space if and only if each of the strata N_i is 1-dimensional. Theorem 7.1 makes clear how to parametrize the space of algebraic restrictions, see sections 7.2, 7.3, 7.4.

The minimal dimension of a non-singular manifold containing N is the sum of the dimensions of the strata. Therefore Theorem 2.19 implies:

Proposition 7.2. *Let $m = \dim N_1 + \cdots + \dim N_s$. If $m \leq n$ then $\left[\mathrm{Symp}(\mathbb{R}^{2n})\right]_N = [\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})]_N$. If $m > n$ then an algebraic restriction $[\omega]_N \in [\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})]_N$ belongs to $\left[\mathrm{Symp}(\mathbb{R}^{2n})\right]_N$ if and only if* $\mathrm{rank}\ \omega(0) \geq 2(m-n)$.

Note that Theorem **A**, Theorem 7.1 and Proposition 7.2 reduce the problem of classification of symplectic regular union singularities with isotropic strata to simple linear algebra problems, see sections 7.2 and 7.3.

Theorem 7.1 and Theorem **C** imply the following corollary on the index of isotropness of a regular union singularity.

Proposition 7.3. *Let N be a regular union singularity (7.1) in a symplectic space $(\mathbb{R}^{2n}, \omega)$. Let $W \subset T_0\mathbb{R}^{2n}$ be the space (7.2). If $\omega|_W \neq 0$ then the index of isotropness of N is equal to* 0. *If $\omega|_W = 0$ then it is equal to the minimum of orders of vanishing of the* 2*-forms $\omega|_{TN_i}$, $i = 1, \ldots, s$. In particular, if the strata N_i are isotropic then the index is either* 0 *(if $\omega|_W \neq 0$) or ∞ (if $\omega|_W = 0$). .*

Proof of Theorem 7.1. Fix a non-singular submanifold M containing N of dimension $\dim N_1 + \cdots + \dim N_s$. Theorem 7.1 follows from Proposition 2.4 and the following statement: a closed 2-form σ on M has zero algebraic restriction to N if and only if (a) $\sigma|_{TN_i} = 0$ $i = 1, \ldots, s$ and (b) $\sigma(0) = 0$. The implication $[\sigma]_N = 0 \implies$ (a), (b) follows from Lemma 2.20. In what follows we prove the implication (a), (b) $\implies$ $[\sigma]_N = 0$. It is easy to show that (a) and (b) imply that σ is a differential of a 1-form α such that (c) α has zero 1-jet and (d) $\alpha|_{TN_i} = 0, i = 1, \ldots, s$. Therefore it suffices to prove that (c) and (d) imply $[\alpha]_N = 0$. To prove this statement take local coordinates $x_1^{(1)}, \ldots, x_{m_1}^{(1)}, \ldots, x_1^{(s)}, \ldots, x_{m_s}^{(s)}$ on M such that the stratum N_i is described by vanishing of all coordinates except $x_1^{(i)}, \ldots, x_{m_i}^{(i)}$ (here $m_i = \dim N_i$). It is easy to see that any 1-form α satisfying (c) and (d) belongs to the ideal in the external algebra of differential forms generated by 0-forms (functions)

$$x_p^{(i)} \cdot x_q^{(j)}, \ \ j \neq i, \ p = 1, \ldots, m_i, \ q = 1, \ldots, m_j,$$

which vanish at any point of N, and 1-forms

$$x_{p_1}^{(i)} \cdot x_{p_2}^{(i)} dx_q^{(j)}, \ \ j \neq i, \ p_1, p_2 = 1, \ldots, m_i, \ q = 1, \ldots, m_j. \tag{7.3}$$

By Proposition 2.3 it remains to prove that the 1-forms (7.3) have zero algebraic restriction to N. This follows from the relation

$$x_{p_1}^{(i)} \cdot x_{p_2}^{(i)} dx_q^{(j)} = d\left(x_{p_1}^{(i)} \cdot x_{p_2}^{(i)} \cdot x_q^{(j)}\right) - x_{p_1}^{(i)} \cdot x_q^{(j)} dx_{p_2}^{(i)} - x_{p_2}^{(i)} \cdot x_q^{(j)} dx_{p_1}^{(i)}.$$

