Transitive Lie algebras

Michail Zhitomirskii

April 18, 2013

- Local homogeneous subsets of $T\mathbb{R}^n$ and $T\mathbb{C}^n$

$$\Sigma = \{\Sigma_x \subset T_x\mathbb{M}^n\}_{x\in M^n}\,, \quad M^n : \text{ analytic manifold}$$

- Local homogeneous subsets of $T\mathbb{R}^n$ and $T\mathbb{C}^n$

$$\Sigma = \{\Sigma_x \subset T_x\mathbb{M}^n\}_{x\in M^n}\,, \quad M^n: \text{ analytic manifold}$$

- Def. Σ is homogeneous if for any $p_1, p_2 \in M^n$ there exists a local analytic diffeo $\Phi : (M^n, p_1) \to (M^n, p_2)$ such that

$$\Phi_{*,x}\Sigma_x = \Sigma_{\Phi(x)}, \quad x \text{ close to } p_1.$$

- Local homogeneous subsets of $T\mathbb{R}^n$ and $T\mathbb{C}^n$

$$\Sigma = \{\Sigma_x \subset T_x\mathbb{M}^n\}_{x \in M^n}, \quad M^n: \text{ analytic manifold}$$

- Def. Σ is homogeneous if for any $p_1, p_2 \in M^n$ there exists a local analytic diffeo $\Phi : (M^n, p_1) \to (M^n, p_2)$ such that

$$\Phi_{*,x}\Sigma_x = \Sigma_{\Phi(x)}, \quad x \text{ close to } p_1.$$

- A local homogeneous subset $T\mathbb{R}^n$ is the germ at $0 \in \mathbb{R}^n$ of a homogeneous subset of TU, U is a nbhd of $0 \in \mathbb{R}^n$.

  Germ wrt to $x \in \mathbb{R}^n$, not wrt to a tangent vector.

- Local homogeneous subsets of $T\mathbb{R}^n$ and $T\mathbb{C}^n$

$$\Sigma = \{\Sigma_x \subset T_x\mathbb{M}^n\}_{x \in M^n}\,, \quad M^n : \text{ analytic manifold}$$

- Def. Σ is homogeneous if for any $p_1, p_2 \in M^n$ there exists a local analytic diffeo $\Phi : (M^n, p_1) \to (M^n, p_2)$ such that

$$\Phi_{*,x}\Sigma_x = \Sigma_{\Phi(x)}, \quad x \text{ close to } p_1.$$

- A local homogeneous subset $T\mathbb{R}^n$ is the germ at $0 \in \mathbb{R}^n$ of a homogeneous subset of TU, U is a nbhd of $0 \in \mathbb{R}^n$.

  Germ wrt to $x \in \mathbb{R}^n$, not wrt to a tangent vector.

- A local homogeneous subset $T\mathbb{C}^n$ (holomorphic tangent bundle):

  replace $\mathbb{R}^n$ by $\mathbb{C}^n$ and local diffeos by local biholomorphisms.

- Riemannian metric on $\mathbb{R}^2$ = elliptic structure in $T\mathbb{R}^2$

- Riemannian metric on $\mathbb{R}^2$ = elliptic structure in $T\mathbb{R}^2$
- Hyperbolic structure in $T\mathbb{R}^2$

- Riemannian metric on $\mathbb{R}^2$ = elliptic structure in $T\mathbb{R}^2$
- Hyperbolic structure in $T\mathbb{R}^2$
- $\mathfrak{a}_2$ and $\mathfrak{b}_2$ (Abelian and non-Abelian 2-dim Lie algebras)

- Riemannian metric on $\mathbb{R}^2$ = elliptic structure in $T\mathbb{R}^2$
- Hyperbolic structure in $T\mathbb{R}^2$
- $\mathfrak{a}_2$ and $\mathfrak{b}_2$ (Abelian and non-Abelian 2-dim Lie algebras)
- Claim. There is a natural way to identify:

Local homogeneous subset of $T\mathbb{R}^n$ up to diffeos	Subset of an n-dim Lie algebra up to automorphisms
Local homogeneous Riemannian metric on $\mathbb{R}^2$ with zero, resp. negative curvature	ellipse in $\mathfrak{a}_2$, resp. $\mathfrak{b}_2$
Local homogeneous hyperbolic structure in $T\mathbb{R}^2$ with zero, positive or negative curvature	hyperbola in $\mathfrak{b}_2$ if zero curvature also hyperbola in $\mathfrak{a}_2$

- From an ellipse E, resp. hyperbola H in $\mathfrak{g} = \mathfrak{a}_2, \mathfrak{b}_2$ to elliptic, resp. hyperbolic structure in $T\mathbb{R}^2$

- From an ellipse E, resp. hyperbola H in $\mathfrak{g} = \mathfrak{a}_2, \mathfrak{b}_2$ to elliptic, resp. hyperbolic structure in $T\mathbb{R}^2$

- Let G be a Lie group with the Lie algebra $\mathfrak{g}$.
Push forward E, resp. H by the flows
of left-invariant vector fields to a nbhd of $id \in G$.

- From an ellipse E, resp. hyperbola H in $\mathfrak{g} = \mathfrak{a}_2, \mathfrak{b}_2$ to elliptic, resp. hyperbolic structure in $T\mathbb{R}^2$
- Let G be a Lie group with the Lie algebra $\mathfrak{g}$.
  Push forward E, resp. H by the flows
  of left-invariant vector fields to a nbhd of $id \in G$.
- All ellipses, resp. hyperbolas in $\mathfrak{a}_2$ are automorphic.
  If $g = \mathfrak{a}_2$ we obtain homogeneous "flat" elliptic or hyperbolic structure, "flat" = zero curvature.

- From an ellipse E, resp. hyperbola H in $\mathfrak{g} = \mathfrak{a}_2, \mathfrak{b}_2$ to elliptic, resp. hyperbolic structure in $T\mathbb{R}^2$

- Let G be a Lie group with the Lie algebra $\mathfrak{g}$.
Push forward E, resp. H by the flows
of left-invariant vector fields to a nbhd of $id \in G$.

- All ellipses, resp. hyperbolas in $\mathfrak{a}_2$ are automorphic.
If $g = \mathfrak{a}_2$ we obtain homogeneous "flat" elliptic or hyperbolic structure, "flat" = zero curvature.

- The classification of ellipses and hyperbolas in $\mathfrak{b}_2$
and the corresponding homogeneous elliptic
and hyperbolic structures are as follows:

Ellipses and hyperbolas in $\mathfrak{b}_2 : [e_1, e_2] = e_2$ up to automorphisms	Local homogeneous subsets of $T\mathbb{R}^2$ up to diffeomorphisms
Ellipse $x_1^2 + \theta x_2^2 = 1$ $\theta > 0$	Riemannian metrics with negative curvature $-\theta$
Hyperbola $x_1^2 - \theta x_2^2 = 1$ $\theta > 0$	Hyperbolic structure with negative curvature $-\theta$
Hyperbola $x_2^2 - \theta x_1^2 = 1$ $\theta > 0$	Hyperbolic structure with positive curvature θ
Hyperbola $x_1 x_2 = 1$	Hyperbolic structure with zero curvature

- Take a Lie group G with the Lie algebra $\mathfrak{g} = \mathfrak{a}_2$ or $\mathfrak{g} = \mathfrak{b}_2$ and push forward an ellipse or hyperbola in $\mathfrak{g}$ by the flows of left-invariant vector fields to a nbhd of $id \in G$.

