Talk in International Workshop, Warsaw, June 2014

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Question: which local homogeneous subsets S of the tangent bundle to an n-manifold M are induced by a subset of an n-dimensional Lie algebra and which are not?

Homogeneous: the same germ (wrt to a point in M, not wrt to a tangent vector) at any two points $p_1, p_2 \in M$ an analytic (holomorphic) local diffeo $\Phi : (M, p_1) \to (M, p_2)$:
$\Phi_{*,x} S_x = S_{\Phi(x)}$, $S_x = S \cap T_x M$, x close to p_1.

Local: with respect to a point in M (not a tangent vector) so that $M = (\mathbb{R}^n, 0)$ or $M = (\mathbb{C}^n, 0)$

(if $\mathbb{C}^n$: holomorphic tangent bundle)

Motivation for the question

personal, from the "(2,5) variables paper" by E. Cartan

Take any 5-dim Lie algebra real or complex Lie algebra $\mathfrak{g}$ and a 2-plane (2-dim subspace) $P \subset g$. Push P forward by left-invariant vector fields to a neighbourhood of *id* of the Lie group of $\mathfrak{g}$ to obtain a homogeneous 2-distribution germ D on $T\mathbb{R}^5$ or $T_h\mathbb{C}^5$ whose symmetry algebra obviously contains $\mathfrak{g}$, but might be bigger.

In order that D has max growth vector (2,3,5) the plane P must be generating: $P = span(v_1, v_2)$,
$\mathfrak{g} = span\{v_1, v_2, [v_1, v_2], [v_1, [v_1, v_2]], [v_2, [v_1, v_2]]\}$

One can expect that taking in this construction the most symmetric $(P, \mathfrak{g})$ (the biggest group of automorphisms) we obtain the most symmetric (2,3,5) distribution germ D.

the most symmetric $(P, \mathfrak{g})$ is the graded nilpotent Lie algebra
$\mathfrak{g}$: $[a_1, a_2] = a_3, [a_1, a_3] = a_4, [a_2, a_3] = a_5$, other $[a_1, a_j] = 0$,
and any generating 2-plane in $\mathfrak{g}$
(all such P are automorphic to $span\{a_1, a_2\}$).

Take any 5-dim Lie algebra real or complex Lie algebra $\mathfrak{g}$ and a generating 2-plane $P \subset g$. Push P forward to by left-invariant vector fields a nbhd of *id* of the Lie group of $\mathfrak{g}$ to obtain a homogeneous (2,3,5)-distribution germ D on $T\mathbb{R}^5$ or $T_h\mathbb{C}^5$ whose symmetry algebra obviously contains $\mathfrak{g}$, but might be bigger. We will say that D is induced by $(P, \mathfrak{g})$.

Most known result of the "(2,5) variables paper". Let $\mathfrak{g}$ be the graded nilpotent (2,3,5) Lie algebra and let $P \subset \mathfrak{g}$ be a generating plane. The homogeneous (2,3,5) distribution germ D induced by $(P, \mathfrak{g})$ is the most symmetric: its symmetry algebra is the 1-4dim Lie algebra g_2. For this D and for this D only (up to diffeos) the Cartan tensor is identically zero.

Question. Is ANY homogeneous (2,3,5) distribution germ germ induced by some $(P, \mathfrak{g})$ for some 5-dim Lie algebra $\mathfrak{g}$ and some generating 2-plane $P \subset \mathfrak{g}$?

Equivalent question. Is it true that any homogeneous (2,3,5) distribution germ can be described by two vector fields generating a 5-dim Lie algebra?

Take any 5-dim Lie algebra real or complex Lie algebra $\mathfrak{g}$ and a generating 2-plane $P \subset g$. Push P forward by left-invariant vector fields to a nbhd of *id* of the Lie group of $\mathfrak{g}$ to obtain a homogeneous (2,3,5)-distribution germ D on $T\mathbb{R}^5$ or $T_h\mathbb{C}^5$ whose symmetry algebra obviously contains $\mathfrak{g}$, but might be bigger. We will say that D is induced by $(P, \mathfrak{g})$.

Question. Is ANY homogeneous (2,3,5) distribution germ germ induced by some $(P, \mathfrak{g})$ for some 5-dim Lie algebra $\mathfrak{g}$ and some generating 2-plane $P \subset \mathfrak{g}$?