7.2. **Regular union of 3 one-dimensional submanifolds.** By Theorem 7.1 the algebraic restrictions of closed 2-forms to a regular union N of three 1-dimensional submanifolds can be identified with 2-forms on the 3-space W spanned by the tangent lines ℓ_1, ℓ_2, ℓ_3 to the strata of N. The action of the group of symmetries of N reduces to the action of the group of linear transformations of W preserving the set $\ell_1 \cup \ell_2 \cup \ell_3$. Therefore the problem of classification of algebraic restrictions to N of closed 2-forms reduces to the following simple problem of linear algebra:

Let ℓ_1, ℓ_2, ℓ_3 be linearly independent 1-dimensional subspaces of a 3-dimensional space W. One has to classify 2-forms σ on W with respect to the group of linear transformations preserving $\ell_1 \cup \ell_2, \cup \ell_3$.

It is easy to prove that in this problem there are exactly 4 orbits, of codimension $0, 1, 2, 3$. The orbit of codimension 0 consists of non-zero 2-forms whose kernel does not belong to any of the 2-spaces $\ell_1 + \ell_2, \ell_1 + \ell_3, \ell_2 + \ell_3$. The orbit of codimension 1 consists of non-zero 2-forms whose kernel belongs to one of these 2-spaces but does not coincide with any of the lines ℓ_1, ℓ_2, ℓ_3. The orbit of codimension 2 consists of non-zero 2-forms whose kernel coincides with one of the lines ℓ_1, ℓ_2, ℓ_3. The orbit of codimension 3 is one "point" – the zero 2-form.

Theorem 7.1 allows to bring this simple classification to the classification of algebraic restrictions given in the first column of Table 8, where

$$N^*: \quad x_1x_2 = x_1x_3 = x_2x_3 = x_{\geq 4} = 0$$

is the normal form with respect to the group diffeomorphisms serving for all regular unions of three 1-dimensional submanifolds. The algebraic restriction to N of any closed 2-form ω is diffeomorphic to one and only one of the algebraic restrictions a^i. The normal form a^i holds if and only if the pair (ω, N) satisfies the condition given in the last column of Table 8. The orbit of a^i with respect to the group of symmetries of N^* has codimension i in the space $\left[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})\right]_{N^*}$.

Normal forms for algebraic restrictions	Symplectic normal forms	Geometric condition
$a^0 = [dx_2dx_3 + dx_3dx_1 + dx_1dx_2]_{N^*}$	$N^0: q_2 = p_1 + p_2,$ $p_1q_1 = q_1p_2 = p_2q_2 = 0,$ $p_{\geq 3} = q_{\geq 3} = 0$	$\omega\vert_W \neq 0,$ $\ker\omega\vert_Q \not\subset T_0N_i + T_0N_j,$ for any $i, j \in \{1,2,3\}$;
$a^1 = [dx_3dx_1 + dx_1dx_2]_{N^*}$	$N^1: q_2 = p_1,$ $p_1q_1 = q_1p_2 = p_2p_1 = 0,$ $p_{\geq 3} = q_{\geq 3} = 0$	$\omega\vert_Q \neq 0,$ $\ker\omega\vert_W \subset T_0N_i + T_0N_j,$ $\ker\omega\vert_W \neq T_O N_i, T_0N_j$ for some $i, j \in \{1,2,3\}$;
$a^2 = [dx_1dx_2]_{N^*}$	$N^2: p_1q_1 = q_1p_2 = p_2p_1 = 0,\ p_{\geq 3} = q_{\geq 2} = 0$	$\omega\vert_Q \neq 0,$ $\ker\omega\vert_W = T_0N_i$ for some $i \in \{1,2,3\}$
$a^3 = [0]_{N^*}$	$N^3: p_1p_2 = p_2p_3 = p_3p_1 = 0,\ p_{\geq 4} = q_{\geq 1} = 0$	$\omega\vert_W = 0.$

Table 8. Classification of symplectic regular union singularities with three 1-dimensional strata. W denotes the 3-space spanned by the tangent lines at 0 to the strata.