- Take a Lie group G with the Lie algebra $\mathfrak{g} = \mathfrak{a}_2$ or $\mathfrak{g} = \mathfrak{b}_2$ and push forward an ellipse or hyperbola in $\mathfrak{g}$ by the flows of left-invariant vector fields to a nbhd of $id \in G$.

- Generalization (not modest)
  - Replace $T\mathbb{R}^2$ by $T\mathbb{R}^n$, resp. $T\mathbb{C}^n$

    $T\mathbb{C}^n$: holomorphic tangent bundle
  - replace $\mathfrak{a}_2$ or $\mathfrak{b}_2$ by any real, resp. complex n-dim Lie algebra $\mathfrak{g}$
  - Replace an ellipse or hyperbola by any subset of $\mathfrak{g}$.

  We obtain a local homogeneous subset of $T\mathbb{R}^n$, resp. $T\mathbb{C}^n$.

- Take a Lie group G with the Lie algebra $\mathfrak{g} = \mathfrak{a}_2$ or $\mathfrak{g} = \mathfrak{b}_2$ and push forward an ellipse or hyperbola in $\mathfrak{g}$ by the flows of left-invariant vector fields to a nbhd of $id \in G$.

- Generalization (not modest)
  - Replace $T\mathbb{R}^2$ by $T\mathbb{R}^n$, resp. $T\mathbb{C}^n$

    $T\mathbb{C}^n$: holomorphic tangent bundle
  - replace $\mathfrak{a}_2$ or $\mathfrak{b}_2$ by any real, resp. complex n-dim Lie algebra $\mathfrak{g}$
  - Replace an ellipse or hyperbola by any subset of $\mathfrak{g}$.

  We obtain a local homogeneous subset of $T\mathbb{R}^n$, resp. $T\mathbb{C}^n$.

- A local homogeneous subset of $T\mathbb{R}^n$, resp. $T\mathbb{C}^n$ will be called special if it cannot be obtained in this way, up to an analytic diffeo, resp. biholomorphism.

  Problem. To find all special local homogeneous subsets of $T\mathbb{R}^n$, resp. $T\mathbb{C}^n$, if any.

- Some results (M.Zh, 2012-13)

- Some results (M.Zh, 2012-13)
- **1.** In $T\mathbb{C}^2$ there are no special local homogeneous subsets.

- Some results (M.Zh, 2012-13)
- **1.** In $T\mathbb{C}^2$ there are no special local homogeneous subsets.
- **2.** In $T\mathbb{R}^2$ an example of a special local homogeneous subset is a Riemannian metrics with positive curvature.

- Some results (M.Zh, 2012-13)
- **1.** In $T\mathbb{C}^2$ there are no special local homogeneous subsets.
- **2.** In $T\mathbb{R}^2$ an example of a special local homogeneous subset is a Riemannian metrics with positive curvature.
- In some sense it is the only example:
  any special local homogeneous subset of $T\mathbb{R}^2$ has symmetry algebra $\mathfrak{so}_3(\mathbb{R})$.

- Some results (M.Zh, 2012-13)
- **1.** In $T\mathbb{C}^2$ there are no special local homogeneous subsets.
- **2.** In $T\mathbb{R}^2$ an example of a special local homogeneous subset is a Riemannian metrics with positive curvature.
- In some sense it is the only example:
  any special local homogeneous subset of $T\mathbb{R}^2$ has symmetry algebra $\mathfrak{so}_3(\mathbb{R})$.
- **3.** In $T\mathbb{C}^3$ within local homogeneous linear subsets (plane fields, affine plane fields, line fields, affine line fields) there are no special ones.

- Some results (M.Zh, 2012-13)
- **1.** In $T\mathbb{C}^2$ there are no special local homogeneous subsets.
- **2.** In $T\mathbb{R}^2$ an example of a special local homogeneous subset is a Riemannian metrics with positive curvature.
- In some sense it is the only example:
  any special local homogeneous subset of $T\mathbb{R}^2$ has symmetry algebra $\mathfrak{so}_3(\mathbb{R})$.
- **3.** In $T\mathbb{C}^3$ within local homogeneous linear subsets (plane fields, affine plane fields, line fields, affine line fields) there are no special ones.
- **4.** In $\mathbb{T}C^4$ within local homogeneous linear subsets the only special one is the most involved class of affine line fields.

  The symmetry algebra for this class is ∞-dim.

- **5.** In $T\mathbb{R}^3$ and in $T\mathbb{C}^3$:

  there are special local homogeneous (non-linear) subsets

  their symmetry algebra can be explicitly
  classified up to diffeos.

  In the case $T\mathbb{C}^3$ the symmetry algebras of special
  local homogeneous subsets are, up to biholomorphisms:

  two single symmetry algebras and two 1-parameter families;
  all are ∞-dim.

- **5.** In $T\mathbb{R}^3$ and in $T\mathbb{C}^3$:

  there are special local homogeneous (non-linear) subsets

  their symmetry algebra can be explicitly
  classified up to diffeos.

  In the case $T\mathbb{C}^3$ the symmetry algebras of special
  local homogeneous subsets are, up to biholomorphisms:

  two single symmetry algebras and two 1-parameter families;
  all are ∞-dim.

- **5.** In $T\mathbb{R}^3$ and in $T\mathbb{C}^3$:

  there are special local homogeneous (non-linear) subsets

  their symmetry algebra can be explicitly
  classified up to diffeos.

  In the case $T\mathbb{C}^3$ the symmetry algebras of special
  local homogeneous subsets are, up to biholomorphisms:

  two single symmetry algebras and two 1-parameter families;
  all are ∞-dim.

- **5.** In $T\mathbb{R}^3$ and in $T\mathbb{C}^3$:

  there are special local homogeneous (non-linear) subsets

  their symmetry algebra can be explicitly
  classified up to diffeos.

  In the case $T\mathbb{C}^3$ the symmetry algebras of special
  local homogeneous subsets are, up to biholomorphisms:

  two single symmetry algebras and two 1-parameter families;
  all are ∞-dim.

- **5.** In $T\mathbb{R}^3$ and in $T\mathbb{C}^3$:

  there are special local homogeneous (non-linear) subsets

  their symmetry algebra can be explicitly classified up to diffeos.

  In the case $T\mathbb{C}^3$ the symmetry algebras of special local homogeneous subsets are, up to biholomorphisms:

  two single symmetry algebras and two 1-parameter families; all are ∞-dim.

- I do not know an example of a special local homogeneous subset of $T\mathbb{C}^n$ with finite-dim symmetry algebra.