Equivalent question. Is it true that any homogeneous (2,3,5) distribution germ can be described by two vector fields generating a 5-dim Lie algebra?

Equivalent question. Is it true that the symmetry algebra of any homogeneous (2,3,5) distribution germ has the splitting property?

Answer(M.Zh). Over $\mathbb{C}$: yes, over $\mathbb{R}$ not always.

The splitting property of a transitive Lie algebra of vector fields.

The symmetry algebra of any local homogeneous subset of $T\mathbb{R}^n$ or $T_h\mathbb{C}^n$, in particular homogeneous (2,3,5) distibution germ on $\mathbb{R}^n$ or $\mathbb{C}^n$ is a transitive Lie algebra $\mathcal{A}$ of vector fields germs at 0:
$dim\,\mathcal{A}(0) = n$
(i.e. $\mathcal{A}$ contains vector fields $\frac{\partial}{\partial x_i} + h.o.t.,\ \ i = 1, ..., n$).

The isotropy subalgebra of a transitive Lie algebra $\mathcal{A}$ of vector field germs at 0 is $\{V \in \mathcal{A} :\ \ V(0) = 0\}$.

The splitting property of of a transitive Lie algebra $\mathcal{A}$ of vector field germs at $0 \in \mathbb{K}^n$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$, with the isotropy subalgebra $\mathcal{I}$:

$$\mathcal{A} = \mathcal{I} + \mathcal{C} \text{ (direct sum of vector spaces)}$$

for some n-dimensional Lie algebra $\mathcal{C}$,
i.e. $\mathcal{I}$ admits a Lie algebra vector-space complement in $\mathcal{A}$.
(We do not require $[\mathcal{I}, \mathcal{C}] = 0$).

Question. Is ANY homogeneous (2,3,5) distribution germ germ induced by some $(P, \mathfrak{g})$ for some 5-dim Lie algebra $\mathfrak{g}$ and some generating 2-plane $P \subset \mathfrak{g}$?

Equivalent question. Is it true that any homogeneous (2,3,5) distribution germ can be described by two vector fields generating a 5-dim Lie algebra?

Equivalent question. Is it true that the symmetry algebra of any homogeneous (2,3,5) distribution germ has the splitting property?

Answer(M.Zh). Over $\mathbb{C}$: yes, over $\mathbb{R}$ not always.

To get this answered I checked the splitting property of all possible symmetry algebras of dim > 5 of local homogeneous (2,3,5) distributions (for dim =5 the splitting property is obvious).

It required intensive study the second, less known part of the "(2,5) variables paper" .

Homogeneous (2,3,5)-distributions with a symmetry algebra of dim > 5 are a part of (2,3,5) distributions with constant and symmetric Cartan tensor.

Constant and symmetric Cartan tensor: the second part of the "(2,5) variables paper". Cartan analyzed the tensors $x_1^2 x_2^2$ and x_1^4 which are all possible symmetric Cartan tensors over $\mathbb{C}$ ($x_1^3 x_2$ is not realizable).

Over $\mathbb{R}$: $\pm x_1^2 x_2^2, \pm(x_1^2 + x_2^2)^2, \pm x_1^4)$

Cartan's results plus a certain work to understand and extend them lead to the explicit classification of the symmetry algebras of dimension > 5 of homogeneous (2,3,5) distribution germs. For each of them the splitting property holds over $\mathbb{C}$, for some of them it does not hold over $\mathbb{R}$.

Classification of symmetry algebras of dim > 5 of local homogeneous (2,3,5) distributions, on top of g_2

• four -1parameter families $S^6_{i,\lambda}, \lambda \neq \lambda^*$ of 6-dim semi-simple symmetry algebras, λ is a modulus wrt diffeos, isomorphic to

$\mathfrak{sl}_2 \oplus \mathfrak{sl}_2,\ \mathfrak{sl}_2 \oplus \mathfrak{so}_3,\ \mathfrak{so}_{3,1},\ \mathfrak{so}_3 \oplus \mathfrak{so}_3,$

• two -1parameter families $S^7_{i,\lambda}$, $\lambda \neq \lambda^*$ of solvable 7-dim symmetry algebra, λ is a modulus wrt isomorphisms.