This classification of algebraic restrictions can be transferred to the following symplectic classification using Theorems **A** and **D** and Proposition 7.2, 7.3.

Theorem 7.4. *Any regular union singularity N with three 1-dimensional strata in the symplectic space $(\mathbb{R}^{2n}, \omega_0 = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n)$, $n \geq 3$ (resp. $n = 2$) is symplectomorphic to one and only one of the varieties N^0, N^1, N^2, N^3 (resp. N^0, N^1, N^2) given in Table 8. The normal form N^i has symplectic multiplicity i. It holds if and only if the pair $(\omega = \omega_0, N)$ satisfies the condition in the last column of the table. The index of isotropness of N^0, N^1, N^2 is equal to 0, of N^3 – to ∞.*

7.3. **Regular union of two 2-dimensional isotropic submanifolds.** In this subsection we obtain symplectic classification of all regular union singularities N with two 2-dimensional *isotropic* strata. (In this case we will say that N is isotropic). Like in the previous subsection, Theorem 7.1 reduces the classification of algebraic restrictions to the following problem of linear algebra:

Let L_1, L_2 be transversal 2-dimensional subspaces of a 4-dimensional space Q. One has to classify 2-forms σ on Q which annihilate L_1 and L_2 with respect to the group of linear transformations preserving $L_1 \cup L_2$.

It is easy to show that in this classification problem the rank of σ is a complete invariant – two 2-forms with the given above properties are equivalent if and only if they have the same rank. By Theorem 7.1 we obtain the classification of algebraic restrictions in Table 9, where

$$N^* : \ x_1x_3 = x_1x_4 = x_2x_3 = x_2x_4 = x_{\geq 5} = 0 \tag{7.4}$$

is the normal form with respect to the group diffeomorphisms serving for all regular unions of two 2-dimensional submanifolds. The algebraic restriction to N of any closed 2-form annihilating the tangent bundles to the strata of N is diffeomorphic to one and only one of the algebraic restrictions a^i. The orbit of a^i has codimension i in the space of algebraic restrictions to N^* of closed 2-forms annihilating the tangent bundles to the strata of N^*. The normal form a^i holds if and only if the pair (ω, N) satisfies the condition in the third column of Table 9.

Normal forms for algebraic restrictions	Symplectic normal forms	Geometric condition	codim
$a^0 = [dx_1dx_3 + dx_2dx_4]_{N^*}$	$N^0 : \{p_{\geq 3} = q_{\geq 1} = 0\} \cup$ $\{p_{\geq 1} = q_{\geq 3} = 0\}$	rank $\omega\vert_W = 4$	0
$a^1 = [dx_1dx_3]_{N^*}$	N^1: (for $2n \geq 6$ only) $\{p_{\geq 3} = q_{\geq 1} = 0\} \cup$ $\{p_{\geq 1} = q_2 = q_{\geq 4} = 0\}$	rank $\omega\vert_W = 2$	1
$a^4 = [0]_{N^*}$	N^4: (for $2n \geq 8$ only) $\{p_{\geq 3} = q_{\geq 1} = 0\} \cup$ $\{p_1 = p_2 = p_{\geq 5} = q_{\geq 1} = 0\}$	$\omega\vert_W = 0$	4

Table 9. Classification of symplectic regular union singularities with two 2-dimensional isotropic strata. W denotes the 4-space spanned by the tangent planes at 0 to the strata.

Using Theorem **A** and Proposition 7.2, 7.3 we can transfer the obtained classification of algebraic restrictions to the following symplectic classification.