  Working doubtful conjecture: such example does not exist.

A local affine line field in $T\mathbb{K}^n$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$
can be defined by two vector field germs A, B
where B shows direction and A is "drift".
A and B are defined up to $B \to H_1 B$, $A \to A + H_2 B$, $H_1(0) \neq 0$.

Within local homogeneous affine line fields:

the simplest class can be called "flat", it is defined by:
$[A, B](x) \in span(B(x))$ for any x.

The most involved class is defined by

$[B,[A,B]](x) \in span\{B(x),[A,B](x)\}$,
$[A,[A,[A,B]]](0) \notin span\{B(0),[A,B](0),[A,[A,B]](0)\}$,
$[B,[A,[A,[A,B]]]](x) \in span\{B(x),[A,B](x),[A,[A,B]](x)\}$

The symmetry algebra is

$$f(x_1)\frac{\partial}{\partial x_1} + f'(x_1)\frac{\partial}{\partial x_2} + e^{x_3} f''(x_1)\frac{\partial}{\partial x_3} + \left(e^{2x_2} f'''(x_1) + e^{x_2} f''(x_1)\right)\frac{\partial}{\partial x_4}.$$

It is isomorphic to the Lie algebra of all vector field germs on $\mathbb{K}$.

- Infinitesimal symmetries
  and transitive Lie algebras of vector fields

- Infinitesimal symmetries
  and transitive Lie algebras of vector fields
- Σ is a local subset of $T\mathbb{K}^n$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$.
  All vector fields are analytic ($\mathbb{K} = \mathbb{R}$) or holomorphic ($\mathbb{K} = \mathbb{C}$).

- Infinitesimal symmetries
  and transitive Lie algebras of vector fields
- Σ is a local subset of $T\mathbb{K}^n$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$.
  All vector fields are analytic ($\mathbb{K} = \mathbb{R}$) or holomorphic ($\mathbb{K} = \mathbb{C}$).
- A vector field germ V at $0 \in \mathbb{K}^n$
  is an infinitesimal symmetry of Σ if

$$\Phi_*^t \Sigma_x = \Sigma_{\Phi(x)}, \quad x \text{ close to } 0 \in \mathbb{K}^n, \; t \text{ close to } 0 \in \mathbb{K}$$

- Infinitesimal symmetries
  and transitive Lie algebras of vector fields

- Σ is a local subset of $T\mathbb{K}^n$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

  All vector fields are analytic ($\mathbb{K} = \mathbb{R}$) or holomorphic ($\mathbb{K} = \mathbb{C}$).

- A vector field germ V at $0 \in \mathbb{K}^n$
  is an infinitesimal symmetry of Σ if

$$\Phi_*^t \Sigma_x = \Sigma_{\Phi(x)}, \quad x \text{ close to } 0 \in \mathbb{K}^n, \; t \text{ close to } 0 \in \mathbb{K}$$

- All infinitesimal symmetries of Σ = Lie algebra of vector fields
  = symmetry algebra of $\Sigma = sym(\Sigma)$.

  Thm. Σ is homogeneous if and only if $dim\,(sym(\Sigma)(0)) = n$.

- Infinitesimal symmetries
  and transitive Lie algebras of vector fields

- Σ is a local subset of $T\mathbb{K}^n$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

  All vector fields are analytic ($\mathbb{K} = \mathbb{R}$) or holomorphic ($\mathbb{K} = \mathbb{C}$).

- A vector field germ V at $0 \in \mathbb{K}^n$
  is an <u>infinitesimal symmetry</u> of Σ if

$$\Phi^t_* \Sigma_x = \Sigma_{\Phi(x)}, \quad x \text{ close to } 0 \in \mathbb{K}^n, \ t \text{ close to } 0 \in \mathbb{K}$$

- All infinitesimal symmetries of Σ = Lie algebra of vector fields = symmetry algebra of Σ = $sym(\Sigma)$.

  Thm. Σ is homogeneous if and only if $dim\,(sym(\Sigma)(0)) = n$.

- Def. A Lie algebra L of vector field germs at $0 \in \mathbb{K}^n$ is called transitive if $dim L(0) = n$.

- Infinitesimal symmetries
  and transitive Lie algebras of vector fields
- Σ is a local subset of $T\mathbb{K}^n$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$.
  All vector fields are analytic ($\mathbb{K} = \mathbb{R}$) or holomorphic ($\mathbb{K} = \mathbb{C}$).
- A vector field germ V at $0 \in \mathbb{K}^n$
  is an infinitesimal symmetry of Σ if

$$\Phi_*^t \Sigma_x = \Sigma_{\Phi(x)}, \quad x \text{ close to } 0 \in \mathbb{K}^n, \ t \text{ close to } 0 \in \mathbb{K}$$

- All infinitesimal symmetries of Σ = Lie algebra of vector fields = symmetry algebra of $\Sigma = sym(\Sigma)$.

  Thm. Σ is homogeneous if and only if $dim\,(sym(\Sigma)(0)) = n$.
- Def. A Lie algebra L of vector field germs at $0 \in \mathbb{K}^n$ is called transitive if $dimL(0) = n$.
- Thus the symmetry algebra of any local homogeneous subset of $T\mathbb{K}^n$ is transitive. But not any transitive algebra of vector field germs is the symmetry algebra of some local homogeneous subset of $T\mathbb{K}^n$.

- Local homogeneous subsets of $T\mathbb{K}^n$
  and splitting property of transitive algebras

- Local homogeneous subsets of $T\mathbb{K}^n$ and splitting property of transitive algebras
- Any Lie algebra of vector field germs at $0 \in \mathbb{K}^n$ has a very important subalgebra consisting of vector field germs vanishing at 0. It is called isotropy subalgebra.

- Local homogeneous subsets of $T\mathbb{K}^n$ and splitting property of transitive algebras
- Any Lie algebra of vector field germs at $0 \in \mathbb{K}^n$ has a very important subalgebra consisting of vector field germs vanishing at 0. It is called isotropy subalgebra.
- Recall the basic construction of local homogeneous subsets $\Sigma \subset T\mathbb{K}^n$:

Take an n-dim Lie algebra $\mathfrak{g}$, a subset of $\mathfrak{g}$, and push this subset forward to a nbhd of id of $G = exp(\mathfrak{g})$ by the flows of left-invariant vector fields. Bring the obtained local subset of TG to a local subset $\Sigma \subset \mathbb{K}^n$ by a local analytic diffeo ($\mathbb{K} = \mathbb{C}$: biholomorphism) $(G, id) \to (\mathbb{K}^n, 0)$.