The symmetry algebra $S^6_{4,\lambda}$, isomorphic to $\mathfrak{so}_3 \oplus \mathfrak{so}_3$, is the symmetry algebra for two rolling balls,
the case $\lambda = \lambda^*$ (when it is not realizable, leads to g_2) corresponds to the ratio of radii $1 : 3$ (R. Bryant).

Thm. Let $n = 5$. Any local homogeneous subset $S \subset T_h\mathbb{C}^n$ which is a (2,3,5) distribution is induced by $(U, \mathfrak{g})$ where $\mathfrak{g}$ is some n-dim complex Lie algebra and U is a subset of $\mathfrak{g}$ which is a generating 2-plane in $\mathfrak{g}$.

Equivalent statement: the symmetry algebra of S has the splitting property.

Question (French style). Is it possible to remove the red color in this theorem?

Question. Is it true that any local homogeneous subset $S \subset T_h\mathbb{C}^n$ is induced by some subset of some complex n-dimensional complex Lie algebra?

Equivalent question: is it true that the symmetry algebra of S has the splitting property?

Question. Is it true that any local homogeneous subset $S \subset T_h\mathbb{C}^n$ is induced by some subset of some complex n-dimensional complex Lie algebra?

Equivalent question: is it true that the symmetry algebra of S has the splitting property?

Here S is arbitrary, including potato field,
but for potato field and any "non-symmetric" S,
namely S such that $S \cap T_0\mathbb{C}^n$, and consequently $S \cap T_p\mathbb{C}^n$ for p close to 0 has no infinitesimal symmetries in $\mathfrak{gl}_n$,
the answer is always yes by the following reason:

the splitting property holds because the central part of the symmetry algebra is $\{0\}$.

The central part of a transitive Lie algebra $\mathcal{A}$ of vector field germs at 0 is, by def, the Lie algebra of the linearizations at 0 of the vector fields in the isotropy subalgebra of $\mathcal{A}$.

General property of transitive Lie algebras $\mathcal{A}$ of vector field germs on $\mathbb{R}^n$ or $\mathbb{C}^n$: if the central part of $\mathcal{A}$ is $\{0\}$ then $\mathcal{A}$ is n-dimensional (and the splitting property is obvious).

Question. Is it true that any local homogeneous subset $S \subset T_h\mathbb{C}^n$ is induced by some subset of some complex n-dimensional complex Lie algebra?

Equivalent question: is it true that the symmetry algebra of S has the splitting property?

Thm.

For $n = 2$: yes

For $n \geq 4$: no, a counterexample can be found already within affine line fields

For $n = 3$ the only counterexample is the symmetry algebra spanned by the vector fields

$$\frac{\partial}{\partial x_1}, \ e^{x_1}\frac{\partial}{\partial x_2}, \ e^{x_2}f(x_1)\frac{\partial}{\partial x_3}$$

where $f(x_1)$ is an arbitrary function. It is the symmetry algebra of (for example) the couple consisting of semi-integrable affine plane field and transversal semi-integrable affine line field.

Affine plane field in $T_h\mathbb{C}^3$, $T\mathbb{R}^3$

in each tangent space an affine plane (a plane which does not contain 0).

Can be described by $\alpha = 1$ where α is a -1form

Homogeneous:

integrable: $d\alpha \equiv 0$,

semi-integrable: $d\alpha(0) \neq 0$, $\alpha \wedge d\alpha \equiv 0$

contact: $(\alpha \wedge d\alpha)(0) \neq 0$

Affine line fields in $T_h\mathbb{C}^n$, $T\mathbb{R}^n$:

in each tangent space a straight line which does not contain 0.

Can be described by $V + (W)$, V, W are vector fields

Homogeneous:

integrable: $[V, W] \equiv 0 \ mod \ (W)$

semi-integrable: $[V, W] \equiv \theta V \ mod(W)$, $\theta \neq 0$

contact: $[V, W](0) \notin span\{V(0), W(0)\}$

(there are many types of contact)

Affine line fields on $\mathbb{C}^n$, $n \geq 4$

which are not induced by an affine line in any n- dim Lie algebra

It is the most degenerate of many types of homogeneous contact affine line fields. For this type the symmetry algebra is parameterized by one function of one variable and it is isomorphic to the Lie algebra $Vect(1)$ of vector fields on $\mathbb{C}$.

The Lie algebra $Vect(1)$ does not contain subalgebras of dim ≥ 4, therefore the splitting is impossible.