Theorem 7.5. *Any regular union singularity N with two isotropic 2-dimensional strata in a symplectic space $(\mathbb{R}^{2n}, \omega_0 = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n)$ is symplectomorphic to one and only one of the varieties N^0, N^1, N^4 in Table 9. The orbit of N^i has codimension i in the class of all regular union singularities with two 2-dimensional isotropic strata. The normal form N^i holds if and only if the pair $(\omega = \omega_0, N)$*

satisfies the condition given in the last column of Table 9. The index of isotropness of N^0, N^1 *is equal to* 0*, of* N^4 *–* ∞*.*

7.4. **Regular union of two 2-dimensional symplectic submanifolds.** In this subsection we classify regular union singularities with two 2-dimensional *symplectic* strata in a symplectic space $(\mathbb{R}^{2n}, \omega)$. Note that in this case the index of isotropness of N is equal to 0. The symplectic classification of such N involves the following invariant. Recall that two germ of submanifolds N_1, N_2 of a symplectic space $(\mathbb{R}^{2n}, \omega)$ are called ω-orthogonal if $\omega(v, u) = 0$ for any vectors $v \in T_0N_1, u \in T_0N_2$.

Definition 7.6. The index of non-orthogonality between 2-dimensional symplectic submanifolds N_1 and N_2 of a symplectic space $(\mathbb{R}^{2n}, \omega)$ is the number

$$\alpha = \alpha(N_1, N_2) = 1 - \frac{(\omega \wedge \omega)(v_1, v_2, u_1, u_2)}{2 \cdot \omega(v_1, v_2) \cdot \omega(u_1, u_2)}$$

where v_1, v_2 is a basis of T_0N_1 and u_1, u_2 is a basis of T_0N_2.

The following obvious statement explains this definition.

Proposition 7.7. *The index* $\alpha(N_1, N_2)$ *is well-defined, i.e. it does not depend on the choice of the bases of* T_0N_1 *and* T_0N_2*. It is equal to* 0 *if and only if there exists a non-zero vector* $u \in T_0N_1$ *such that* $\omega(v, u) = 0$ *for any* $v \in T_0N_2$*. It is equal to* 1 *if and only if the* 4*-form* $\omega \wedge \omega$ *has zero restriction to the space* $Q = T_0N_1 + T_0N_2$*.*

In other words, $\alpha(N_1, N_2) = 0$ if the space T_0N_1 has non-trivial intersection with the ω-orthogonal complement to T_0N_2 in the space Q. In particular, if N_1 and N_2 are ω-orthogonal then $\alpha(N_1, N_2) = 0$.

Proposition 7.8. *Let* $N = N_1 \cup N_2$ *be the regular union of two* 2*-dimensional symplectic submanifolds of a symplectic space* $(\mathbb{R}^{2n}, \omega)$*. Let* α *be the index of non-orthogonality between* N_1 *and* N_2*. If* N_1 *and* N_2 *are not* ω*-orthogonal then the algebraic restriction* $[\omega]_N$ *is diffeomorphic to the algebraic restriction*

$$a^{\alpha} = [dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + dx_1 \wedge dx_3 + \alpha dx_2 \wedge dx_4]_{N^*},$$

where $N^* =$ *(7.4). If* N_1 *and* N_2 *are* ω*-orthogonal then* $[\omega]_N$ *is diffeomorphic to the algebraic restriction*

$$a^{\perp} = [dx_1 \wedge dx_2 + dx_3 \wedge dx_4]_{N^*}.$$

The orbit of $a^{\perp}$ *has codimension* 4 *in* $\left[\Lambda^{2,\mathrm{closed}}(\mathbb{R}^{2n})\right]_{N^*}$*.*

Theorems **A**, **D** and Propositions 7.7, 7.8 imply the following corollary.