- Local homogeneous subsets of $T\mathbb{K}^n$
  and splitting property of transitive algebras
- Any Lie algebra of vector field germs at $0 \in \mathbb{K}^n$ has a very important subalgebra consisting of vector field germs vanishing at 0. It is called isotropy subalgebra.
- Recall the basic construction of local homogeneous subsets $\Sigma \subset T\mathbb{K}^n$:

Take an n-dim Lie algebra $\mathfrak{g}$, a subset of $\mathfrak{g}$, and push this subset forward to a nbhd of id of $G = exp(\mathfrak{g})$ by the flows of left-invariant vector fields. Bring the obtained local subset of TG to a local subset $\Sigma \subset \mathbb{K}^n$ by a local analytic diffeo ($\mathbb{K} = \mathbb{C}$: biholomorphism) $(G, id) \to (\mathbb{K}^n, 0)$.

- Consider the symmetry algebra $sym(\Sigma)$ and its isotropy subalgebra $I \subset sym(Sigma)$. The construction implies

$$sym(\Sigma) = \mathfrak{g} + I.$$

- Local homogeneous subsets of $T\mathbb{K}^n$ and splitting property of transitive algebras

  Let L be a transitive Lie algebra
  of vector field germs at $0 \in \mathbb{K}^n$.
  Let I be its isotropy subalgebra.
  Note that I is a subspace of L of codimension n.

  Def. I will say that L has the splitting property
  if I has a complement in L which is a Lie algebra,
  i.e. $L = \mathfrak{g} + I$ for some n-dim Lie algebra g.

- Local homogeneous subsets of $T\mathbb{K}^n$ and splitting property of transitive algebras

  Let L be a transitive Lie algebra
  of vector field germs at $0 \in \mathbb{K}^n$.
  Let I be its isotropy subalgebra.
  Note that I is a subspace of L of codimension n.

  Def. I will say that L has the splitting property
  if I has a complement in L which is a Lie algebra,
  i.e. $L = \mathfrak{g} + I$ for some n-dim Lie algebra g.

- Recall that a local homogeneous subset of $T\mathbb{K}^n$ is called special if it cannot be obtained by the given construction.

  Prop. A local homogeneous subset $\Sigma T\mathbb{K}^n$ is special
  if and only if its symmetry algebra $sym(\Sigma)$
  does not have the splitting property.

- Examples of special local homogeneous subsets of $T\mathbb{K}^n$

- Examples of special local homogeneous subsets of $T\mathbb{K}^n$

- **1.** Local Riemannian metrics (elliptic structure) on $\mathbb{R}^2$ with positive curvature.

  The symmetry algebra, as an abstract Lie algebra, is $\mathfrak{so}_3(\mathbb{R})$. The isotropy subalgebra is 1-dimensional. The splitting property does not hold simply because $\mathfrak{so}_3(\mathbb{R})$ does not contain any 2-dim subalgebras.

- Examples of special local homogeneous subsets of $T\mathbb{K}^n$
- **1.** Local Riemannian metrics (elliptic structure) on $\mathbb{R}^2$ with positive curvature.

  The symmetry algebra, as an abstract Lie algebra, is $\mathfrak{so}_3(\mathbb{R})$. The isotropy subalgebra is 1-dimensional. The splitting property does not hold simply because $\mathfrak{so}_3(\mathbb{R})$ does not contain any 2-dim subalgebras.
- **2.** The mentioned above class of local homogeneous affine line fields on $\mathbb{K}^4$, with the ∞-dim symmetry algebra

$$f(x_1)\frac{\partial}{\partial x_1}+f'(x_1)\frac{\partial}{\partial x_2}+e^{x_3}f''(x_1)\frac{\partial}{\partial x_3}+\left(e^{2x_2}f'''(x_1)+e^{x_2}f''(x_1)\right)\frac{\partial}{\partial x_4}$$

  As an abstract Lie algebra it is isomorphic to the Lie algebra $Vect(1)$ of all vector field germs on $\mathbb{K}$. The splitting property does not hold simply because $Vect(1)$ does not contain any 4-dim subalgebras (exercise).

To go on need

- **1**. Which transitive Lie algebras of vector field germs are symmetry algebras of some local homog. subset of $T\mathbb{K}^n$?

To go on need

- **1**. Which transitive Lie algebras of vector field germs are symmetry algebras of some local homog. subset of $T\mathbb{K}^n$?
- **2**. Is it possible to deal with transitive Lie algebras of vector field germs as with abstract Lie algebras endowed with a subalgebra (corresponding to the isotropy subalgebra)?

To go on need

- **1**. Which transitive Lie algebras of vector field germs are symmetry algebras of some local homog. subset of $T\mathbb{K}^n$?
- **2**. Is it possible to deal with transitive Lie algebras of vector field germs as with abstract Lie algebras endowed with a subalgebra (corresponding to the isotropy subalgebra)?
- **2.1** Let (L, I) and $(\widetilde{L}, \widetilde{I})$ are transitive Lie algebras of vector field germs on $\mathbb{K}^n$, where I and $\widetilde{I}$ are isotropy sublagebras.

  Is it true that L and $\widetilde{L}$ are diffeomorphic

  (i.e. there exists a local diffeo Φ of $\mathbb{K}^n$ such that $\Phi_* L = \widetilde{L}$)

  if and only if (L, I) and $(\widetilde{L}, \widetilde{I})$ are isomorphic,

  i.e. there exists an isomorphism from L to $\widetilde{L}$ sending I to $\widetilde{I}$?

To go on need

- **1**. Which transitive Lie algebras of vector field germs are symmetry algebras of some local homog. subset of $T\mathbb{K}^n$?
- **2**. Is it possible to deal with transitive Lie algebras of vector field germs as with abstract Lie algebras endowed with a subalgebra (corresponding to the isotropy subalgebra)?
- **2.1** Let (L, I) and $(\widetilde{L}, \widetilde{I})$ are transitive Lie algebras of vector field germs on $\mathbb{K}^n$, where I and $\widetilde{I}$ are isotropy sublagebras. Is it true that L and $\widetilde{L}$ are diffeomorphic (i.e. there exists a local diffeo Φ of $\mathbb{K}^n$ such that $\Phi_* L = \widetilde{L}$) if and only if (L, I) and $(\widetilde{L}, \widetilde{I})$ are isomorphic, i.e. there exists an isomorphism from L to $\widetilde{L}$ sending I to $\widetilde{I}$?
- **2.2** Under which condition, if any, an abstract Lie algebra L endowed with a subalgebra I of codimension n can be realized as a transitive Lie algebra of vector field germs on $\mathbb{K}^n$ with the isotropy subalgebra I?

- Transitive Lie algebras of vector field germs
  and abstract transitive Lie algebras. Nagano principle

- Transitive Lie algebras of vector field germs and abstract transitive Lie algebras. Nagano principle

- Prop. The isotropy subalgebra I of a transitive algebra L of vector field germs does not contain non-trivial ideals of L.

- Transitive Lie algebras of vector field germs and abstract transitive Lie algebras. Nagano principle
- Prop. The isotropy subalgebra I of a transitive algebra L of vector field germs does not contain non-trivial ideals of L.
- Def. An abstract transitive Lie algebra is a Lie algebra $\mathfrak{g}$ endowed with a subalgebra I (called isotropy subalgebra) containing no non-trivial ideals of the whole algebra.