Question. Which local homogeneous subset Σ of $T\mathbb{R}^n$ with $n = 2, 3$ are induced by some subset of some real n-dimensional complex Lie algebra?

Equivalent question: is it true that the symmetry algebra of Σ has the splitting property?

Thm. For $n = 2$ the only counterexample is the symmetry algebra $(\mathfrak{so}_3, \mathcal{I})$ of a Riemannian metric (field of ellipses) with constant positive curvature.

Here $\mathcal{I}$ denotes a direction (1-dim subspace) in $\mathfrak{so}_3$ corresponding to the isotropy subalgebra (over $\mathbb{R}$ all directions are automorphic so that the choice of $\mathcal{I}$ is irrelevant).

It is a counterexample since $\mathfrak{so}_3$, unlike $\mathfrak{sl}_2$, does not contain 2-dim subalgebras, therefore the splitting is impossible.

For finite dimensional transitive Lie algebras of vector fields it is worth to use the Nagano-Sussmann principle (in combination with geometry and normal forms)

It allows to replace any finite-dim transitive Lie algebra of vector fields by a abstract transitive Lie algebra which is $(\mathcal{A}, \mathcal{I})$ where $\mathcal{A}$ is a Lie algebra and $\mathcal{I}$ is a subalgebra of $\mathcal{A}$ such that $\mathcal{I}$ contains no non-trivial ideals of the whole $\mathcal{A}$.

The subalgebra $\mathcal{I}$ is the isotropy subalgebra of the transitive Lie algebra of vector fields.

Thm. (H. Sussmann) Two transitive Lie algebras $\mathcal{A}_1, \mathcal{A}_2$ of vector fields germs, with isotropy subalgebras $\mathcal{I}_1, \mathcal{I}_2$ are diffeomorphic if and only if $(\mathcal{A}_1, \mathcal{I}_1)$ and $(\mathcal{A}_2, \mathcal{I}_2)$ are isomorphic.

For $T\mathbb{R}^2$ the $(\mathfrak{so}_3, \mathcal{I})$ is the only counterexample. For all other symmetry algebras of local homogeneous subsets of $T\mathbb{R}^2$ we have the splitting property.

Simplest examples

Example: the symmetry algebra of a field of ellipses in $\mathbb{R}^2$ (=Riemannina metric) with negative constant curvature is $(\mathfrak{sl}_2, \mathcal{I}_e)$ where $\mathcal{I}_e$ is elliptic direction. We have the splitting $\mathfrak{sl}_2 = \mathcal{I}_e + (\mathbb{R} \ltimes \mathbb{R})$ where $\mathbb{R} \ltimes \mathbb{R}$ is non-Abelian 2-dim Lie algebra. Therefore:

- a Riemannian metrics on $\mathbb{R}^2$ with constant negative curvature can be identified with an ellipse in non-Abelian 2-dim Lie algebra

(the curvature is the parameter in the classification of ellipses wrt automorphisms)

Example: the symmetry algebra of the field of hyperbolas in $T\mathbb{R}^2$ (=hyperbilic structure) with any non-zero constant curvature is $(\mathfrak{sl}_2, \mathcal{I}_h)$ where $\mathcal{I}_h$ is hyperbolic direction. We have the splitting $\mathfrak{sl}_2 = \mathcal{I}_h + (\mathbb{R} \ltimes \mathbb{R})$ where $\mathbb{R} \ltimes \mathbb{R}$ is non-Abelian 2-dim Lie algebra. Therefore:

• a hyperbolic structure in $T\mathbb{R}^2$ with a constant non-zero curvature can be identified with a hyperbola in non-Abelian 2-dim Lie algebra $\mathcal{A}$

(the curvature is the parameter in the classification of hyperbolas wrt automorphisms; the cases that the hyperbola intersects or not the line $\mathcal{A}^2$ distinguish positive and negative curvature.)

Question. Which local homogeneous subsets $S \subset T\mathbb{R}^3$ with are induced by some subset of some real n-dimensional complex Lie algebra?

Equivalent question: is it true that the symmetry algebra of S has the splitting property?

We have the same counterexample as the only counterexample over $\mathbb{C}$, the symmetry algebra

$$\frac{\partial}{\partial x_1},\ e^{x_1}\frac{\partial}{\partial x_2},\ e^{x_2}f(x_1)\frac{\partial}{\partial x_3}.$$

Other counterexamples?