Theorem 7.9. *Let* $\omega_0 = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n$*. Let* $N = N_1 \cup N_2$ *be the regular union singularity with two* 2*-dimensional symplectic strata in the symplectic space* $(\mathbb{R}^{2n}, \omega_0)$*. If* N_1 *and* N_2 *are not* ω_0*-orthogonal then* N *has symplectic multiplicity* 1 *and is symplectomorphic to the variety*

$$N^{\alpha}: \ \{q_1 = p_2, p_1 = p_{\geq 3} = q_{\geq 3} = 0\} \cup \{p_2 = \alpha q_1, p_{\geq 3} = q_{\geq 2} = 0\},$$

where α *is the index of non-orthogonality between* N_1 *and* N_2*. If* N_1 *and* N_2 *are* ω_0*-orthogonal then* N *has symplectic multiplicity* 4 *and is symplectomorphic to*

$$N^{\perp}: \ \{p_1 = q_1 = p_{\geq 3} = q_{\geq 3} = 0\} \cup \{p_{\geq 2} = q_{\geq 2} = 0\}.$$

If $n \geq 3$ *then any of the normal forms is realizable and if* $n = 2$ *– any except the normal form* N^1*.*

It follows that the index of non-orthogonality distinguishes all normal forms except $N^{\perp}$ and N^0 – for each of them the index is equal to 0. These normal forms can be distinguished as follows. Intersect the ω-orthogonal complement to the tangent space to N_1 with the tangent space to N_2. If the index of non-orthogonality is equal to 0 then the dimension of the intersection is either 1 or 2. It is 1 if N is symplectomorphic to N^0 and it is 2 if N is symplectomorphic to $N^{\perp}$.

Proof of Proposition 7.8. By Theorem 7.1 the algebraic restriction to $N^* = (7.4)$ of any closed 2-form can be expressed in the form

$$[\omega]_{N^*} = [f(x_1,x_2)dx_1 \wedge dx_2 + g(x_3,x_4)dx_3 \wedge dx_4 + \\ +c_1dx_1 \wedge dx_3 + c_2dx_1 \wedge dx_4 + c_3dx_2 \wedge dx_3 + c_4dx_2 \wedge dx_4]_{N^*}. \tag{7.5}$$

Therefore $[\omega]_N$ is diffeomorphic to (7.5). The condition that the strata are symplectic with respect to ω depends only on the algebraic restriction $[\omega]_N$ and is equivalent to the condition $f(0) \neq 0, g(0) \neq 0$. This condition allows to reduce $f(x_1,x_2)$ and $g(x_3,x_4)$ to 1 by a symmetry of N^* of the form $(x_1,x_2) \to (\phi_1(x_1,x_2), \phi_2(x_1,x_2))$, $(x_3,x_4) \to (\psi_1(x_3,x_4), \psi_2(x_3,x_4))$. We obtain the normal form

$$[dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + c_1dx_1 \wedge dx_3 + c_2dx_1 \wedge dx_4 + c_3dx_2 \wedge dx_3 + c_4dx_2 \wedge dx_4]_{N^*}$$

with real parameters c_1, c_2, c_3, c_4. The condition that the strata are ω-orthogonal is also a property of the algebraic restriction $[\omega]_{N^*}$. It holds if and only if $c_1 = c_2 = c_3 = c_4 = 0$. In this case we obtain the normal form $a^{\perp}$. If the strata are not ω-orthogonal then at least one of the numbers $c_1, \cdots, c_4$ is different from 0. The case $c_1 = 0$ can be transferred to the case $c_1 \neq 0$ by one of the symmetries $(x_1,x_2,x_3,x_4) \mapsto (-x_2,x_1,x_3,x_4)$, $(x_1,x_2,x_3,x_4) \mapsto (x_1,x_2,-x_4,x_3)$. The scale symmetry $(x_1,x_2,x_3,x_4) \to (c_1^{-1}x_1, c_1x_2, x_3, x_4)$ reduces c_1 to 1. Now we can reduce c_2 and c_3 to 0 by the symmetry $(x_1,x_2,x_3,x_4) \to (x_1-c_3x_2, x_2, x_3-c_2x_4, x_4)$. We obtain the normal form a^{α}, and it remains to note that in this normal form α is exactly the index of non-orthogonality between the strata of N. □

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