  Two abstract transitive Lie algebras $(\mathfrak{g}, I)$ and $(\widetilde{\mathfrak{g}}, \widetilde{I})$ are isomorphic if there exists an isomorphism $T : \mathfrak{g} \to \widetilde{\mathfrak{g}}$ such that $T(I) = \widetilde{I}$.

- Transitive Lie algebras of vector field germs
  and abstract transitive Lie algebras. Nagano principle

- Prop. The isotropy subalgebra I of a transitive algebra L of vector field germs does not contain non-trivial ideals of L.

- Def. An abstract transitive Lie algebra is a Lie algebra $\mathfrak{g}$ endowed with a subalgebra I (called isotropy subalgebra) containing no non-trivial ideals of the whole algebra.

  Two abstract transitive Lie algebras $(\mathfrak{g}, I)$ and $(\widetilde{\mathfrak{g}}, \widetilde{I})$ are isomorphic if there exists an isomorphism $T : \mathfrak{g} \to \widetilde{\mathfrak{g}}$ such that $T(I) = \widetilde{I}$.

- Prop. Any finite-dim abstract transitive Lie algebra $(\mathfrak{g}, I)$ can be realized by a transitive Lie algebra of vector field germs on $\mathbb{K}^n$, where $n = codim\ I$.

- Transitive Lie algebras of vector field germs and abstract transitive Lie algebras. Nagano principle
- Prop. The isotropy subalgebra I of a transitive algebra L of vector field germs does not contain non-trivial ideals of L.
- Def. An abstract transitive Lie algebra is a Lie algebra $\mathfrak{g}$ endowed with a subalgebra I (called isotropy subalgebra) containing no non-trivial ideals of the whole algebra.

  Two abstract transitive Lie algebras $(\mathfrak{g}, I)$ and $(\widetilde{\mathfrak{g}}, \widetilde{I})$ are isomorphic if there exists an isomorphism $T : \mathfrak{g} \to \widetilde{\mathfrak{g}}$ such that $T(I) = \widetilde{I}$.
- Prop. Any finite-dim abstract transitive Lie algebra $(\mathfrak{g}, I)$ can be realized by a transitive Lie algebra of vector field germs on $\mathbb{K}^n$, where $n = codim\ I$.
- Thm (H.J. Sussmann, 1974) Two finite-dim transitive Lie algebras of vector field germs on $\mathbb{K}^n$ are diffeomorphic if and only if they are isomorphic as abstract transitive Lie algebras.

Prop. The isotropy subalgebra I of a transitive algebra L of vector field germs does not contain non-trivial ideals of L.

Proof. Let $R \in I$ be an ideal of L. Assume $R \neq \{0\}$. Take in R a vector field V whose Taylor series starts with terms of minimal possible order $d \geq 1$ (here we use analyticity):

$$R \supset V = V^{(d)} + h.o.t., \;\; V^{(d)} \text{ homogeneous of order } d$$

Since L is transitive, L contains vector fields

$$a_i = \frac{\partial}{\partial x_i} + h.o.t.$$

Since R is an ideal, $[a_i, V] \in R$. We have

$$[a_i, V] = \left[\frac{\partial}{\partial x_i}, V^{(d)}\right] + h.o.t.$$

If at least one of $\left[\frac{\partial}{\partial x_i}, V^{(d)}\right]$ is not zero we get contradiction to the assumption that d is minimal possible. Therefore

$$\left[\frac{\partial}{\partial x_i}, V^{(d)}\right] = 0, \quad i = 1, ..., n.$$

Since $d \geq 1$ it follows $V^{(d)} = 0$. Contradiction.

Prop. Any finite-dim abstract transitive Lie algebra $(\mathfrak{g}, I)$ can be realized by a transitive Lie algebra of vector field germs on $\mathbb{K}^n$, where $n = codim\ I$.

Proof. Let G be a nbhd of id of a Lie group of $\mathfrak{g}$. Factorize G by the equivalence $g_1 \sim g_2$ if $g_2 = g_1 e^{\lambda}$ for some $\lambda \in I$. We obtain an n-dimensional manifold M. The left-invariant vector fields on G define vector fields on M. We obtain a Lie algebra L of vector fields on M.

Let $\lambda \in I$. It defines a flow on M: $g \to e^{t\lambda} g$. We have $id \to e^{t\lambda} \sim id$. Therefore the left invariant vector field defined by vectors in I give, after the factorization, vector field on M which vanish at $id \in M$. Therefore they belong to the isotropy subalgebra of L. The fact that the isotropy subalgebra of L contains nothing else and $dimL(id) = n$ follows from the condition that I contains no non-trivial ideals of $\mathfrak{g}$.

- Example: $(\mathfrak{so}_3(\mathbb{C}), I)$, $dim\ I = 1$
  There are two such transitive Lie algebras, up to isomorphisms, corresponding to singular and non-singular directions $span(b)$ in $\mathfrak{so}_3(\mathbb{C})$. The singular directions are defined by: $ad(b)$ is a nilpotent operator.

  In the standard basis $[e_1, e_2] = e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2$ the singular directions are spanned by vectors $r_1 e_1 + r_2 e_2 + r_3 e_3$ where $r_1^2 + r_2^2 + r_3^2 = 0$.

- Example: $(\mathfrak{sl}_2(\mathbb{R}), I)$, $dim\ I = 1$
  There are three such transitive Lie algebras, up to isomorphisms, corresponding to singular, hyperbolic and elliptic directions. The singular directions $span(b)$ are defined as for $\mathfrak{so}_3(\mathbb{C})$, but now they form a cone separating

  $ad(b)$ has eigenvalues $0, \pm 1$ (up to a factor): hyperbolic
  $ad(b)$ has eigenvalues $0, \pm i$ (up to a factor): elliptic

- Claim. Any transitive Lie algebra $(\mathfrak{sl}_2(\mathbb{R}), I)$ has the splitting property.

  If *dim* $I = 2$ it is obvious. If *dim* $I = 1$ one has

$$\mathfrak{sl}_2(\mathbb{R}) = \mathfrak{b}_2 + I$$

- Claim. Any transitive Lie algebra $(\mathfrak{sl}_2(\mathbb{R}), I)$ has the splitting property.

  If *dim* $I = 2$ it is obvious. If *dim* $I = 1$ one has

$$\mathfrak{sl}_2(\mathbb{R}) = \mathfrak{b}_2 + I$$

- Example:

  for $\mathfrak{sl}_2 : \; [a_1, a_2] = b, \; [a_1, b] = a_1, \; [a_2, b] = -a_2$

  and $I = span(b)$

  one has $\mathfrak{b}_2 = span(a_1 + r_1 b, \; a_2 + r_2 b)$ where $r_1 r_2 = 1/2$.

- Claim. Any transitive Lie algebra $(\mathfrak{sl}_2(\mathbb{R}), I)$ has the splitting property.