For $n = 3$ the only other counterexamples are

• one of 6 "natural liftings" from $\mathbb{R}^2$ to $\mathbb{R}^3$ of $(\mathfrak{so}_3, \mathcal{I})$ on $\mathbb{R}^2$

• The Lie algebra $(\mathbb{R} \ltimes \mathfrak{h}_1, \mathcal{I})$ where $\mathfrak{h}_1$ is the Heisenberg Lie algebra, the semi-direct product $\mathbb{R} \ltimes \mathfrak{h}_1$ is defined by a 2×2 matrix with non-real eigenvalues, and the -1dim isotropy subalgebra $\mathcal{I} \subset \mathbb{R} \ltimes \mathfrak{h}_1$ has zero-component in $\mathbb{R}$.

Liftings from $\mathbb{R}^2$ to $\mathbb{R}^3$

Given a Lie algebra $\mathcal{A}$ of vector fields on $\mathbb{R}^2(x_1, x_2)$ define a Lie algebra $\hat{\mathcal{A}}$ as the span of vector fields in $\mathcal{A}$ and the vector fields:

- L-lifting: $h(x_1, x_2, x_3)\frac{\partial}{\partial x_3}$
- L*-lifting: $h(x_1, x_2)\frac{\partial}{\partial x_3}$
- LP-lifting: $h(x_3)\frac{\partial}{\partial x_3}$
- L^*P^*-lifting: $\frac{\partial}{\partial x_3}$
- P-lifting: $h_1(x_3)V$ and $h_2(x_3)\frac{\partial}{\partial x_3}, \quad V \in \mathcal{A}$
- P^*-lifting: $h_1(x_3)V$ and $\frac{\partial}{\partial x_3}$

In P-lifting and P^*-lifting $h_1(x_3)$ does not depend on $V \in \mathcal{A}$

In order to obtain these results I classified, wrt diffeos, all possible symmetry algebras of local homogeneous sybsets of $T\mathbb{K}^n$ with $n = 2, 3, \ \ \mathbb{K} = \mathbb{C}, \mathbb{R}$.

We have:

1. Symmetry algebras of local homogeneous subsets of $\mathbb{K}^n$
2. Transitive Lie algebras of vector fields on $\mathbb{K}^n$
3. Lie algebras of vector fields on $\mathbb{K}^n$.

Certainly $1 \subset 2 \subset 3$.

Classification of 3.:
for $n = 2$ available:
$\mathbb{K} = \mathbb{C}$: S. Lie
$\mathbb{K} = \mathbb{R}$: S.Lie + many revisions, the last in 1990 by Gonzales-Lopez, Kamran, Olver.

for $n = 3$: probably not doable

1. Symmetry algebras of local homogeneous subsets of $\mathbb{K}^n$
2. Transitive Lie algebras of vector fields on $\mathbb{K}^n$
3. Lie algebras of vector fields on $\mathbb{K}^n$.

Certainly $1 \subset 2 \subset 3$.

Classification of 2. (which is a small part of 3.):
for $n = 3$: probably not available, probably doable, around 1000 normal forms

Claim. 1. is much less of 2. (for $n = 3$: around $1/5$ of 2.)

The first reason why 1. is much less than 2

Prop. If two transitive Lie algebras $\mathcal{A}_1 \subset \mathcal{A}_2$ have the same central part then $\mathcal{A}_1$ cannot be the symmetry algebra of any local homogeneous subset of the tangent bundle: if the infinitesimal symmetries include $\mathcal{A}_1$ they also include $\mathcal{A}_2$.

The second reason why 1. is much less than 2

• In the proposition above the words have the same central part can be replaced by have the central parts with the same geometry

Central parts with the same geometry

In algebraic terms, the central part of a transitive Lie algebra of vector fields on $\mathbb{K}^n$ is a representation in $\mathfrak{gl}_n$ of an arbitrary Lie algebra which admits such representation.

Equivalently, it is a Lie algebra of linear vector fields V on $T_0\mathbb{K}^n$, $V(0) = 0$.

The central parts $\mathcal{C}_1 \subset \mathcal{C}_2$ have the same geometry if any $\mathcal{C}_1$-invariant subset of $T_0\mathbb{K}^n$ is also $\mathcal{C}_2$-invariant.

Trivial example:
$\mathfrak{sl}_n \subset \mathfrak{gl}_n$ and $\mathfrak{gl}_n$ have the same geometry.