  If *dim* $I = 2$ it is obvious. If *dim* $I = 1$ one has

$$\mathfrak{sl}_2(\mathbb{R}) = \mathfrak{b}_2 + I$$

- Example:

  for $\mathfrak{sl}_2 : \ [a_1, a_2] = b, \ [a_1, b] = a_1, \ [a_2, b] = -a_2$

  and $I = span(b)$

  one has $\mathfrak{b}_2 = span(a_1 + r_1 b, \ a_2 + r_2 b)$ where $r_1 r_2 = 1/2$.

- In $\mathfrak{so}_3(\mathbb{R})$, unlike $\mathfrak{so}_3(\mathbb{C})$, all directions I are equal (automorphic);

  they are equally bad because $(\mathfrak{so}_3(\mathbb{C}), I)$ does not have the splitting property: $\mathfrak{so}_3(\mathbb{C})$ does not have any 2-dim subalgbras.

Ellipses and hyperbolas in $\mathfrak{b}_2 : [e_1, e_2] = e_2$ up to automorphisms	Local homog. subsets of $T\mathbb{R}^2$ up to diffeos	symm. algebra is transitive Lie algebra
Ellipse $x_1^2 + \theta x_2^2 = 1$ $\theta > 0$	Riemannian metric with negative curvature $-\theta$	$(\mathfrak{sl}_2(\mathbb{R}), I)$ elliptic 1-dim I
Hyperbola $x_1^2 - \theta x_2^2 = 1$ $\theta > 0$	Hyperbolic struct. with negative curvature $-\theta$	$(\mathfrak{sl}_2(\mathbb{R}), I)$ hyperb. 1-dim I
Hyperbola $x_2^2 - \theta x_1^2 = 1$ $\theta > 0$	Hyperbolic struct. with positive curvature θ	$(\mathfrak{sl}_2(\mathbb{R}), I)$ hyperb. 1-dim I
Hyperbola $x_1 x_2 = 1$	Hyperbolic struct. with zero curvature	$\left((\mathbb{R} + \mathfrak{a}^2)_{\pm 1}, \mathbb{R} \right)$ [z,x]=x, [z,y]= -y, [x,y]=0

- Not any transitive Lie algebra
  is a symmetry algebra of a local homogeneous subset of $T\mathbb{K}^n$

- Not any transitive Lie algebra
  is a symmetry algebra of a local homogeneous subset of $T\mathbb{K}^n$
- The table contains transitive Lie algebras $(\mathfrak{sl}_2(\mathbb{R}), I)$ with hyperbolic and with elliptic, but <u>not</u> singular 1-dim I. Why?

- Not any transitive Lie algebra
  is a symmetry algebra of a local homogeneous subset of $T\mathbb{K}^n$

- The table contains transitive Lie algebras $(\mathfrak{sl}_2(\mathbb{R}), I)$ with hyperbolic and with elliptic, but not singular 1-dim I. Why?

- Claim: if $I \in \mathfrak{sl}_2(\mathbb{K})$ is singular,
  there is no local homogeneous subset of $T\mathbb{K}^2$
  with the symmetry algebra $(\mathfrak{sl}_2(\mathbb{K}), I)$.

- Not any transitive Lie algebra is a symmetry algebra of a local homogeneous subset of $T\mathbb{K}^n$
- The table contains transitive Lie algebras $(\mathfrak{sl}_2(\mathbb{R}), I)$ with hyperbolic and with elliptic, but <u>not</u> singular 1-dim I. Why?
- Claim: if $I \in \mathfrak{sl}_2(\mathbb{K})$ is singular, there is no local homogeneous subset of $T\mathbb{K}^2$ with the symmetry algebra $(\mathfrak{sl}_2(\mathbb{K}), I)$.
- Prop. If Σ is a local homogeneous subset of $T\mathbb{K}^n$ and $sym(\Sigma)$ contains a transitive Lie algebra (L, I) which belongs to a bigger transitive Lie algebra $(\widetilde{L}, \widetilde{I})$ and the central part of these two transitive Lie algebras are the same or geometrically-same then $sim(\Sigma)$ also contains $(\widetilde{L}, \widetilde{I})$.

- Not any transitive Lie algebra
  is a symmetry algebra of a local homogeneous subset of $T\mathbb{K}^n$
- The table contains transitive Lie algebras $(\mathfrak{sl}_2(\mathbb{R}), I)$ with hyperbolic and with elliptic, but not singular 1-dim I. Why?
- Claim: if $I \in \mathfrak{sl}_2(\mathbb{K})$ is singular,
  there is no local homogeneous subset of $T\mathbb{K}^2$
  with the symmetry algebra $(\mathfrak{sl}_2(\mathbb{K}), I)$.
- Prop. If Σ is a local homogeneous subset of $T\mathbb{K}^n$ and $sym(\Sigma)$ contains a transitive Lie algebra (L, I) which belongs to a bigger transitive Lie algebra $(\widetilde{L}, \widetilde{I})$ and the central part of these two transitive Lie algebras are the same or geometrically-same then $sim(\Sigma)$ also contains $(\widetilde{L}, \widetilde{I})$.
- For $(\mathfrak{sl}_2(\mathbb{R}), I)$ with singular I: $\widetilde{L} = \left\{ f'(x_2)\frac{\partial}{\partial x_1} + f(x_2)\frac{\partial}{\partial x_2} \right\}$.

- Central part of a transitive Lie algebra

  Def. The central part of a Lie algebra (L, l) of vector field germs at $0 \in \mathbb{K}^n$ is the linear approximations of the vector fields $V \in l$ at 0.

  These linear approximation are <u>linear</u> vector fields <u>on $T_0\mathbb{K}^n$</u> and form a subalgebra of $\mathfrak{gl}_n(\mathbb{K})$.

- Central part of a transitive Lie algebra

  Def. The central part of a Lie algebra (L, l) of vector field germs at $0 \in \mathbb{K}^n$ is the linear approximations of the vector fields $V \in l$ at 0.

  These linear approximation are linear vector fields on $T_0\mathbb{K}^n$ and form a subalgebra of $\mathfrak{gl}_n(\mathbb{K})$.

- The same definition in terms of abstract transitive Lie algebra $(\mathfrak{g}, l)$:

  The central part is the factor-algebra $l/\sim$ where $b_1, b_2 \in l$ are equivalent if $[(b_1 - b_2), \mathfrak{g}] \subseteq l$.

- Central part of a transitive Lie algebra

  Def. The central part of a Lie algebra (L, l) of vector field germs at $0 \in \mathbb{K}^n$ is the linear approximations of the vector fields $V \in l$ at 0.

  These linear approximation are linear vector fields on $T_0\mathbb{K}^n$ and form a subalgebra of $\mathfrak{gl}_n(\mathbb{K})$.

- The same definition in terms of abstract transitive Lie algebra $(\mathfrak{g}, l)$:

  The central part is the factor-algebra $l/\sim$ where $b_1, b_2 \in l$ are equivalent if $[(b_1 - b_2), \mathfrak{g}] \subseteq l$.

- If two abstract transitive Lie algebras are isomorphic then their central parts $\mathcal{A}_1, \mathcal{A}_2 \subset \mathfrak{gl}_n(\mathbb{K})$ are conjugate: $T^{-1}\mathcal{A}_1 T = \mathcal{A}_2$ for some non-singular T, which is much stronger then isomorphic.

- Why $(\mathfrak{sl}_2(\mathbb{K}), I)$ with <u>singular</u> I belongs <u>namely to</u> $f'(x_2)\frac{\partial}{\partial x_1} + f(x_2)\frac{\partial}{\partial x_2}$ and why they have the same central part?

- Why $(\mathfrak{sl}_2(\mathbb{K}), I)$ with singular I belongs namely to $f'(x_2)\frac{\partial}{\partial x_1} + f(x_2)\frac{\partial}{\partial x_2}$ and why they have the same central part?
- The central part of $(\mathfrak{sl}_2(\mathbb{K}), I)$ with singular I is the vector field $x_2\frac{\partial}{\partial x_1}$ on $T_0\mathbb{K}^2$, defined up to $GL(2)$.

- Why $(\mathfrak{sl}_2(\mathbb{K}), I)$ with singular I belongs namely to $f'(x_2)\frac{\partial}{\partial x_1} + f(x_2)\frac{\partial}{\partial x_2}$ and why they have the same central part?
- The central part of $(\mathfrak{sl}_2(\mathbb{K}), I)$ with singular I is the vector field $x_2\frac{\partial}{\partial x_1}$ on $T_0\mathbb{K}^2$, defined up to $GL(2)$.
- The key point is that the vector field $x_2\frac{\partial}{\partial x_1}$ on $T_0\mathbb{K}^2$ has an invariant affine line $l^{af} \subset T_0\mathbb{K}^2$, for example $\frac{\partial}{\partial x_2} + span(\frac{\partial}{\partial x_1})$.

- Why $(\mathfrak{sl}_2(\mathbb{K}), l)$ with singular l belongs namely to $f'(x_2)\frac{\partial}{\partial x_1} + f(x_2)\frac{\partial}{\partial x_2}$ and why they have the same central part?
- The central part of $(\mathfrak{sl}_2(\mathbb{K}), l)$ with singular l is the vector field $x_2\frac{\partial}{\partial x_1}$ on $T_0\mathbb{K}^2$, defined up to $GL(2)$.
- The key point is that the vector field $x_2\frac{\partial}{\partial x_1}$ on $T_0\mathbb{K}^2$ has an invariant affine line $l^{af} \subset T_0\mathbb{K}^2$, for example $\frac{\partial}{\partial x_2} + span(\frac{\partial}{\partial x_1})$.
- Push forward ℓ^{af} by the flows of sl_2. We obtain homogeneous affine line field L^{af} in $T\mathbb{K}^2$ with a certain transitive Lie algebra which contains $(\mathfrak{sl}_2(\mathbb{K}), l)$.

- Why $(\mathfrak{sl}_2(\mathbb{K}), I)$ with singular I belongs namely to $f'(x_2)\frac{\partial}{\partial x_1} + f(x_2)\frac{\partial}{\partial x_2}$ and why they have the same central part?
- The central part of $(\mathfrak{sl}_2(\mathbb{K}), I)$ with singular I is the vector field $x_2\frac{\partial}{\partial x_1}$ on $T_0\mathbb{K}^2$, defined up to $GL(2)$.
- The key point is that the vector field $x_2\frac{\partial}{\partial x_1}$ on $T_0\mathbb{K}^2$ has an invariant affine line $l^{af} \subset T_0\mathbb{K}^2$, for example $\frac{\partial}{\partial x_2} + span(\frac{\partial}{\partial x_1})$.
- Push forward ℓ^{af} by the flows of sl_2. We obtain homogeneous affine line field L^{af} in $T\mathbb{K}^2$ with a certain transitive Lie algebra which contains $(\mathfrak{sl}_2(\mathbb{K}), I)$.
- L^{af} can be described by a differential 1-form α, $\alpha(0) \neq 0$: $L_x = \{v \in T_x\mathbb{K}^n : \ \alpha(x) = 1\}$. Due to homogeneity either $d\alpha(0) \neq 0$ or $d\alpha \equiv 0$. If $d\alpha(0) \neq 0$ then the central part of $sym(L^{af})$ is conjugate to $x_2\frac{\partial}{\partial x_1}$.

- Why $(\mathfrak{sl}_2(\mathbb{K}), I)$ with singular I belongs namely to $f'(x_2)\frac{\partial}{\partial x_1} + f(x_2)\frac{\partial}{\partial x_2}$ and why they have the same central part?
- The central part of $(\mathfrak{sl}_2(\mathbb{K}), I)$ with singular I is the vector field $x_2\frac{\partial}{\partial x_1}$ on $T_0\mathbb{K}^2$, defined up to $GL(2)$.
- The key point is that the vector field $x_2\frac{\partial}{\partial x_1}$ on $T_0\mathbb{K}^2$ has an invariant affine line $l^{af} \subset T_0\mathbb{K}^2$, for example $\frac{\partial}{\partial x_2} + span(\frac{\partial}{\partial x_1})$.
- Push forward ℓ^{af} by the flows of sl_2. We obtain homogeneous affine line field L^{af} in $T\mathbb{K}^2$ with a certain transitive Lie algebra which contains $(\mathfrak{sl}_2(\mathbb{K}), I)$.
- L^{af} can be described by a differential 1-form α, $\alpha(0) \neq 0$: $L_x = \{v \in T_x\mathbb{K}^n : \ \alpha(x) = 1\}$. Due to homogeneity either $d\alpha(0) \neq 0$ or $d\alpha \equiv 0$. If $d\alpha(0) \neq 0$ then the central part of $sym(L^{af})$ is conjugate to $x_2\frac{\partial}{\partial x_1}$.
- All α such that $\alpha(0) \neq 0$ and $d\alpha(0) \neq 0$ are locally diffeomorphic, for example to e $\alpha = e^{x_1}dx_2$. The symmetry algebra of this α, i.e. the vector fields Z such that $\mathcal{L}_Z\alpha = d(Z\rfloor\alpha) + Z\rfloor d\alpha = 0$ is exactly $f'(x_1)\frac{\partial}{\partial x_1} + f(x_1)\frac{\partial}{\partial x_2}$.

- As mentioned, I know few examples of special local homogeneous subset of $T\mathbb{C}^n$ (which is not defined up to biholomorphisms by a subset of an n-dim complex Lie algebra) whose symmetry algebra is ∞-dimensional.

- As mentioned, I know few examples of special local homogeneous subset of $T\mathbb{C}^n$ (which is not defined up to biholomorphisms by a subset of an n-dim complex Lie algebra) whose symmetry algebra is ∞-dimensional.
- I do not know such examples with a finite-dimensional complex symmetry algebra because I do not know examples of complex finite-dim transitive Lie algebras which:

  (a) do not have the splitting property AND

  (b) are geometrically-maximal = symmetry algebras of some local homogeneous subset of the tangent bundle.