Example: fix $\lambda_1, \lambda_2 \neq 0$.

$$span \left\{ \begin{pmatrix} * & * & 0 \\ 0 & 0 & 0 \\ 0 & * & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \right\} \text{same geometry as } \begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & * & * \end{pmatrix}$$

There is a LOT of other examples.

Finite-dim symmetry algebras of dim ≥ 3 of local homogeneous subset of $T\mathbb{R}^2$:

• translation symmetry algebra:

$$span\left\{\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial x_2},(a_{11}x_1+a_{12}x_2)\frac{\partial}{\partial x_1}+(a_{11}x_1+a_{12}x_2)\frac{\partial}{\partial x_2}\right\}$$

where (a_{ij}) is a non-singular matrix

• symmetry algebra of a Riemannian metric (field of ellipses) with constant negative curvature $(sl_2, \mathcal{I}_e)$

• symmetry algebra of a field of hyperbolas with constant non-zero curvature $(sl_2, \mathcal{I}_h)$

• symmetry algebra of a Riemannian metric (field of ellipses) with constant positive curvature $(so_3, \mathcal{I})$

Infinite-dim symmetry algebras of dim ≥ 3 of local homogeneous subset of $T\mathbb{R}^2$:

a symmetry algebra of:

- a line field: $f(x_1)\frac{\partial}{\partial x_1} + g(x_1, x_2)\frac{\partial}{\partial x_2}$
- a vector field: $f(x_1)\frac{\partial}{\partial x_1} + g(x_1)\frac{\partial}{\partial x_2}$
- integrable affine line field: $c\frac{\partial}{\partial x_1} + f(x_1, x_2)\frac{\partial}{\partial x_2}, \ \ c \in \mathbb{R}$
- non-integrable affine line field: $f(x_1)\frac{\partial}{\partial x_1} + f'(x_1)\frac{\partial}{\partial x_2}$
- two transversal line fields: $f(x_1)\frac{\partial}{\partial x_1} + g(x_2)\frac{\partial}{\partial x_2}$
- integrable affine line field and a vector field parallel to it: $f(x_1)\frac{\partial}{\partial x_1} + c\frac{\partial}{\partial x_2}, \ \ c \in \mathbb{R}$
- integrable affine line field and a vector field transversal to it ($=$ integrable field of affine semi-lines): $f(x_2)\frac{\partial}{\partial x_1} + c\frac{\partial}{\partial x_2}, \ c \in \mathbb{R}$

Examples of symmetry algebras of local homogeneous subsets of $T\mathbb{R}^3$

1. $f(x_1)\frac{\partial}{\partial x_1} + f'(x_1)\frac{\partial}{\partial x_2} + (f''(x_1) - x_3 f'(x_1))\,\frac{\partial}{\partial x_3}$

2. $f(x_1)\frac{\partial}{\partial x_1} + (f''(x_1) + x_2 f'(x_1))\,\frac{\partial}{\partial x_2} +$
$+ (f'''(x_1) + x_2 f''(x_1) + 2f'(x_1))\,\frac{\partial}{\partial x_3}$

Both 1. and 2. are isomorphic to $Vect(1)$, the Lie algebra of vector fields on $\mathbb{R}$, but, after applying the isomorphisms, the isotropy subalgebra are as follows:

for 1. $\mathcal{I}_1 = \left\{ f(x)\frac{\partial}{\partial x} :\ f(0) = f'(0) = f''(0) = 0 \right\}$

for 2. $\mathcal{I}_2 = \left\{ f(x)\frac{\partial}{\partial x} :\ f(0) = f''(0) = f'''(0) = 0 \right\}$

and $(Vect(1), \mathcal{I}_1)$ and $(Vect(1), \mathcal{I}_2)$ are not isomorphic.

1. $f(x_1)\frac{\partial}{\partial x_1} + f'(x_1)\frac{\partial}{\partial x_2} + \left(f''(x_1) - x_3 f'(x_1)\right)\frac{\partial}{\partial x_3}$

2. $f(x_1)\frac{\partial}{\partial x_1} + \left(f''(x_1) + x_2 f'(x_1)\right)\frac{\partial}{\partial x_2} +$
$+ \left(f'''(x_1) + x_2 f''(x_1) + 2f'(x_1)\right)\frac{\partial}{\partial x_3}$

The way to obtain these and other ∞-dim symmetry algebras is to classify central parts and to work with invariant objects defined by the central part.