- As mentioned, I know few examples of special local homogeneous subset of $T\mathbb{C}^n$ (which is not defined up to biholomorphisms by a subset of an n-dim complex Lie algebra) whose symmetry algebra is ∞-dimensional.
- I do not know such examples with a finite-dimensional complex symmetry algebra because I do not know examples of complex finite-dim transitive Lie algebras which:

  (a) do not have the splitting property AND

  (b) are geometrically-maximal = symmetry algebras of some local homogeneous subset of the tangent bundle.
- Exercise. Give an example of a finite-dimensional complex transitive Lie algebra which does not have the splitting property (refusing from (b)). The simplest I know is $span(a_1, a_2, a_3, b)$, $[a_2, b] = a_1$, $[a_2, a_3] = b$, all other brackets are zero, the isotropy subalgebra is $span(b)$.

- As mentioned, I know few examples of special local homogeneous subset of $T\mathbb{C}^n$ (which is not defined up to biholomorphisms by a subset of an n-dim complex Lie algebra) whose symmetry algebra is ∞-dimensional.
- I do not know such examples with a finite-dimensional complex symmetry algebra because I do not know examples of complex finite-dim transitive Lie algebras which:

  (a) do not have the splitting property AND

  (b) are geometrically-maximal = symmetry algebras of some local homogeneous subset of the tangent bundle.
- Exercise. Give an example of a finite-dimensional complex transitive Lie algebra which does not have the splitting property (refusing from (b)). The simplest I know is $span(a_1, a_2, a_3, b)$, $[a_2, b] = a_1$, $[a_2, a_3] = b$, all other brackets are zero, the isotropy subalgebra is $span(b)$.
- But unfortunately (or fortunately?) this transitive Lie algebra is not the symmetry algebra of any local homogeneous subset of $T\mathbb{C}^4$.

- Prop. If Σ is a local homogeneous subset of $T\mathbb{K}^n$ and $sym(\Sigma)$ contains a transitive Lie algebra (L, I) which belongs to a bigger transitive Lie algebra $(\widetilde{L}, \widetilde{I})$ and the central part of these two transitive Lie algebras are the same or geometrically-same then $sim(\Sigma)$ also contains $(\widetilde{L}, \widetilde{I})$.

- Prop. If Σ is a local homogeneous subset of $T\mathbb{K}^n$ and $sym(\Sigma)$ contains a transitive Lie algebra (L, l) which belongs to a bigger transitive Lie algebra $(\widetilde{L}, \widetilde{l})$ and the central part of these two transitive Lie algebras are the same or geometrically-same then $sim(\Sigma)$ also contains $(\widetilde{L}, \widetilde{l})$.

- Definition of geometrically-same

  Two Lie subalgebras $\mathfrak{g}_1, \mathfrak{g}_2 \subset \mathfrak{gl}_n(\mathbb{K})$, viewed as <u>linear vector fields on $T_0\mathbb{K}^n$</u> (and therefore defined up to conjugacy rather then isomorphisms) are geometrically same if any $\mathfrak{g}_1$-invariant subset of $T_0\mathbb{K}^n$ is also $\mathfrak{g}_2$-invariant.

- Prop. If Σ is a local homogeneous subset of $T\mathbb{K}^n$ and $sym(\Sigma)$ contains a transitive Lie algebra (L, I) which belongs to a bigger transitive Lie algebra $(\widetilde{L}, \widetilde{I})$ and the central part of these two transitive Lie algebras are the same or geometrically-same then $sim(\Sigma)$ also contains $(\widetilde{L}, \widetilde{I})$.

- Definition of geometrically-same

  Two Lie subalgebras $\mathfrak{g}_1, \mathfrak{g}_2 \subset \mathfrak{gl}_n(\mathbb{K})$, viewed as linear vector fields on $T_0\mathbb{K}^n$ (and therefore defined up to conjugacy rather then isomorphisms) are geometrically same if any $\mathfrak{g}_1$-invariant subset of $T_0\mathbb{K}^n$ is also $\mathfrak{g}_2$-invariant.

- For example $\mathfrak{sl}_n(\mathbb{K})$ is geometrically the same as $\mathfrak{gl}_n(\mathbb{K})$ since the singular locus of $\mathfrak{sl}_n(\mathbb{K})$, defined as the set of points in $T_0\mathbb{K}^n$ at which $dim\mathfrak{sl}_n(\mathbb{K})$ is less then n, is simply $\{0\}$.

- Prop. If Σ is a local homogeneous subset of $T\mathbb{K}^n$ and $sym(\Sigma)$ contains a transitive Lie algebra (L, I) which belongs to a bigger transitive Lie algebra $(\widetilde{L}, \widetilde{I})$ and the central part of these two transitive Lie algebras are the same or geometrically-same then $sim(\Sigma)$ also contains $(\widetilde{L}, \widetilde{I})$.

- Definition of geometrically-same

  Two Lie subalgebras $\mathfrak{g}_1, \mathfrak{g}_2 \subset \mathfrak{gl}_n(\mathbb{K})$, viewed as linear vector fields on $T_0\mathbb{K}^n$ (and therefore defined up to conjugacy rather then isomorphisms) are geometrically same if any $\mathfrak{g}_1$-invariant subset of $T_0\mathbb{K}^n$ is also $\mathfrak{g}_2$-invariant.

- For example $\mathfrak{sl}_n(\mathbb{K})$ is geometrically the same as $\mathfrak{gl}_n(\mathbb{K})$ since the singular locus of $\mathfrak{sl}_n(\mathbb{K})$, defined as the set of points in $T_0\mathbb{K}^n$ at which $dim\mathfrak{sl}_n(\mathbb{K})$ is less then n, is simply $\{0\}$.

- There are many other examples with much more involved subalgebras of $gl(n)$ defined up to conjugacy.

- The big problem in terms of transitive Lie algebras

- The big problem in terms of transitive Lie algebras
- Def. A transitive Lie algebra is geometrically-maximal if it is not contained in a bigger transitive Lie algebra with the same or geometrically same central part.

  Prop. A transitive Lie algebra is the symmetry algebra of a local homogeneous subset of $T\mathbb{R}^n$ if and only if it is geometrically-maximal.

- The big problem in terms of transitive Lie algebras
- Def. A transitive Lie algebra is geometrically-maximal if it is not contained in a bigger transitive Lie algebra with the same or geometrically same central part.

  Prop. A transitive Lie algebra is the symmetry algebra of a local homogeneous subset of $T\mathbb{R}^n$ if and only if it is geometrically-maximal.
- Problem. To find geometrically transitive Lie algebras which do not have the splitting property.

- Thanks to Amos Nevo for a number of conversations introducing me to the magic of Lie algebras and proving for me several lemmas I could not prove myself.

  Thanks to Uri Bader who answered many my questions.

- Thanks to Amos Nevo for a number of conversations introducing me to the magic of Lie algebras and proving for me several lemmas I could not prove myself.

  Thanks to Uri Bader who answered many my questions.

- Thanks to the listeners.

  Any comment will be strongly appreciated.

  This talk is published in my homepage.