Central part of 1.: $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & 0 & 0 \end{pmatrix}$, of 2.: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ * & 0 & -2 \end{pmatrix}$

1. is the symmetry algebra of one of two types of homogeneous contact affine line field.

2. is the symmetry algebra of a couple of a contact plane field and transversal line field, one of non-flat types:
$dx_2 - x_3 dx_1 = 0,\ \left(\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_3}\right)$.

The classification of symmetry algebras
of local homogeneous subsets of $T\mathbb{R}^3$ required:

Classification of all realizable central parts
(realizable representations of Lie algebras in $\mathfrak{gl}_3$)

(a) without rank 1 matrices

(b) containing rank 1 matrices

In the case (a) the central part is either not prolongable or has a finite-dim prolongation
(proved by Ottazi and Waphurst, 2009)
and consequently the symmetry algebra is finite-dimensional.

Here prolongation of the central part: classical-before-Tanaka, the Singer-Sternberg prolongation.

Example. The prolongation of the representation of $\mathfrak{gl}_2$ in $\mathfrak{gl}_3$ given by

$$\begin{pmatrix} a & c & 0 \\ d & \frac{a+b}{2} & c \\ 0 & d & b \end{pmatrix}$$

is

$$(\mathfrak{so}_{3,2},\ \mathcal{I})$$

which has the biggest dimension 10 within finite-dim symmetry algebras of local homogeneous subsets of $T\mathbb{R}^3$.

This subset of $T\mathbb{R}^3$ is the cone $x_1^2 + x_2^2 - x_3^2 = 0$ in $T_0\mathbb{R}^3$ translated to $T\mathbb{R}^3$ by $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}$.

The fact that the symmetry algebra of this field of cones is $\mathfrak{so}_{3,2}$ is classical. The complexification of the symmetry algebra is diffeomorphic to the complexification of the symmetry algebra of flat conformal structure on $\mathbb{R}^3$.

Required:

Computation of the prolongations and geometric interpretation of results.

Solving non-trivial problems
(requiring Cartan method or a posteriori equivalent tools)
on classification of homogeneous couples or triples including:

- two contact plane fields
- contact plane field and transversal line field
- vector field in a contact plane field

the classical results on the classification of homogeneous couples consisting of two line fields which span a contact plane field = classification of ODEs $y'' = f(x, y, y')$.

Required:

Liftings from $\mathbb{K}^2$ to $\mathbb{K}^3$ of transitive Lie algebras. On top of $L, L^*, L^*P^*, LP, P, P^*$ liftings: contact LP lifting which gives a subalgebra of the symmetry algebra of a couple consisting of a contact plane field and a transversal line field.

Example.

The symmetry algebra of the couple of two line fields which span a contact plane field, the "flat" case, is

$$(\mathfrak{sl}_3, \mathfrak{h}_1): \ \mathfrak{sl}_3 = \begin{pmatrix} a & * & * \\ * & b & * \\ * & * & -(a+b) \end{pmatrix}, \quad \mathfrak{h}_1 = \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{pmatrix}$$

and it is the contact LP lifting of

$$\mathfrak{sl}_3 = span\Big\{ \frac{\partial}{\partial x_1}, \ \frac{\partial}{\partial x_2}, \ x_1\frac{\partial}{\partial x_1}, \ x_1\frac{\partial}{\partial x_2}, \ x_2\frac{\partial}{\partial x_1}, \ x_2\frac{\partial}{\partial x_2}, \\ x_1\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}\right), \ x_2\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}\right) \Big\}.$$

which is the Lie algebra of the group of projective transformations.

Required:

Distinguishing finite-dim and infinite-dim symmetry algebras.

The symmetry algebras whose central parts contain no rank 1 matrices are finite-dim. I had to distinguishing the cases that the symmetry algebra whose central part contains rank 1 matrices is finite-dim or infinite-dim.

Thm. A symmetry algebra of local homogeneous subset of $T\mathbb{R}^3$ whose central part contains rank 1 matrices is finite dimensional if and only if it is a subalgebra of the symmetry algebra of "flat" couple of two line fields in $T\mathbb{R}^3$ which span a contact plane field: the symmetry algebra $(\mathfrak{sl}_3, \mathfrak{h}_1)$ given in the previous page.